



CONGRUENCES FOR m -REGULAR PARTITIONS MODULO 4

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Abstract

We prove several congruences modulo 4 for m -regular partitions with several different m , and discuss some investigative tools toward a general theorem on the widespread existence of similar congruences. In particular, we give an algorithmic even-odd dissection for any $B_m(q)$ into a small number of terms, which may be repeated to indefinitely large powers of 2. We close with some comments on the parity of the partition numbers and 4-regular partitions.

1. Introduction

The m -regular partitions are those in which parts are not divisible by m . Their generating function is

$$B_m(q) = \sum_{n=0}^{\infty} b_m(n)q^n = \prod \frac{1 - q^{mk}}{1 - q^k}.$$

Two contrasting facts describe a curious phenomenon in m -regular partitions which it is the purpose of this paper to explore. The first, proved by Radu [8] (completing work by Ono on Subbarao's conjecture), is that there exists no arithmetic progression $Ak + B$, A, B nonzero integers, for which the usual partition function is constantly even, i.e., there is no A, B such that for all k ,

$$p(Ak + B) \equiv 0 \pmod{2}.$$

Indeed, the parity of the partition function gives every appearance of being random, and a longstanding open conjecture is that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n < x | p(n) \equiv 0 \pmod{2}\} = \frac{1}{2}.$$

On the other hand, for many m there are numerous distinct (i.e., non-nested) arithmetic progressions $Ak + B$ for which

$$b_m(Ak + B) \equiv 0 \pmod{2^j}$$

for $j \geq 1$. Most of these are mod 2, and advancing the analysis past parity is one of the main motivations of this paper. Furthermore, while congruences modulo primes r for the usual partition function in arithmetic progression $Ak + B$ always have A a multiple of r – also proved by Radu, quite recently [7], as a consequence of the structure of the space of modular forms the generating function inhabits, and an expected consequence of the usual proof techniques in the area – for m -regular partition functions this is not necessarily the case.

For instance, Rødseth, in [9], finds that

$$b_2(25k + 6) \equiv 0 \pmod{4}, \quad \text{and} \quad b_2(5^8k + 94401) \equiv 0 \pmod{8},$$

and in fact a larger family of identities containing these.

Our immediate goal is to prove several new individual congruences mod 4, including one conjectured in an earlier paper of the author [4], using a fairly standard set of techniques. One aim of this section is to show that such congruences can now be produced with relative ease with the field's current technology. In suggestion of a next step to advance the literature, we will give an algorithmic even-odd dissection of $B_m(q)$ for any m , and make several observations that hold for general modulus.

We close with speculation on the relationship between parity in m -regular partitions and the parity of the partition function, in particular the very predictable parity behavior of the 4-regular partitions, and a resulting triangular-number recursing algorithm to calculate the parity of the partition numbers.

1.1. Notation and Definitions

We use the standard notation

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \quad , \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n.$$

An η -quotient is a quotient of products of functions $(a_i; q^j)_\infty$. Later, when the number of η -quotients in identities gets large, we will employ the notation

$$(a_1, \dots, a_k; t) = (q^{a_1}; q^t)_\infty \cdots (q^{a_k}; q^t)_\infty.$$

When we say of two functions that

$$\sum_{n=0}^{\infty} b(n)q^n \equiv_m \sum_{n=0}^{\infty} c(n)q^n,$$

we mean that $b(n) \equiv c(n) \pmod{m}$ for all n .

We will assume familiarity with the standard vocabulary of modular forms as used in the partition literature, particularly the theorems of Newman (on conditions under which an η -quotient is a modular form) and Sturm (on congruences and equality for modular forms). For an overview of the necessary theorems, see [4].

2. Congruences

Our first theorem is a short extension of previous work which will illustrate two useful principles.

