GENERATING d-COMPOSITE SANDWICH NUMBERS

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Abstract

Let $d \in \mathcal{D} = \{1, \ldots, 9\}$, and let k be a positive integer with $\gcd(k, 10d) = 1$. Define a sequence $\{s_n(k, d)\}_{n=1}^{\infty}$ by

$$s_n(k,d) := k\underbrace{dd\ldots d}_{n}k.$$

We say k is a *d-composite sandwich number* if $s_n(k, d)$ is composite for all $n \geq 1$. For a d-composite sandwich number k, we say k is *trivial* if $s_n(k, d)$ is divisible by the same prime for all $n \geq 1$, and *nontrivial* otherwise. In this paper, we develop a simple criterion to determine when a d-composite sandwich number is nontrivial, and we use it to establish many results concerning which types of integers can be d-composite sandwich numbers. For example, we prove that there exist infinitely many primes that are simultaneously trivial d-composite sandwich numbers for all $d \in \mathcal{D}$. We also show that there exist infinitely many positive integers that are simultaneously nontrivial d-composite sandwich numbers for all $d \in D$, where $D \subset \mathcal{D}$ with $|D| = 4$ and $D \neq \{3, 6, 7, 9\}$.

1. Introduction

In [22], the first author proved that for any given fixed digit $d \in \{1, 3, 5, 7\}$, there exist infinitely many positive integers k, such that $\gcd(k, d) = 1$ and every integer in the sequence

$$kd, \quad kdd, \quad kddd, \quad kdddd, \quad kddddd, \quad \ldots,$$

is composite. Other authors [16, 23, 24] have handled modifications of this appending-digits problem. An inserting-digits problem was treated in [13]. In this paper, we investigate the following variation. Let $d \in \mathcal{D} = \{1, \ldots, 9\}$, and let k be a positive integer with $\gcd(k, 10d) = 1$. Define a sequence $\{s_n(k,d)\}_{n=1}^{\infty}$ by

$$s_n(k,d) := k\underbrace{dd\ldots d}_{n}k. \tag{1.1}$$

We say k is a *d-composite sandwich number* if $s_n(k,d)$ is composite for all $n \geq 1$. If there exists a prime p such that $s_n(k,d) \equiv 0 \pmod{p}$ for all $n \geq 1$, we say k is *trivial*, otherwise we say k is *nontrivial*. The restriction that $\gcd(k, 10d) = 1$ has been imposed to discard obvious trivial situations. If $d = 0$, we observe that $\gcd(k, 10d) = k$, so that $d = 0$ is ruled out except possibly in the case of $k = 1$. However, since $s_1(1, 0) = 101$ is prime, we see that $k = 1$ is not a 0-composite sandwich number, and we can exclude the digit $d = 0$ from all consideration. Therefore, we assume throughout this article that $d \in \mathcal{D}$ and k is a positive integer with $\gcd(k, 10d) = 1$. We prove results concerning what types of integers can be d-composite sandwich numbers, and we consider both trivial and nontrivial situations. In particular, we prove the following.

Theorem 1.1. *For any $d \in \mathcal{D}$, there exist infinitely many primes that are trivial d-composite sandwich numbers.*

Theorem 1.2. *There exist infinitely many primes q that are simultaneously trivial d-composite sandwich numbers for all $d \in \mathcal{D}$.*

Theorem 1.3. *For any $d \in \mathcal{D}$, there exist infinitely many nontrivial d-composite sandwich numbers.*

Theorem 1.4. *There exist infinitely many positive integers that are simultaneously nontrivial d-composite sandwich numbers for all $d \in D$, where $D \subset \mathcal{D}$ with $|D| = 4$ and $D \neq \{3, 6, 7, 9\}$.*

Theorem 1.5. *For any $d \in \mathcal{D}$, there are infinitely many sets of 13 consecutive positive integers that are all d-composite sandwich numbers.*

Theorem 1.6. *For any $d \in \mathcal{D}$, there are infinitely many positive integers k such that k^2 is simultaneously a d-composite sandwich number, a Sierpiński number and a Riesel number.*

Remark 1.7. Although Theorem 1.1 follows directly from Theorem 1.2, we nevertheless treat it independently since we find the smallest trivial d-composite sandwich number for each individual digit $d \in \mathcal{D}$.

2. Preliminaries

This section contains some basic definitions and concepts that are useful in this paper. Other preliminary concepts that are needed for the proof of only one theorem are presented in the appropriate section.

Definition 2.1. Let $a > 1$ be an integer. A prime divisor p of $a^n - 1$ is called a *primitive divisor* of $a^n - 1$ if $a^m \not\equiv 1 \pmod{p}$ for all positive integers $m < n$.

The following theorem concerning the existence of primitive divisors is due to Bang [1].

Theorem 2.2. *Let a and n be positive integers with $a \geq 2$. Then $a^n - 1$ has a primitive divisor with the following exceptions:*

- *$a = 2$ and $n = 6$*
- *$a + 1$ is a power of 2 and $n = 2$.*

In terms of group theory, the prime p is a primitive divisor of $a^n - 1$ if and only if n is the order of a in the group of units modulo p. We denote this order as $\mathrm{ord}_p(a)$.

The following concept, which is due to Erdős [9], plays an essential role in the proofs of many of our results.

Definition 2.3. A *(finite) covering system*, or simply a *covering*, of the integers is a system of congruences $x \equiv a_i \pmod{m_i}$, with $1 \leq i \leq t$ such that every integer n satisfies at least one of the congruences. To avoid a trivial situation, we require $m_i > 1$ for all i. We let $\mathcal{M} = \mathrm{lcm}_i(m_i)$ for all moduli m_i in a covering.

Many applications of coverings require an associated set of primes, where each of these primes corresponds in some way to a particular modulus in the covering. It will be convenient throughout this article to represent a covering and the associated set of primes using a set $\mathcal{C}$ of ordered triples (a_i, m_i, p_i) (or simply ordered pairs (a_i, m_i) if the primes p_i are too large to display conveniently), where $x \equiv a_i \pmod{m_i}$ is a congruence in the covering and p_i is the corresponding prime. When a covering is used for a proof of a theorem in this article, the correspondence between the prime p_i and the modulus m_i is that p_i is either a primitive divisor of $10^{m_i} - 1$ (with the exception of $p_i = 3$), or that p_i is a primitive divisor of $2^{m_i} - 1$ for all i in the covering. For certain moduli m_i, the numbers $10^{m_i} - 1$ and $2^{m_i} - 1$ have more than one primitive divisor. In those cases, the corresponding modulus m_i can be used repeatedly in the covering – once for each primitive divisor. Abusing notation slightly, we refer to $\mathcal{C}$ as a "covering".

For a positive integer k, we let $\ell(k)$ denote the number of digits in the decimal representation of k. In this paper, we are concerned with the sequence $\{s_n(k,d)\}_{n=1}^{\infty}$ defined in (1.1). It will be convenient to use the following easily-derived formula for $s_n(k,d)$:

$$s_n(k,d) = k\left(10^{\ell(k)+n} + 1\right) + d \cdot 10^{\ell(k)}\left(\frac{10^n - 1}{9}\right). \tag{2.1}$$

Computer computations in this paper were performed by the authors using Maple and MAGMA.

3. Trivial Situations

Recall that k is a trivial d-composite sandwich number if there exists a prime p such that $s_n(k,d) \equiv 0 \pmod{p}$ for all $n \geq 1$. It may not be immediately apparent that such numbers even exist, but in fact, we shall see that they are quite abundant. In this section we determine necessary and sufficient conditions on k, such that k is a trivial d-composite sandwich number.