Theorem 1. *For $k \geq 1$, $n \geq 0$ integers, $b_4(9^k n + \frac{57 \cdot 9^{k-1} - 1}{8}) \equiv 0 \pmod{4}$ and $b_4(9^k n + \frac{33 \cdot 9^{k-1} - 1}{8}) \equiv 0 \pmod{4}$.*

Proof. The 4-regular partitions, described as partitions with even parts distinct, were examined by Andrews, Hirschhorn and Sellers in [1]. They found, by completely dissecting B_4 , that

$$b_4(9n + 4) \equiv 0 \pmod{4} \quad \text{and} \quad b_4(9n + 4) \equiv 0 \pmod{12}.$$

In the course of dissecting $B_4(q)$ they showed that

$$\sum_{n=0}^{\infty} b_4(9n + 1)q^n \equiv_{24} \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}^2 (q^3; q^3)_{\infty}^4}{(q; q)_{\infty}^4 (q^6; q^6)_{\infty}^2}.$$

We note that $(q^2; q^2)_{\infty}^2 \equiv (q; q)_{\infty}^4 \pmod{4}$, and so, since $B_4(q) = \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty}}$, we have that $\sum_{n=0}^{\infty} b_4(9n + 1)q^n \equiv_4 B_4(q)$.

Thus the $n = 9k + 4$ and $n = 9k + 7$ subprogressions of $9n + 1$ will be 0 modulo 4, and by taking repeated subprogressions we obtain the infinite family claimed. \square

Here we illustrate that when a congruence is desired, a full dissection is not always necessary. It would probably be possible to dissect $\sum_{n \geq 0} b_4(9n + 1)q^n$ modulo 9, but working with a congruent function can much simplify terms and shorten the work – here, to a single step.

$B_m(q)$ is frequently self-similar over various moduli, a characteristic which becomes useful in dissections such as the previous proof. Both of these notions are useful in our next theorem:

Theorem 2. *For integer $n \geq 0$ $b_5(20n + 12) \equiv 0 \pmod{4}$ and $b_5(20n + 16) \equiv 0 \pmod{4}$.*

Proof. We begin by decomposing $B_5 \pmod{5}$, which is easy with Mike Hirschhorn’s [2] identity for the partition function,

$$\frac{1}{(q; q)_\infty} = \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6} (R^4(q^5) + qR^3(q^5) + 2q^2R^2(q^5) + 3q^3R(q^5) + 5q^4 - 3q^5R^{-1}(q^5) + 2q^6R^{-2}(q^5) - q^7R^{-3}(q^5) + q^8R^{-4}(q^5)) \quad (1)$$

where $R(q) = \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty}{(q; q^5)_\infty (q^4; q^5)_\infty}$. Since $B_5(q) = \frac{(q^5; q^5)_\infty}{(q; q)_\infty}$, this gives us

$$\begin{aligned} \sum_{n \geq 0} b_5(5n + 1)q^n &= \frac{(q^5; q^5)_\infty^5}{(q; q)_\infty^5} (R^3(q) + 2qR^{-2}(q)) \\ &= \frac{1}{(q^2; q^5)_\infty^2 (q^3; q^5)_\infty^2 (q; q^5)_\infty^8 (q^4; q^5)_\infty^8} + \frac{2q}{(q; q^5)_\infty^3 (q^4; q^5)_\infty^3 (q^2; q^5)_\infty^7 (q^3; q^5)_\infty^7} \\ &\equiv_4 \frac{(q^{20}; q^{20})_\infty^2}{(q^4; q^4)_\infty^2} \left(\frac{(q^8; q^{20})_\infty^2 (q^{12}; q^{20})_\infty^2}{(q^2; q^5)_\infty^2 (q^3; q^5)_\infty^2} + 2q(q^4; q^{20})_\infty (q^{16}; q^{20})_\infty \frac{(q; q)_\infty}{(q^5; q^5)_\infty} \right). \quad (2) \end{aligned}$$

To obtain the last line from the previous we made repeated use of various forms of the identities $(q; q)_\infty^4 \equiv_4 (q^2; q^2)_\infty^2$ and $2(q; q)_\infty^2 \equiv_4 2(q^2; q^2)_\infty$.