Theorem 3.1. *Let $d \in \mathcal{D}$ and let $k \geq 1$ be an integer such that* $\gcd(k, 10d) = 1$*. Then k is a trivial d-composite sandwich number if and only if* $\gcd(9k+d, 10^{\ell(k)}+1) > 1$.

Proof. Assume first that k is a trivial d-composite sandwich number. Then there exists a prime p such that $s_n(k,d) \equiv 0 \pmod{p}$ for all $n \geq 1$. Note that if $p = 3$, then

$$s_n(k,d) = k\left(10^{n+\ell(k)}+1\right)+d\left(10^{n-1}+10^{n-2}+\cdots+1\right)10^{\ell(k)} \equiv 2k+nd \pmod{3}.$$

Hence, $s_n(k,d) \equiv 0 \pmod 3$ for all $n \geq 1$ if and only if $\gcd(k,d) \equiv 0 \pmod 3$, which we have excluded from consideration here. Thus, $p \geq 7$, since the condition $\gcd(k, 10d) = 1$ also excludes the possibility that $p = 2$ or $p = 5$. Since $s_n(k,d) \equiv 0 \pmod{p}$, we have that

$$0 \equiv 9 \cdot s_n(k,d) \equiv A \cdot 10^n + B \pmod{p},$$

where

$$A \equiv (9k+d)\,10^{\ell(k)} \pmod{p} \quad \text{and} \quad B \equiv 9k - d \cdot 10^{\ell(k)} \pmod{p}. \tag{3.1}$$

Since $s_1(k,d) \equiv s_2(k,d) \equiv 0 \pmod{p}$, we deduce that $90A \equiv 0 \pmod{p}$. Since $p \geq 7$, it follows that $A \equiv B \equiv 0 \pmod{p}$, and so $9k+d \equiv 0 \pmod{p}$. Also, solving the second congruence in (3.1) for $9k$ and substituting into the first congruence in (3.1) gives

$$d \cdot 10^{\ell(k)}\left(10^{\ell(k)}+1\right) \equiv 0 \pmod{p},$$

which implies that $10^{\ell(k)}+1 \equiv 0 \pmod{p}$, unless $p = d = 7$. But in this case, since $A \equiv 0 \pmod 7$, we have that $k \equiv 0 \pmod 7$, which we have excluded. Thus, $\gcd(9k+d, 10^{\ell(k)}+1) \equiv 0 \pmod{p}$.

Conversely, suppose that p is a prime such that $\gcd(9k+d, 10^{\ell(k)}+1) \equiv 0 \pmod{p}$. Then, in (3.1), we have $A \equiv 0 \pmod{p}$, and since $10^{\ell(k)} \equiv -1 \pmod{p}$, we also get that

$$B \equiv 9k+d \equiv 0 \pmod{p}.$$

Therefore,

$$9 \cdot s_n(k,d) \equiv A \cdot 10^n + B \equiv 0 \pmod{p},$$

and since $p \geq 7$, we conclude that $s_n(k,d) \equiv 0 \pmod{p}$ for all $n \geq 1$, and the proof is complete. □

Example 3.2. Let $k = 260487394697203$ and $d = 2$. Then $\ell(k) = 15$, $\gcd(k, 10d) = 1$ and

$$\gcd(9 \cdot 260487394697203 + 2, 10^{15} + 1) = 211,$$

so that $s_n(k,2) \equiv 0 \pmod{211}$ for all $n \geq 1$. Hence, k is a trivial 2-composite sandwich number.

Remark 3.3. We caution the reader that if $\gcd(9k + d, 10^{\ell(k)} + 1) = 1$, then it does not follow that k is a nontrivial d-composite sandwich number, since not all integers are composite sandwich numbers. However, this condition can be used to detect if a known d-composite sandwich number is trivial or not. We apply this condition in Section 4.

3.1. Theorem 1.1

To prove Theorem 1.1, we first need some preliminary results. The first result, which we state without proof, is a well-known version of the prime number theorem for arithmetic progressions [27]. We use the standard Landau little-o notation.

Theorem 3.4. *Let* $\gcd(a,m) = 1$ *and let* $\pi(x;m,a)$ *be the number of primes* $p \leq x$ *such that* $p \equiv a \pmod{m}$. *Then*

$$\pi(x;m,a) = \frac{x\,(1 + o(1))}{\phi(m)\log x}.$$

The following corollary is immediate from Theorem 3.4.

Corollary 3.5. *Let* $f(z) > 1$ *and* $g(z) > 1$ *be strictly increasing functions with* $f(z) < g(z)$ *for all sufficiently large* z. *Then the number of primes* p *with* $f(z) < p \leq g(z)$ *such that* $p \equiv a \pmod{m}$ *is*

$$\pi\left(g(z);m,a\right) - \pi\left(f(z);m,a\right) = \frac{(1 + o(1))}{\phi(m)}\left(\frac{g(z)}{\log g(z)} - \frac{f(z)}{\log f(z)}\right).$$

We need the following lemma.

Lemma 3.6. *Let* x *be a positive integer and let* p *be a prime such that* $10^x + 1 \equiv 0 \pmod{p}$. *If* y *is an integer with* $1 \leq y \leq p-1$, *then there exist nonnegative integers* N *and* z *such that* $k = pN + y$ *is odd with* $\ell(k) = (2z+1)x$.

Proof. Note that $\ell(p) \leq x + 1$. Let $z \geq p$ and let $w = (2z+1)x - \ell(p)$. Then $w > \ell(p)$. If y is odd, let $N = 10^w$, and if y is even, let $N = 10^w + 1$. Let $k = pN + y$. Then k is odd and

$$\ell(k) = \ell(pN + y) = \ell(p) + w = (2z+1)x.$$

□

3.1.1. Proof of Theorem 1.1

Let $p \geq 7$ be a prime such that $\operatorname{ord}_p(10) \equiv 0 \pmod 2$. For the sake of brevity of notation, we let $x = \operatorname{ord}_p(10)/2$. Then

$$10^x + 1 \equiv 0 \pmod p.$$

Let $y \leq p-1$ be a positive integer such that $y \equiv -d/9 \pmod p$. By Lemma 3.6, there exists a nonnegative integer z and an odd positive integer k such that $k \equiv y \pmod p$ and $\ell(k) = (2z+1)x$. Apply Corollary 3.5 with $a = y$, $m = p$, $f(z) = 10^{(2z+1)x-1}$ and $g(z) = 10^{(2z+1)x}$ to get that

$$\lim_{z\to\infty} \left(\pi\left(g(z); p, y\right) - \pi\left(f(z); p, y\right)\right) = \infty.$$

Thus, for any sufficiently large integer z, there is a prime q such that $q \equiv y \equiv -d/9 \pmod p$ and $\ell(q) = (2z+1)x$. Then $\gcd(9q+d, 10^{\ell(q)}+1) \equiv 0 \pmod p$, and hence, by Theorem 3.1, it follows that q is a trivial d-composite sandwich number, and the proof of Theorem 1.1 is complete.