We now dissect this progression mod 4. We require the even-odd dissection of $\frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}$, which was given by G. N. Watson [10]:

$$\begin{aligned} \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty} &= \frac{(q^8; q^8)_\infty}{(q^2; q^2)_\infty} \\ &\times \left(\frac{q^3}{(q^{32}; q^{80})_\infty (q^{48}; q^{80})_\infty} + \frac{(q^4; q^{40})_\infty (q^{36}; q^{40})_\infty}{(q^{16}; q^{40})_\infty (q^{24}; q^{40})_\infty (q^8; q^{80})_\infty (q^{72}; q^{80})_\infty} \right) \end{aligned}$$

and of $\frac{(q; q)_\infty}{(q^3; q^5)_\infty}$:

$$\frac{(q; q)_\infty}{(q^5; q^5)_\infty} = \frac{(q^2; q^4)_\infty (q^8; q^8)_\infty}{(q^{10}; q^{20})_\infty^3 (q^{40}; q^{40})_\infty} - q \frac{(q^4; q^4)_\infty (q^4; q^8)_\infty (q^{40}; q^{40})_\infty}{(q^{10}; q^{10})_\infty^2}.$$

The latter dissection is easily verifiable by the theorems of Newman and Sturm, as all terms are modular forms ($k = 2, \Gamma_0(40)$) after multiplication by $q^{-1/6} \eta(z)^4$ and noting that $(q; q^2)_\infty = (q; q)_\infty / (q^2; q^2)_\infty$. Calculation verifies that all coefficients of both sides are equal up to their Sturm bounds. (The referee notes that the dissection of $\frac{(q; q)_\infty}{(q^3; q^5)_\infty}$ has also appeared before: [12, p. 129].)

At this point, we will begin using the abbreviated notation described earlier.

Substituting these expressions into (2) and taking odd terms, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} b_5(10n+6)q^n &\equiv_4 2 \frac{(10; 10)^2}{(2; 2)^2} \\ &\times \left(q \frac{(4, 6; 10)^2(4; 4)^2(2, 18, 20; 20)}{(1; 1)^2(4; 4)} + \frac{(2, 8; 10)(1; 2)(4; 4)}{(5; 10)^3(20; 20)} \right) \\ &\equiv_4 2 \frac{(10; 10)^2}{(2; 2)^3} \left(q(4, 6; 10)^2(4; 4)(2, 18, 20; 20) + (2, 8; 10)(4; 4) \frac{(1; 1)}{(5; 5)} \right). \end{aligned} \tag{3}$$

Applying the dissection of $(1; 1)/(5; 5)$ again and taking the odd terms, we find that

$$\sum_{n=0}^{\infty} b_5(20n+16)q^n \equiv_4 2 \frac{(5; 5)^2}{(1; 1)^3} \left((2; 2)(2, 3; 5)^2(1, 9; 10) - \frac{(1, 4; 5)(2; 2)^2(2; 4)(20; 20)}{(5; 5)^2} \right). \tag{4}$$

The two terms inside the parentheses are congruent mod 2, as can be seen by canceling factors and making repeated use of the identity $(1; 1)^2 \equiv_2 (2; 2)$. Thus the entire expression is 0 mod 4.

The proof that $b_5(20n + 12) \equiv 0 \pmod{4}$ starts with the same dissection modulo 5 and takes the two terms comprising the $5n + 2$ progression. We require Watson’s dissection of $1/(1, 4; 5)$ from the same paper [10]. As the techniques are otherwise similar, we omit it here. \square

Remark: This congruence is almost certainly not part of an infinite family of the form in the previous theorem. Progressions $b_m(An + B)$ similar to $b_m(n) \pmod{N}$, in numerical investigations to date, have only appeared when $N = p_i^j$ for $p_i^{e_i} \parallel m$, $p_i^{e_i} \geq \sqrt{m}$. We conjecture that this behavior is governed by hypotheses similar to those which Basil Gordon and Ken Ono [5] used in describing when it is the case that a high power of a prime divides the coefficients of $B_m(q)$ with probability approaching 1. To be precise,