For each $d \in \mathcal{D}$, we use Theorem 3.1 to give in Table 1 the smallest prime q that is a trivial d-composite number with $\gcd(q, 10d) = 1$. The primes p such that $s_n(q, d) \equiv 0 \pmod p$ for all $n \geq 1$ are also given.

d	q	p
1	101	7, 13
2	11	101
3	7	11
4	101	11
5	107	11
6	109	7
7	89	101
8	101	7
9	103	13

Table 1: Smallest Trivial d-Composite Sandwich Primes q with $\gcd(q, 10d) = 1$

3.2. Proof of Theorem 1.2

Suppose that m is a positive integer such that $10^m + 1$ has at least 9 distinct prime factors $p_1, p_2, \ldots, p_9$. Using the Chinese remainder theorem, we solve the system of congruences

$$k \equiv -d/9 \pmod{p_d}.$$

Thus, for sufficiently large such m, by Lemma 3.6 and Corollary 3.5 (as in the proof of Theorem 1.1), there is a prime q in the resulting arithmetic progression such that

$\ell(q) = m$. Since $\gcd(q, 10d) = 1$ and $\gcd\left(9q + d, 10^{\ell(q)} + 1\right) = p_d$, it follows from Theorem 3.1 that q is a trivial d-composite sandwich prime for all $d \in \mathcal{D}$.

We illustrate the techniques with an example.

Example 3.7. The smallest value of m such that $10^m + 1$ has at least 9 distinct prime factors is $m = 39$. These distinct primes are

$$P = [1058313049, 859, 7, 6397, 157, 388847808493, 13, 11, 216451]. \tag{3.2}$$

Using the Chinese remainder theorem, we solve the system of congruences

$$k \equiv -d/9 \pmod{P[d]}, \qquad d \in \mathcal{D},$$

to get

$$k \equiv 230951458321744873361409944253643808822 \pmod{\prod_d P[d]}.$$

The smallest prime in this arithmetic progression with 39 digits is

$$q = 100018222755251410413064071348441303899.$$

Since $\gcd(q, 10d) = 1$ and $\gcd\left(9q + d, 10^{\ell(q)} + 1\right) = P[d]$, it follows from Theorem 3.1 that q is a trivial d-composite sandwich number for all $d \in \mathcal{D}$.

Remark 3.8. The particular order of the primes in the list (3.2) gives the smallest prime q that is simultaneously a trivial d-composite sandwich number for all $d \in \mathcal{D}$.

4. Nontrivial Situations

Recall that nontrivial d-composite sandwich numbers are positive integers k such $s_n(k, d)$ is composite for all $n \geq 1$, but not every term $s_n(k, d)$ is divisible by the same prime. We use a covering to find such numbers. Given $d \in \mathcal{D}$, we want to construct a covering $\mathcal{C} = \mathcal{C}(d)$, such that for each triple $(a_i, m_i, p_i) \in \mathcal{C}$, we have that $s_n(k, d) \equiv 0 \pmod{p_i}$ whenever $n \equiv a_i \pmod{m_i}$. Here the primes p_i, with the exception of $p_i = 3$, are chosen to be primitive divisors of $10^{m_i} - 1$ (but we still assume that 3 is a divisor of $(10^{m_i} - 1)/9$ if $p_i = 3$). To find a number k that satisfies these conditions, we first set $s_n(k, d)$ congruent to 0 modulo p_i and solve (2.1) for k modulo p_i for each i, noting that $10^n \equiv 10^{a_i} \pmod{p_i}$. Then we use the Chinese remainder theorem to piece together this information and get an infinite arithmetic progression of such values of k that simultaneously solve all congruences $s_n(k, d) \equiv 0 \pmod{p_i}$. Two additional congruences

$$k \equiv 1 \pmod 2 \quad \text{and} \quad k \equiv z \pmod 5, \quad \text{where } z \not\equiv 0 \pmod 5, \tag{4.1}$$

can be added to the system, if necessary, to ensure that $\gcd(k, 10) = 1$. Note that these two new congruences do not conflict with any congruences in $\mathcal{C}$ since 2 and 5 are never primitive divisors of numbers of the form $10^z - 1$. Also, it is usually easy to guarantee that $\gcd(k, d) = 1$ by adding additional congruences for k if necessary, or by choosing appropriate residues associated with the primes 3 and 7, if they appear in $\mathcal{C}$. Once k is found, we can easily check that $\gcd(k, 10d) = 1$, and then use Theorem 3.1 to determine if k is nontrivial.

Certain criteria must be satisfied to accomplish this task. To begin, we set $s_n(k, d) = 0$ and formally solve for k in (2.1) to get

$$k = -\frac{d \cdot 10^{\ell(k)} \left(10^n - 1\right)}{9 \cdot \left(10^{\ell(k)+n} + 1\right)}. \tag{4.2}$$

However, modulo p_i there are two difficulties here: (4.2) makes no sense if $p_i = 3$ or if $10^{\ell(k)+n} + 1 \equiv 0 \pmod{p_i}$.

The first difficulty is easily overcome with a modest price. If $p_i = 3$, then we can expand $\left(10^n - 1\right)/9$ to get

$$\frac{10^n - 1}{9} = 10^{n-1} + 10^{n-2} + \cdots + 10 + 1 \equiv n \pmod{3}.$$

We would like to replace n here with a_i since $n \equiv a_i \pmod{m_i}$ in $\mathcal{C}$. To do this, we must have that $m_i \equiv 0 \pmod{3}$ to guarantee that 3 is a divisor of $\left(10^{m_i} - 1\right)/9$. In other words, if $p_i = 3$ in $\mathcal{C}$, then $m_i \equiv 0 \pmod{3}$. Hence, since $10^{\ell(k)+a_i} + 1 \equiv 2 \not\equiv 0 \pmod{3}$, we can rewrite (4.2) modulo p_i, when $p_i = 3$, as

$$k \equiv -\frac{d \cdot 10^{\ell(k)} a_i}{10^{\ell(k)+a_i} + 1} \equiv d a_i \pmod{3}. \tag{4.3}$$

The second difficulty is slightly more annoying. Here we have $p_i \neq 3$ and we can reduce (4.2) modulo p_i to

$$k \equiv -\frac{d \cdot 10^{\ell(k)} \left(10^{a_i} - 1\right)}{9 \cdot \left(10^{\ell(k)+a_i} + 1\right)} \pmod{p_i}, \tag{4.4}$$

which makes sense provided $10^{\ell(k)+a_i} + 1 \not\equiv 0 \pmod{p_i}$. Since we are looking for k, we obviously do not know the value of $\ell(k)$. But, using a suitable specific value x in place of $\ell(k)$, we can construct a corresponding value of k. A "suitable" x is a number that satisfies two conditions. The first condition is that $10^{x+a_i} + 1$ must be invertible modulo p_i for each i. The second condition we require is that $x \geq \ell(\mathcal{P})$, where $\mathcal{P} = 10 \prod_{i=1}^{t} p_i$ and t is the total number of elements in $\mathcal{C}$. The factor of 10 here arises from the fact that we have added the two additional congruences for k modulo 2 and 5. This second condition can be achieved since if one value of x exists that satisfies the first condition, then there exist infinitely many values of x satisfying the first condition. This follows from the fact that

$$10^{x+a_i} + 1 \equiv 10^{x+s\mathcal{M}+a_i} + 1 \pmod{p_i},$$

for any positive integer s. Then, since $x \geq \ell(\mathcal{P})$, we can "jack-up" the value of k in the arithmetic progression we get from solving the system of congruences for k using the Chinese remainder theorem by adding multiples of $\mathcal{P}$, and hence produce a value of k with exactly $\ell(k) = x$. Moreover, if $k \equiv z \pmod{\mathcal{P}}$, where $0 \leq z \leq \mathcal{P}-1$, and we choose $x \geq 2\ell(\mathcal{P})$, then we can choose $k = 10^{x-\ell(\mathcal{P})}\mathcal{P} + z$. Observe then that $\ell(k) = x$. Hence, we have

$$k = 10^{\ell(k)-\ell(\mathcal{P})}\mathcal{P} + z. \tag{4.5}$$

In many situations in this article, it is convenient to restrict our attention to values of k in the form of (4.5). Note that, in practice, it is not always necessary to choose a value of $x \geq 2\ell(\mathcal{P})$ to achieve the form (4.5).