Conjecture 1. *If for some $A, B \in \mathbb{N}$ it holds that $\sum_{n=0}^{\infty} b_m(An + B)q^n \equiv_N B_m(q)$, then $N = p^j$ for a unique prime $p \mid m$ for which $p^k \parallel m$ and $p^k \geq \sqrt{m}$.*

Last in this section, we prove a conjecture raised by the author in an earlier paper [4]:

Theorem 3. *For integer $k \geq 0$, $b_9(32k + 13) \equiv 0 \pmod{12}$.*

Proof. We begin with the known dissection of $B_9 \pmod{4}$ from that paper:

$$B_9(q) = \frac{(12; 12)^2}{(2; 2)^2(6, 30; 36)} + q \frac{(12; 24)^2(36; 36)}{(4; 4)(4; 8)^6} + 3q^3 \frac{(24; 24)^2(36; 36)}{(4; 4)^3(4; 8)^2}.$$

(To verify, multiply both sides by $q^{1/3}\eta(4z)^4$, obtaining modular forms of weight 2 on $\Gamma_0(216)$, and compare coefficients.)

This gives us

$$\sum_{n=0}^{\infty} b_9(4n + 1)q^n = \frac{(3; 6)^2(9; 9)}{(1; 2)^6(1; 1)}.$$

We need the identities

$$\frac{(3; 6)}{(1; 2)^3} = \frac{1}{(2; 4)^4} \left(\frac{1}{(2, 10; 12)^2} + \frac{3q}{(4, 8; 12)^2} \right)$$

and

$$\frac{1}{(1, 5; 6)^2} = \frac{(4; 8)(12; 24)}{(2; 4)^3(6; 12)} + \frac{2q(16; 32)^2}{(4; 8)^4(12; 24)(8; 16)^3} + \frac{4q^3}{(4; 8)^4(12; 24)(8; 16)(16; 32)^2}.$$

Both are identities of modular forms after multiplication by suitable factors, and can be verified by Sturm.

Applying the dissection of B_9 again, we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_9(4n + 1)q^n &= \left[\frac{1}{(2; 4)^4} \left(\frac{1}{(2, 10; 12)^2} + \frac{3q}{(4, 8; 12)^2} \right) \right]^2 \\ &\times \left[\frac{(12; 12)^2}{(2; 2)^2(6, 30; 36)} + q \frac{(12; 24)^2(36; 36)}{(4; 4)(4; 8)^6} + 3q^3 \frac{(24; 24)^2(36; 36)}{(4; 4)^3(4; 8)^2} \right] \quad (5) \end{aligned}$$

Expanding out and taking odd terms, we get

$$\sum_{n=0}^{\infty} b_9(8n + 5)q^n = 6g_1 + g_2 + 9qg_3 + 3qg_4 + 27q^2g_5$$

where the g_i are various η -products. Three of them have no $n = 4k + 1$ terms that are nonzero mod 4:

$$\begin{aligned} 6g_1 &= 6 \frac{(3; 6)^2(6; 6)^4}{(1; 1)^4(1; 2)^8(3, 15; 18)} \equiv_4 6 \frac{(12; 12)^2}{(4; 8)} (3, 15, 18, 21, 33; 36) \\ &= 6 \frac{(12; 12)^2}{(4; 8)} \left(\frac{1}{(24, 36, 108, 120; 144)} - q^3 \frac{(24, 120; 144)}{(12; 24)} \right), \end{aligned}$$

$$\begin{aligned}
 g_2 &= \frac{(6; 12)^2(18; 18)(3; 6)^4}{(2; 2)(2; 4)^6(1; 2)^{12}} \equiv_4 \frac{(6; 12)^4(18; 18)}{(2; 2)(2; 4)^{12}}, \quad \text{and} \\
 27q^2g_5 &= 27q^2 \frac{(4; 4)^2(6; 6)^4(12; 12)^2(18; 18)}{(1; 1)^8(2; 2)} \\
 &\equiv_4 27q^2 \frac{(4; 4)^2(6; 6)^4(12; 12)^2(18; 18)}{(2; 2)^5}.
 \end{aligned}$$

In going from the first to the second line in the congruence for g_1 we employ the identity

$$(3, 15, 18, 21, 33; 36) = \frac{1}{(24, 36, 108, 120; 144)} - q^3 \frac{(24, 120; 144)}{(12; 24)}$$

which can be verified by rewriting as η -quotients, multiplying through by $q^{-1/2}\eta(6)^2$, and confirming the equality of modular forms in $\Gamma_0(576)$.