Finally, in order to verify that we have not created a trivial situation, we use Theorem 3.1 and check that $\gcd(9k + d, 10^{\ell(k)} + 1) = 1$. We illustrate this method with an example.

Example 4.1. Let $d = 1$ and let $\mathcal{C} = \{(2, 3, 3), (1, 3, 37), (0, 6, 7), (3, 6, 13)\}$. Let $x = 7$. Then $x \geq \ell(\mathcal{P}) = \ell\left(10\prod_i p_i\right) = \ell(101010) = 6$, and it is straightforward to check that $10^{x+a_i} + 1 \not\equiv 0 \pmod{p_i}$, for all i. Using (4.3) and (4.4), we get that the resulting system of congruences for k is:

$$\begin{aligned} k &\equiv 2 \pmod{3} \\ k &\equiv 1 \pmod{37} \\ k &\equiv 0 \pmod{7} \\ k &\equiv 2 \pmod{13} \\ k &\equiv 1 \pmod{2} \\ k &\equiv 1 \pmod{5}. \end{aligned} \tag{4.6}$$

Using the Chinese remainder theorem to solve (4.6) gives $k \equiv 85841 \pmod{101010}$. Then $k = 10 \cdot 101010 + 85841 = 1095941$ is in this arithmetic progression and $\ell(k) = 7$. Clearly, $\gcd(k, 10) = 1$, and using a computer we verify that $\gcd(9k + 1, 10^{\ell(k)} + 1) = 1$. Hence, by Theorem 3.1, we conclude that k is a nontrivial 1-composite sandwich number.

While it is true that the infinitely many values of x described in the discussion prior to Example 4.1 give rise to infinitely many d-composite sandwich numbers k, it is not true that k is necessarily nontrivial. In Example 4.1, we chose to let $x = 7$. Since $\mathcal{M} = 6$, every value of $x \equiv 1 \pmod{6}$, with $x \geq 7$, will generate 1-composite sandwich numbers k, although k may be trivial. For example, if $x = 13$ and $k = 10000371 \cdot 101010 + 85841 = 1010137560551$, then $\ell(k) = 13$ and k is a 1-composite sandwich number. However, k is trivial by Theorem 3.1 since $\gcd(k, 10) = 1$ and $\gcd(9k + 1, 10^{\ell(k)} + 1) = 859$. Note that this particular value of k is not in the form (4.5); and indeed every value of k in the form (4.5) with $\ell(k) \equiv 1 \pmod{6}$ is nontrivial. To see this, we let $k = 10^{\ell(k)-6} \cdot 101010 + 85841$ and we suppose that q is a prime divisor of $\gcd(9k+1, 10^{\ell(k)}+1)$. Since $10^{\ell(k)} \equiv -1 \pmod{q}$ and $9k+1 \equiv 0 \pmod{q}$, we deduce that q is a prime divisor of

$$9 \cdot 10^6 \cdot 85841 + 10^6 - 9 \cdot 101010 = 772569090910 = 2 \cdot 5 \cdot 77256909091.$$

Clearly, $q \notin \{2, 5\}$, and so $q = 77256909091$. But then $2\ell(k) = \mathrm{ord}_q(10) = 77256909090$ implies that

$$\ell(k) = 77256909090/2 = 38628454545 \equiv 0 \pmod{3},$$

which contradicts the fact that $\ell(k) \equiv 1 \pmod{6}$.

The question still remains as to whether a suitable value of x can always be found to generate d-composite sandwich numbers using the method described in this section. We provide conditions on $\mathcal{C}$ to guarantee the existence of such an x, but first we need a lemma.

Lemma 4.2. *Let x, a and m be positive integers with $m \equiv 0 \pmod{2}$. Let $\widehat{m} = m/2$, and let p be a primitive divisor of $10^m - 1$. Then*

$$10^{x+a} + 1 \equiv 0 \pmod{p} \Longleftrightarrow x \equiv \widehat{m} - a \pmod{m}.$$

Proof. Suppose first that $10^{x+a} + 1 \equiv 0 \pmod{p}$. Then $10^{2(x+a)} \equiv 1 \pmod{p}$ so that $m = \mathrm{ord}_p(10)$ divides $2(x+a)$. Thus, $x + a \equiv 0 \pmod{\widehat{m}}$. Note that if $x + a \equiv 0 \pmod{m}$, then $0 \equiv 10^{x+a} + 1 \equiv 2 \pmod{p}$, which is impossible since p is odd. Hence, $x + a \not\equiv 0 \pmod{m}$. Therefore, there is an integer b such that

$$x + a = \frac{(2b+1)m}{2} = bm + \widehat{m} \equiv \widehat{m} \pmod{m}.$$

Conversely, suppose that $x \equiv \widehat{m} - a \pmod{m}$. Then $2(x+a) \equiv 0 \pmod{m}$, which implies that $(10^{x+a} - 1)(10^{x+a} + 1) = 10^{2(x+a)} - 1 \equiv 0 \pmod{p}$. Hence, $10^{x+a} + 1 \equiv 0 \pmod{p}$ since $x + a \not\equiv 0 \pmod{m}$. □

Proposition 4.3. *Let $\mathcal{C} = \{(a_i, m_i, p_i)\}$ be a covering with exactly t congruences such that p_i is a primitive divisor of $10^{m_i} - 1$ for each $p_i \neq 3$. Relabel, if necessary, so that $m_1, \ldots, m_s$ are even, where $s \leq t$. Let $\widehat{m}_i = m_i/2$ for each i with $1 \leq i \leq s$. If there exists $y \in \mathbb{Z}$ such that*

$$\widehat{m}_i - a_i \not\equiv y \pmod{m_i} \quad \text{for all } i \text{ with } 1 \leq i \leq s, \tag{4.7}$$

then there exist infinitely many values of x such that $10^{x+a_i} + 1 \not\equiv 0 \pmod{p_i}$ for all i.

Proof. We first claim that $10^{x+a_i}+1 \not\equiv 0 \pmod{p_i}$ for each i with m_i odd. If $p_i = 3$, then this fact is obvious, so assume that $p_i \neq 3$. If $10^{x+a_i} + 1 \equiv 0 \pmod{p_i}$, then $10^{2(x+a_i)} \equiv 1 \pmod{p_i}$. Hence, $2(x + a_i) \equiv 0 \pmod{m_i}$, since by the construction of $\mathcal{C}$ we have that $\mathrm{ord}_{p_i}(10) = m_i$. Thus $x + a_i \equiv 0 \pmod{m_i}$ since m_i is odd. But then, $0 \equiv 10^{x+a_i} + 1 \equiv 2 \pmod{p_i}$, which is impossible since clearly $p_i \neq 2$. Hence, if m_i is odd, any value of x will suffice to ensure that $10^{x+a_i} + 1 \not\equiv 0 \pmod{p_i}$, and the claim is established. This also proves the proposition if $s = 0$.