The other two terms are negatives of each other mod 4:

$$\begin{aligned}
 9qg_4 &= 9q \frac{(6; 12)^2(18; 18)(6; 6)^4}{(2; 2)(2; 4)^6(1; 1)^4(1; 2)^4} \equiv_4 9q \frac{(6; 12)^2(18; 18)(6; 6)^4}{(2; 2)(2; 4)^6} \quad \text{and} \\
 3qg_3 &= 3q \frac{(12; 12)^2(18; 18)(3; 6)^4}{(2; 2)^3(2; 4)^2(1; 2)^{12}} \equiv_4 3q \frac{(6; 12)^2(18; 18)(6; 6)^4}{(2; 2)(2; 4)^6}.
 \end{aligned}$$

Thus the $n = 4k + 1$ terms in $\sum_{n=0}^{\infty} b_9(8n + 5)q^n$ are 0 mod 4, in fact mod 12. \square

Remark: It was also conjectured in [4] that $b_9(64k + 13) \equiv 0 \pmod{24}$. The author was, after his presentation at the Integers Conference in 2013, informed that Olivia Yao had produced a proof of both conjectures, which has now appeared in [13].

3. Algorithmic Dissections

In the m -regular partitions literature there is to date a notable lack of theorems concerning properties of $B_m(q)$ for large classes of m . But congruences for $b_m(4kn + j) \pmod{4}$ appear to be sufficiently numerous that a general theorem on the existence of such congruences seems possible. In this section are offered some initial observations which, it is hoped, may lead to proofs in this direction. Alone, the observations below do not immediately help in proving any congruence properties, but because the dissections discussed are strictly algorithmic, this might allow computer verification of future congruences, and possibly provide a theoretical tool for investigators. The use of theta-function addition identities, such as the main tool of [11], might provide such a theoretical route, for instance.

To begin with, it is possible to construct the even-odd dissection of any $B_m(q)$ employing two identities:

Lemma 1.

$$\frac{1}{(q; q)_\infty} = \frac{(12, 20; 32)(16; 16)}{(2; 2)^2(6, 10; 16)} + q \frac{(4, 28; 32)(16; 16)}{(2; 2)^2(2, 14; 16)}$$

and

Lemma 2. $(q; q)_\infty = (16; 16)((2, 12, 14, 18, 20, 30; 32) - q(4, 6, 10, 22, 26, 28; 32)).$

(Each can be proven through the theory of theta-functions. The author thanks Michael Somos for making the observation in sequences A058695 and A058696 of the Online Encyclopedia of Integer Sequences [6] and discussing them upon request, and discussions with Mike Hirschhorn for the latter.) Put together, the two tell us

Theorem 4. For any integer $m \geq 1$,

$$\begin{aligned} B_m(q) = & \frac{(12, 20; 32)(16; 16)(16m; 16m)(2m, 12m, 14m, 18m, 20m, 30m; 32m)}{(2; 2)^2(6, 10; 16)} \\ & - q^{(m+1)} \frac{(4, 28; 32)(16; 16)(4m, 6m, 10m, 22m, 26m, 28m; 32m)}{(2; 2)^2(2, 14; 16)} \\ & + q \frac{(4, 28; 32)(16; 16)(16m; 16m)(2m, 12m, 14m, 18m, 20m, 30m; 32m)}{(2; 2)^2(2, 14; 16)} \\ & - q^m \frac{(12, 20; 32)(16; 16)(4m, 6m, 10m, 22m, 26m, 28m; 32m)}{(2; 2)^2(6, 10; 16)}. \end{aligned} \tag{6}$$