Suppose now that $s \geq 1$ and consider the list of congruences

$$\begin{aligned} x &\equiv \widehat{m}_1 - a_1 \pmod{m_1} \\ x &\equiv \widehat{m}_2 - a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv \widehat{m}_s - a_s \pmod{m_s}. \end{aligned} \tag{4.8}$$

It follows from (4.7) that (4.8) is not a covering, and therefore there exist infinitely many positive integers x that do not satisfy any of the congruences in (4.8). Hence, for any such value of x, we have by Lemma 4.2 that $10^{x+a_i} + 1 \not\equiv 0 \pmod{p_i}$ for all i, which completes the proof of the proposition. □

4.1. Proof of Theorem 1.3

We first show that if there exists a covering $\mathcal{C}$ that can be used to generate a nontrivial d-composite sandwich number k (as in Example 4.1) with $\gcd(\ell(k), \mathcal{M}) = 1$, then there exist infinitely many nontrivial d-composite sandwich numbers. To complete the proof, we then find a covering $\mathcal{C}$ to construct a nontrivial d-composite sandwich number k with $\gcd(\ell(k), \mathcal{M}) = 1$ for each value of d.

Assume that k is a nontrivial d-composite sandwich number in the form (4.5) that has been constructed from $\mathcal{C} = \{(a_i, m_i, p_i)\}$ using the method described at the beginning of Section 4. Suppose also that $\gcd(\ell(k), \mathcal{M}) = 1$ and $k \equiv z \pmod{\mathcal{P}}$ with $1 \leq z \leq \mathcal{P} - 1$, where $\mathcal{P} = 10 \prod_i p_i$ or $\mathcal{P} = \prod_i p_i$, depending on whether the two additional congruences (4.1) have been added. Then $k = 10^{\ell(k) - \ell(\mathcal{P})}\mathcal{P} + z$, and $\gcd\left(9k + d, 10^{\ell(k)} + 1\right) = 1$, by Theorem 3.1. Let P be the largest prime divisor of

$$\mathcal{Z} = 9\left(z \cdot 10^{\ell(\mathcal{P})} - \mathcal{P}\right) + d \cdot 10^{\ell(\mathcal{P})}.$$

Since $\gcd(\ell(k), \mathcal{M}) = 1$, by Dirichlet's theorem on primes in an arithmetic progression, there exist infinitely many positive integers j such that $\ell(k) + j\mathcal{M}$ is a prime with $\ell(k) + j\mathcal{M} > P$. We claim that

$$k_j := 10^{\ell(k) - \ell(\mathcal{P}) + j\mathcal{M}}\mathcal{P} + z$$

is a nontrivial d-composite sandwich number for each such value of j. Note that k_j is indeed a d-composite sandwich number since

$$s_n(k_j, d) \equiv s_n(k, d) \equiv 0 \pmod{p_i},$$

when $n \equiv a_i \pmod{m_i}$. Also note that $\ell(k_j) = \ell(k) + j\mathcal{M}$. Let

$$A = 9k_j + d \quad \text{and} \quad B = 10^{\ell(k) + j\mathcal{M}} + 1,$$

and assume that k_j is trivial. Then, by Theorem 3.1, we have that $\gcd(A, B) \equiv 0 \pmod{p}$ for some prime p. Since $B \equiv 0 \pmod{p}$, we get that $10^{\ell(k)-\ell(\mathcal{P})+j\mathcal{M}} \equiv -10^{-\ell(\mathcal{P})} \pmod{p}$. Then, since $A \equiv 0 \pmod{p}$, we have

$$A \equiv 9\left(-10^{-\ell(\mathcal{P})}\mathcal{P} + z\right) + d \equiv 0 \pmod{p},$$

which implies that $\mathcal{Z} \equiv 0 \pmod{p}$. In other words, if k_j is trivial, then $s_n(k_j, d) \equiv 0 \pmod{p}$ for all $n \geq 1$, for some prime divisor p of the fixed positive integer $\mathcal{Z}$. However, since $10^{\ell(k)+j\mathcal{M}} + 1 \equiv 0 \pmod{p}$, and $\ell(k) + j\mathcal{M}$ is prime, it follows that $2\left(\ell(k) + j\mathcal{M}\right) = \mathrm{ord}_p(10)$ divides $p - 1$, which is impossible since $\ell(k) + j\mathcal{M} \geq p$. This contradiction proves the existence of infinitely many nontrivial d-composite sandwich numbers k if one is known to exist with $\gcd(\ell(k), \mathcal{M}) = 1$. To complete the proof of the theorem, we construct such a nontrivial d-composite sandwich number for each value of d. We use the following covering for each d:

$$\begin{aligned}\mathcal{C} = &\{(0,3,37),(3,4,101),(2,5,41),(0,5,271),(2,6,13),(4,10,9091),\\ &(5,12,9901),(1,15,31),(28,30,211),(19,30,241),(13,30,2161)\}.\end{aligned}$$

Here $\mathcal{M} = 60$, and $\ell(k) = 29$ for each d, so that $\gcd(\ell(k), \mathcal{M}) = 1$. Also, it is easily verified using a computer that $\gcd(9k + d, 10^{\ell(k)} + 1) = 1$ in each case. We add the two congruences $k \equiv 1 \pmod{2}$ and $k \equiv 1 \pmod{5}$ for k. We omit the details and simply provide in Table 2 the value of k for each d.

d	k
1	17447080826852847307281356871
2	18177573287783379196675646521
3	17252957989711701638754384961
4	17983450450642233528148674611
5	17058835152570555970227413051
6	17789327613501087859621702701
7	16864712315429410301700441141
8	17595204776359942191094730791
9	16670589478288264633173469231

Table 2: Nontrivial d-Composite Sandwich Numbers k with $\gcd(\ell(k), \mathcal{M}) = 1$

5. Proof of Theorem 1.4

In order to prove the existence of infinitely many positive integers that are *simultaneously* nontrivial d-composite sandwich numbers for all $d \in D \subset \mathcal{D}$, where $|D| = 4$

and $D \neq \{3, 6, 7, 9\}$, we use an argument similar to the one used in the proof of Theorem 1.3. In this situation, however, we must construct four separate coverings $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ corresponding to the four digits. As before, in each of these coverings, with the exception of $p_i = 3$, p_i is a primitive divisor of $10^{m_i} - 1$, where m_i is a modulus in the covering. The tricky part is that if we use the same prime p_i in more than one covering, the resulting congruences for k must be the same. For this reason, we have avoided using any prime more than once. The price we pay for this avoidance is that the coverings are more difficult to construct. The four coverings we use are as follows:

$$\begin{aligned}
\mathcal{C}_1 =& \{(1,4,101),(0,6,13),(3,8,73),(7,8,137),(8,12,9901),(10,16,17),(2,16,5882353) \\
& (16,18,19),(14,24,99990001),(28,36,999999000001),(22,48,9999999900000001) \\
& (4,72,3169),(40,72,98641),(46,72,3199044596370769),(94,144,8929)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_2 =& \{(1,3,3),(0,3,37),(5,6,7),(2,18,52579),(26,27,757),(14,27,440334654777631), \\
& (44,54,70541929),(32,54,14175966169),(35,81,163),(77,81,9397),(50,81,2462401), \\
& (62,81,676421558270641),(23,81,130654897808007778425046117), \\
& (8,162,456502382570032651)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_3 =& \{(1,5,41),(4,5,271),(7,10,9091),(13,15,31),(8,15,2906161),(15,20,3541), \\
& (2,20,27961),(20,25,21401),(0,25,25601),(5,25,182521213001),(3,30,211), \\
& (18,30,241),(22,30,2161),(40,50,251),(10,50,5051),(15,50,78875943472201), \\
& (32,60,61),(12,60,4188901),(45,60,39526741),(35,75,151),(10,75,4201)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_4 =& \{(1,7,239),(6,7,4649),(7,9,333667),(0,14,909091),(5,21,43),(17,21,1933), \\
& (9,21,10838689),(11,28,29),(18,28,281),(7,28,121499449), \\
& (17,32,353),(9,32,449),(1,32,641),(25,32,1409),(2,42,127), \\
& (24,42,2689),(12,42,459691),(4,56,7841),(32,56,127522001020150503761), \\
& (31,63,10837),(37,63,23311),(40,63,45613),(19,63,45121231), \\
& (10,63,1921436048294281),(23,84,226549),(3,84,4458192223320340849), \\
& (93,96,97),(69,96,206209),(45,96,66554101249),(21,96,75118313082913), \\
& (53,112,113),(109,112,7376575589640313840\,1), \\
& (21,112,119968369144846370226083377),(58,126,5274739), \\
& (121,126,18977242267323558587448573265\,9),(149,168,603812429055411913), \\
& (243,252,2290624689643723122789957563362013976604469004003960368992\,9), \\
& (75,252,43266855241),(159,252,1009), \\
& (77,336,1433319827159466789806966856379479179916136529424792832495021393), \\
& (301,336,20702700289853417666160090801\,61)\}.
\end{aligned}$$

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$, and let $\mathcal{P}$ be the product of all the primes in $\mathcal{C}$ multiplied by 10 (adding the primes 2 and 5). Then $\ell(\mathcal{P}) = 741$. We give details only for the case of $D = \{1, 2, 3, 4\}$ since the other cases are similar. Here, $\mathcal{C}_i$ corresponds to the digit i. We create a system of congruences for k using (4.3) and (4.4), together with the two added congruences $k \equiv 1 \pmod 2$ and $k \equiv 1 \pmod 5$. Then we search over values of $x \geq 741$ (in place of $\ell(k)$) to find a solution to the system (via the Chinese remainder theorem), in accordance with (4.5), that simultaneously satisfies the criteria from Proposition 4.3 and Theorem 1.3. The smallest value of x that satisfies these conditions is $x = 779$, and a solution with $\ell(k) = 779$ is

$$\begin{aligned} k =&354402919536891231666617478448225865091169486068109830763170018874603521 7\\ &640722236610232114398708244041473381682458890074648536082717193831766619 4\\ &196269772299923324483627469205926500044434954589845840512762885513873116 6\\ &714813094469836034858679080259301220216805136827772892550615640345195093 6\\ &578098473153723140296301972691449952538501778473508984629197587519129826 5\\ &191330832235144744855685292696490268693233138949862013540740415277192305 8\\ &412375144578120505821654096046570967460976505394087849989858001358370552 5\\ &894737923680832195678178558125147398327567204411171706056904368958361061 5\\ &967835508353644193722717432172069133953475703610874735665483794855024364 6\\ &547111781103182209805264776797356574734513464516103445760921466216712252 6\\ &6391841423913444597795160981393953097819656099201. \end{aligned}$$

Indeed, it is straightforward to check that

$$\gcd(9k + d, 10^{\ell(k)} + 1) = \gcd(k, 10d) = 1,$$

for each digit $d \in \{1, 2, 3, 4\}$. In addition, $\mathcal{M} = 453600$ so that $\gcd(\ell(k), \mathcal{M}) = 1$, which completes the proof of the theorem.

Remark 5.1. Note that there are 4! possible ways the coverings $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ can be "assigned" to the digits in a set $D \subset \mathcal{D}$ with $|D| = 4$. For a particular D, some of these correspondences do not yield a solution that satisfies all of the necessary criteria. However, with the exception of $D = \{3, 6, 7, 9\}$, at least one correspondence produces a desired solution using these four coverings. The exceptional case $D = \{3, 6, 7, 9\}$ in Theorem 1.4 is a result of the fact that, regardless of how we assign these four coverings to the digits in D, the congruences arising from the covering $\mathcal{C}_2$ force either $k \equiv 0 \pmod 3$ or $k \equiv 0 \pmod 7$. Although k is still a composite sandwich number in this case, it is trivial.

6. Proof of Theorem 1.5

Given any $d \in \mathcal{D}$, to prove that there are infinitely many sets of 13 consecutive positive integers that are all d-composite sandwich numbers, we utilize the method and coverings that were used in the proof of Theorem 1.4. However, here we use the same d for all four coverings $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$. In addition, we substitute $k + 2$ for k in the congruences generated from $\mathcal{C}_2$, $k + 4$ for k in the congruences for k in the congruences generated from $\mathcal{C}_3$, and $k+6$ for k in the congruences generated from $\mathcal{C}_4$. We also add the congruences $k \equiv 1 \pmod 2$ and $k \equiv 2 \pmod 5$ to the system for k. Since the techniques are similar, we give details only in the case of $d = 1$. The smallest value of $x \geq 741$ here, in accordance with (4.5), that simultaneously satisfies the criteria from Proposition 4.3 and Theorem 1.3 is $x = 811$. Using the Chinese remainder theorem, a solution to the system of congruences for k, with $\ell(k) = 841$, is

$$
\begin{aligned}
k =\,&3544029195368912316666174784482258650908326838647239060200876864650079431\\
&1711630812697324230525538611838646043882766742346069163764391075438928881\\
&3699789159029469059047678298092672792216574482270249499003001497798585976\\
&5344308183364637135810236440486282392927159604420693600828207861578411170\\
&6681186262615978434167536949357633418075684192054467217185023166944874737\\
&1689757054755673346721553522756962087212178947796157521768786309652796994\\
&4685765718557579581798061572469898685785778779323768087150735721199826790\\
&4782770253370103767597480428692295437780589986448693949970580537837368249\\
&5758123653025610979354753176939604748433221775845235692325886596409193285\\
&7213743007630396022221086807104788683968673543948689532792465149164194750\\
&8378705604021354244189937489028854509228736851403542592414929979630436913\\
&87637597.
\end{aligned}
$$

Note that $k + z \equiv 0 \pmod 2$ for all odd integers z with $-3 \leq z \leq 9$. Also, $k + z \equiv 0 \pmod 5$ for $z \in \{-2, 3, 8\}$. Hence, in these cases, the integers $k + z$ are trivial 1-composite sandwich numbers. For the values of z with $z \in \{0, 2, 4, 6\}$, we have used the respective coverings $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$, to ensure that $k + z$ is a 1-composite sandwich number. It is again easy to check that

$$\gcd(9k + 1, 10^{\ell(k)} + 1) = \gcd(k, 10) = 1$$

and $\gcd(\ell(k), \mathcal{M}) = 1$. Thus, we can construct infinitely many nontrivial 1-composite sandwich numbers k such that $k-3, k-2, \ldots, k+9$ are all 1-composite (not necessarily nontrivial) sandwich numbers. As mentioned, the methods are similar for the other digits.

7. Theorem 1.6

Definition 7.1. A *Sierpiński number* k is an odd positive integer such that $k \cdot 2^n + 1$ is composite for all integers $n \geq 1$.

Definition 7.2. A *Riesel number* k is an odd positive integer such that $k \cdot 2^n - 1$ is composite for all integers $n \geq 1$.

In 1956, Riesel [26] proved that there are infinitely many Riesel numbers, and in 1960, Sierpiński [28] proved that there are infinitely many Sierpiński numbers. Since then, other authors have examined extensions and variations of these ideas [2, 4, 5, 6, 7, 8, 10, 12, 11, 12, 14, 15, 18, 19, 20, 21, 22, 23, 24]. Coverings are used quite extensively in these investigations.

7.1. Proof of Theorem 1.6

We provide details only for $d = 2$ since the procedure is identical for other values of $d \in D$.

The approach we use is similar to one used in [23]. We build three coverings $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ with the following properties:

- $\mathcal{C}_1$ is used to generate infinitely many 2-composite sandwich numbers that are also perfect squares. The corresponding primes here are primitive divisors of numbers of the form $10^m - 1$, where m is a modulus in $\mathcal{C}_1$.
- $\mathcal{C}_2$ is used to generate infinitely many squares that are also Sierpiński numbers. The corresponding primes here are primitive divisors of numbers of the form $2^m - 1$, where m is a modulus in $\mathcal{C}_2$.
- $\mathcal{C}_3$ is used to generate infinitely many squares that are also Riesel numbers. The corresponding primes here are primitive divisors of numbers of the form $2^m - 1$, where m is a modulus in $\mathcal{C}_3$.
- The only corresponding prime that appears in more than one covering is $p = 3$.

Even though we use the prime $p_i = 3$ in all three coverings, the resulting congruence for k^2 is the same in all cases. Thus, we can piece together all three coverings and use the Chinese remainder theorem to get squares that are simultaneously 2-composite sandwich, Sierpiński and Riesel.

The first covering is: $\mathcal{C}_1 = \{(2, 3, 3), (1, 3, 37), (3, 6, 7), (0, 6, 13)\}$. Here, we have $\ell\left(\prod_{i=1}^{4} p_i\right) = \ell(10101) = 5$, and so we need to choose x with $x \geq 5$. In addition, by Proposition 4.3, we need to choose x such that $x \not\equiv 0 \pmod 3$. We choose $x = 5$. Replacing k with k^2 in (4.3) and (4.4), we get the following set of congruences for k^2:

$$\begin{aligned} k^2 &\equiv 1 \pmod 3 \\ k^2 &\equiv 11 \pmod{37} \\ k^2 &\equiv 1 \pmod 7 \\ k^2 &\equiv 0 \pmod{13}. \end{aligned} \tag{7.1}$$

Note that each of the residues in (7.1) is a square modulo the corresponding prime.

The second covering is:

$$\mathcal{C}_2 = \{(1, 2, 3), (2, 4, 5), (4, 8, 17), (8, 16, 257), (16, 32, 65537), (32, 64, 641), (0, 64, 6700417)\}.$$

We use $\mathcal{C}_2$ and solve for k^2 in each congruence $k^2 \cdot 2^n + 1 \equiv 0 \pmod{p_i}$. The resulting system of congruences is:

$$\begin{array}{ll} k^2 \equiv 1 \pmod{3} & k^2 \equiv 1 \pmod{65537} \\ k^2 \equiv 1 \pmod{5} & k^2 \equiv 1 \pmod{641} \\ k^2 \equiv 1 \pmod{17} & k^2 \equiv -1 \pmod{6700417}. \\ k^2 \equiv 1 \pmod{257}. & \end{array} \tag{7.2}$$

Again, each of the residues in (7.2) is a square modulo the corresponding prime.

The third covering is:

$$\begin{aligned}\mathcal{C}_3 = \{&(0,2,3),(3,5,31),(6,7,127),(8,9,73),(4,15,151),(7,20,41),(12,21,337),\\ &(9,24,241),(11,25,601),(21,25,1801),(11,27,262657),(17,28,113),(0,35,71),\\ &(15,35,122921),(37,40,61681),(14,45,631),(29,45,23311),(37,56,15790321),\\ &(39,60,1321),(5,63,92737),(25,70,281),(49,72,433),(25,72,38737),\\ &(16,75,10567201),(1,75,100801),(3,84,14449),(41,90,18837001),\\ &(40,105,152041),(100,105,29191),(10,105,106681),(29,108,279073),\\ &(101,108,246241),(69,120,4562284561),(23,126,77158673929),(56,135,271),\\ &(101,135,49971617830801),(135,140,7416361),(31,150,1133836730401),\\ &(143,168,88959882481),(59,168,3361),(116,175,39551),(37,180,54001),\\ &(110,189,207617485544258392970753527),(47,189,1560007),(31,200,401),\\ &(91,200,3173389601),(131,200,2787601),(141,210,664441),(21,210,1564921),\\ &(41,216,138991501037953),(185,216,33975937),(51,225,13861369826299351),\\ &(201,225,1348206751),(126,225,617401),(167,270,15121),\\ &(191,280,84179842077657862011867889681),(81,300,1201)\}.\end{aligned}$$

Here we solve the congruence $k^2 \cdot 2^n - 1 \equiv 0 \pmod{p_i}$ for k^2 in each case to get:

$$\begin{array}{ll} k^2 \equiv 1 \pmod{3} & k^2 \equiv 65738 \pmod{106681} \\ k^2 \equiv 4 \pmod{31} & k^2 \equiv 213401 \pmod{279073} \\ k^2 \equiv 2 \pmod{127} & k^2 \equiv 128 \pmod{246241} \\ k^2 \equiv 2 \pmod{73} & k^2 \equiv 2147481600 \pmod{4562284561} \\ k^2 \equiv 85 \pmod{151} & k^2 \equiv 57868481159 \pmod{77158673929} \\ k^2 \equiv 33 \pmod{41} & k^2 \equiv 138 \pmod{271} \\ k^2 \equiv 175 \pmod{337} & k^2 \equiv 17179869184 \pmod{49971617830801} \\ k^2 \equiv 233 \pmod{241} & k^2 \equiv 32 \pmod{7416361} \\ k^2 \equiv 157 \pmod{601} & k^2 \equiv 549201642000 \pmod{1133836730401} \\ k^2 \equiv 16 \pmod{1801} & k^2 \equiv 33554432 \pmod{88959882481} \\ k^2 \equiv 65536 \pmod{262657} & k^2 \equiv 1792 \pmod{3361} \\ k^2 \equiv 14 \pmod{113} & k^2 \equiv 15631 \pmod{39551} \\ k^2 \equiv 1 \pmod{71} & k^2 \equiv 28376 \pmod{54001} \\ k^2 \equiv 65208 \pmod{122921} & k^2 \equiv 60446290980731458735308 8 \\ k^2 \equiv 8 \pmod{61681} & \qquad \pmod{207617485544258392970753527} \end{array}$$