Therefore the generating function for m -regular partitions, for any m , can be dissected into even and odd parts consisting of no more than 2 terms each. (Which two constitute the even terms depends, of course, on the parity of m .) The resulting sums are no longer η -products, but are products of shifted η -products of the form $(a, p - a; p)$, and these may almost (i.e., up to congruence mod any desired power of 2^k) be indefinitely bisected algorithmically:

Lemma 3. For $1 \leq a < p/2$,

$$\begin{aligned} (a, p - a; p) = & \frac{1}{(p; p)} \left[(-q^{p-a}) \frac{(4p + 8a, 28p - 8a; 32p)(16p; 16p)}{(2p + 4a, 14p - 4a; 16p)} \right. \\ & + \frac{(20p - 8a, 12p + 8a; 32p)(16p; 16p)}{(10p - 4a, 6p + 4a; 16p)} - (q^a) \frac{(12p - 8a, 20p + 8a; 32p)(16p; 16p)}{(6p - 4a, 10p + 4a; 16p)} \\ & \left. + (q^{3p-2a}) \frac{(-4p + 8a, 36p - 8a; 32p)(16p; 16p)}{(-2p + 4a, 18p - 4a; 16p)} \right] \end{aligned} \tag{7}$$

Proof. The Jacobi Triple Product tells us

$$(a, p - a, p; p) = \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{pn(n-1)/2}.$$

By expanding out the right-hand side and grouping terms according to parities of binomial coefficients, we obtain

$$\begin{aligned}
 (a, p - a; p) &= \frac{1}{(p; p)} [(1 - q^a) + q^p(-q^{-a} + q^{2a}) + q^{3p}(q^{-2a} - q^{3a}) + \dots] \\
 &= \frac{1}{(p; p)} \left[\sum_{n=-\infty}^{\infty} -(q^{p-a})^{\binom{4n-1}{2}} (q^a)^{\binom{4n}{2}} + (q^{p-a})^{\binom{4n+1}{2}} (q^a)^{\binom{4n}{2}} \right. \\
 &\quad \left. -(q^{p-a})^{\binom{4n}{2}} (q^a)^{\binom{4n-1}{2}} - (q^{p-a})^{\binom{4n-2}{2}} (q^a)^{\binom{4n-1}{2}} \right] \\
 &= \frac{1}{(p; p)} \left[\sum_{n=-\infty}^{\infty} (q^{16p})^{\binom{n}{2}} (q^{2p+4a})^n (-q^{p-a}) + (q^{16p})^{\binom{n}{2}} (q^{10p-4a})^n \right. \\
 &\quad \left. -(q^{16p})^{\binom{n}{2}} (q^{6p-4a})^n (-q^a) + (q^{16p})^{\binom{n}{2}} (q^{-2p+4a})^n (q^{3p-2a}) \right]. \tag{8}
 \end{aligned}$$

Applying the Jacobi Triple Product to each of the four terms above gives the claimed dissection.

□

If the factor of $\frac{1}{(p;p)}$ is problematic for a desired bisection, simply note that the progressions involved are symmetric mod $2p$ as well. We get twice as many factors but do at least bisect successfully.

Remark: B_3 , B_5 and B_9 possess even-odd dissections which consist of a single finite η -product in each progression (see for instance [11]). It would surprise the author if this happened again.

It is easy to observe that

$$B_{m_1 m_2}(q) = \frac{(m_1; m_1)}{(1; 1)} \frac{(m_1 m_2; m_1 m_2)}{(m_1; m_1)} = B_{m_1}(q) B_{m_2}(q^{m_1}).$$

If dissections for $B_{m_1}(q)$ and $B_{m_2}(q)$ are known, this allows dissections for $B_{m_1 m_2}(q)$. For instance, from [11] we have

$$B_5(q) = \frac{(8; 8)(20; 20)^2}{(2; 2)^2(40; 40)} + q \frac{(4; 4)^3(10; 10)(40; 40)}{(2; 2)^3(8; 8)(20; 20)}$$

and

$$B_9(q) = \frac{(12; 12)^3(18; 18)}{(2; 2)^2(6; 6)(36; 36)} + q \frac{(4; 4)^2(6; 6)(36; 36)}{(2; 2)^3(12; 12)}.$$

Thus we can say

$$\begin{aligned}
 B_{45}(q) = & \left(\frac{(8; 8)(20; 20)^2}{(2; 2)^2(40; 40)} \frac{(60; 60)^3(90; 90)}{(10; 10)^2(30; 30)(180; 180)} \right. \\
 & + q^6 \frac{(4; 4)^3(10; 10)(40; 40)}{(2; 2)^3(8; 8)(20; 20)} \frac{(20; 20)^2(30; 30)(180; 180)}{(10; 10)^3(60; 60)} \\
 & + q \left(q^4 \frac{(8; 8)(20; 20)^2}{(2; 2)^2(40; 40)} \frac{(20; 20)^2(30; 30)(180; 180)}{(10; 10)^3(60; 60)} \right. \\
 & \left. \left. + \frac{(4; 4)^3(10; 10)(40; 40)}{(2; 2)^3(8; 8)(20; 20)} \frac{(60; 60)^3(90; 90)}{(10; 10)^2(30; 30)(180; 180)} \right) \right).
 \end{aligned}$$

Early investigation on the heritability of congruence properties from B_{m_1} to $B_{m_1,k}$ has been less than successful, except for the trivial cases in which a congruence is inherited from the original partition function. It seems plausible that this should occur under some circumstances, however, and this would seem to be a useful general theorem if one could be obtained.

4. Parity Speculation

We close with some thoughts to motivate future research.

If there were really no underlying structure to the parity of $p(n)$ we might expect that a quotient of the partition function and a magnified version of itself would again look structureless. But if the usual partition function possessed some kind of pseudo-regularity modulo 2 or powers of 2, we might then expect that in such a quotient some cancellation would occur and regularities arise. Since this is the phenomenon we see, it seems possible that further investigation of the parity and higher 2-adic structure of the m -regular partitions could yield some insight into the 2-adic behavior of the partition numbers themselves.

That is, if we can reasonably control the parity of $b_m(n)$, possibly by taking advantage of the frequent self-similarity of m -regular partition functions in arithmetic progression, we might be able to produce an efficient algorithm to describe the parity of $p(n)$, or even useful asymptotic analysis.

Consider the following. One easily observes (it was first remarked to the author by Bernard S. Lin) that when m is prime,

$$\prod \frac{1 - q^{m^2 k}}{1 - q^k} \equiv_m \prod \frac{(1 - q^{mk})^m}{1 - q^m} =: \sum_{n=0}^{\infty} a_m(n) q^n$$

by the well-known prime divisibility properties of the binomial coefficients. The numbers $a_m(n)$ are the numbers of m -cores of n , i.e., those partitions in which no hooklengths are of size m . Thus we immediately have

Proposition 1. For m prime, $b_{m^2}(n) \equiv a_m(n) \pmod{m}$.

Since there is a large literature of congruences for m -core partitions, this gives a collection of congruences for m^2 -regular partitions. For instance, from ([3], Corollary 8) on 3-cores, we obtain

Corollary 1. For $p \equiv 2 \pmod{3}$ prime, k a positive even integer,

$$b_9(n) \equiv_3 b_9 \left(p^k n + \frac{p^k - 1}{3} \right).$$

Of particular interest are the 4-regular partitions, for the parity of the number of 2-core partitions of n is exactly 1 when n is a triangular number and 0 otherwise.

Observe that we may describe the normal partition numbers $p(n)$ in terms of m -regular partition numbers: choosing $m = 4$, we obtain

Proposition 2.

$$p(n) = p(0)b_4(n) + p(1)b_4(n-4) + p(2)b_4(n-8) + p(3)b_4(n-12) + \dots$$

We see that this recurrence gives an algorithm for determining the parity of $p(n)$ which is shorter than the pentagonal number theorem, for among the $n/4$ terms $n - 4k$ roughly $\sqrt{n/4}$ will be triangular numbers, and only their coefficients $p(k)$ will need to be recursed. Perhaps further investigation of the theoretical properties of this algorithm would be interesting.

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