$$
\begin{array}{ll}
k^2 \equiv 86 \pmod{631} & k^2 \equiv 1283744 \pmod{1560007} \\
k^2 \equiv 18914 \pmod{23311} & k^2 \equiv 390 \pmod{401} \\
k^2 \equiv 524288 \pmod{15790321} & k^2 \equiv 3173389089 \pmod{3173389601} \\
k^2 \equiv 725 \pmod{1321} & k^2 \equiv 1194865 \pmod{2787601} \\
k^2 \equiv 89839 \pmod{92737} & k^2 \equiv 414908 \pmod{664441} \\
k^2 \equiv 100 \pmod{281} & k^2 \equiv 1228462 \pmod{1564921} \\
k^2 \equiv 99 \pmod{433} & k^2 \equiv 134645869146869 \pmod{138991501037953} \\
k^2 \equiv 36689 \pmod{38737} & k^2 \equiv 6999617 \pmod{33975937} \\
k^2 \equiv 2399130 \pmod{10567201} & k^2 \equiv 1244027990477916 \pmod{1386136982629935 1} \\
k^2 \equiv 50401 \pmod{100801} & k^2 \equiv 16777216 \pmod{1348206751} \\
k^2 \equiv 12643 \pmod{14449} & k^2 \equiv 503159 \pmod{617401} \\
k^2 \equiv 18836985 \pmod{18837001} & k^2 \equiv 13936 \pmod{15121} \\
k^2 \equiv 90500 \pmod{152041} & k^2 \equiv 6189700196426901374495 62112 \\
k^2 \equiv 32 \pmod{29191} & \qquad \pmod{841798420776578620118678 89681} \\
 & k^2 \equiv 18 \pmod{1201}
\end{array} \tag{7.3}
$$

Adding the congruence $k^2 \equiv 1 \pmod{2}$ to the total system comprised of the congruences from $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, and choosing a square root in each congruence, we solve for k. Denote the smallest positive solution arising from this choice of square roots as k_s. Then $\ell\left(k_s^2\right) \equiv 0 \pmod{6}$. To get a desired value of k with $\ell(k^2) \equiv 5 \pmod{6}$, we simply choose an appropriate value in the arithmetic progression. The smallest such value is

$$\kappa = k_s + 100 \cdot 2 \left(\prod_{p_i \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3} p_i \right),$$

where $\ell\left(\kappa^2\right) = 755 \equiv 5 \pmod{6}$. Then

$$
\begin{aligned}
\kappa^2 = & 1019047736805764081984947455806539607247050939720372783864553 6786 \\
& 3814245458582704351902041013046418057986606439796225208769928 3422 \\
& 4492620891886511473841093149621728527411300462367593616610241 4532 \\
& 8833503147456880764322334547755103758877917759497792700028513 7096 \\
& 5395846309777658517172824278088200963540686171949063146235604 3344 \\
& 5355471369350576909107473412666632616920949310511495213282628 7819 \\
& 4645043659797789881267150303810514261940862024426620998351207 9649 \\
& 0270257386424654493438184999061473202766294291403579159467689 6536 \\
& 1688871771931705073853976853765356606744672645208022147829057 7620 \\
& 3177179003896691628226187746200585507396304452842837394091888 9306 \\
& 5464303864862827772397572223445452172322840357735358676572315 6863 \\
& 7833737869781777339143234199470291788881
\end{aligned}
$$

is a 2-composite sandwich square that is also a Sierpiński and Riesel number. Since there are infinitely many such values of k in the arithmetic progression, the theorem is established.

Remark 7.3. Since $\gcd(9\kappa^2+2, 10^{\ell(\kappa^2)}+1)=1$, it follows from Theorem 3.1 that κ^2 is nontrivial. This is not the case for all such values of k^2 produced from the arithmetic progression. For example, the number $\left(k_s + 102 \cdot 2\prod_{p_i \in \mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3} p_i\right)^2$ is a trivial 2-composite sandwich square that is a Sierpiński and Riesel number.

We give in Table 3 a list of coverings to generate d-composite sandwich squares. These coverings can be used in conjunction with $\mathcal{C}_2$ and $\mathcal{C}_3$, as in the given case of $d=2$, to generate d-composite sandwich squares that are also Sierpiński and Riesel. The number of digits modulo $\mathcal{M}$ for these squares is also given in each case.

d	Covering	$\ell\left(k^2\right) \pmod{\mathcal{M}}$
1	$\{(1,3,3),(2,3,37),(3,6,7),(0,6,13)\}$	5
2	$\mathcal{C}_1$	5
3	$\{(0,2,11),(2,3,37),(1,4,101),(3,6,13),(7,12,9901)\}$	8
4	$\{(1,3,3),(2,3,37),(3,6,7),(0,6,13)\}$	5
5	$\{(2,3,3),(1,3,37),(0,6,7),(3,6,13)\}$	5
6	$\{(0,2,11),(0,3,37),(1,6,13),$ $(5,9,333667),(17,18,19),(11,18,52579)\}$	4
7	$\{(0,2,11),(0,3,37),(3,4,101),(5,6,13),(1,12,9901)\}$	8
8	$\mathcal{C}_1$	5
9	$\{(0,2,11),(1,3,37),(3,6,13),$ $(5,9,333667),(17,18,19),(11,18,52579)\}$	4

Table 3: Coverings Used to Generate d-Composite Sandwich Squares

8. Final Comments

In Theorem 1.1 and Theorem 1.2, we found infinitely many primes that are trivial d-composite sandwich numbers. However, in Theorem 1.3, all the nontrivial d-composite numbers are composite. The reason for this is that using a covering argument to produce such numbers inherently produces composite numbers. To see this, note that any covering $\mathcal{C}$ must have at least one zero residue. Suppose that $(0, m_i, p_i) \in \mathcal{C}$. Then, by (4.3) and (4.4), we have that $k \equiv 0 \pmod{p_i}$ so that k is composite. Thus, a covering argument cannot be used to find nontrivial d-composite sandwich primes. Of course, this raises two natural questions: Do any such primes exist; and if so, how do we find them? A computer search has found candidates for such primes. For example, 7 "seems" to be a 1-composite sandwich number, but there is no apparent obstruction to encountering a prime in the sequence $\{s_n(7,1)\}_{n=1}^{\infty}$. A partial covering

$$\{(1,3,3),(2,6,11),(5,6,13),(0,6,7)\}$$

shows that there is indeed a pattern to the prime divisors of $s_n(7,1)$ for these congruence classes modulo n. However, the uncovered class of $n \equiv 3 \pmod 6$ remains somewhat of a mystery; and if a prime occurs in $\{s_n(7,1)\}_{n=1}^{\infty}$, it must occur in this class. Further splitting of this class reveals that there are still patterns to the prime divisors, but ultimately no covering has been found to explain the entire situation. Similar phenomena have been observed before in [11, 18, 22] where the sequence in question defies a covering argument; but, in fact, the sequence is composite. In these situations, the "bad" class yields to a factorization rather than the same prime divisor. It is conjectured that the smallest prime divisor of these "bad" terms in the sequence is unbounded as n approaches infinity. It appears, however, in the case of nontrivial d-composite sandwich numbers, that such a factorization is unlikely, although we have not been able to confirm this belief.

Recently, Bob Hough [17] has given a negative answer to a famous question in covering systems known as the *minimum modulus problem*. This problem, originally posed by Erdős [9], asks if the minimum modulus in a covering with distinct moduli can be arbitrarily large. Previously, the best known result was due to Nielsen [25], who constructed a covering with distinct moduli and minimum modulus 40. To prove Theorem 1.4 and Theorem 1.5, we constructed four coverings with no repeated associated prime. Admittedly, our situation is not as restrictive as the minimum modulus problem since we can use repeated moduli. So, it is conceivable that we might be able to extend our results somewhat by constructing more than four coverings with no repeated corresponding prime, but it seems computationally infeasible, and perhaps impossible, in light of Hough's theorem, to extend this process to nine coverings.

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