NOTES ON PRIMES $P \equiv 1 \bmod D$ AND $A^{P-1/D} \equiv 1 \bmod P$

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Received: 11/14/13, Revised: 12/19/14, Accepted: 3/8/15, Published: 5/29/15

Abstract

Let $d > 0$ be a squarefree integer and a be an integer, which is not -1 nor a square. Let $P_{(a,d)}(x)$ be the number of primes $p \leq x$ such that $p \equiv 1 \bmod d$ and $a^{(p-1)/d} \equiv 1 \bmod p$. Numerical data indicate that the function as approximately equal to a constant multiple of $\pi(x)/(d\varphi(d))$ for sufficiently large x, where $\pi(x)$ is the number of primes up to x and $\varphi(d)$ is the Euler-φ function. The involved constant multiple depends on both a and d. In this paper we obtain an average order of the function and explore some properties of the primes counted by the function.

1. Introduction

For a squarefree integer $d > 0$ and an integer a, which is not -1 nor a perfect square, let $\mathcal{P}_{(a,d)} = \{\text{primes } p : \ p \equiv 1 \bmod d \text{ and } a^{(p-1)/d} \equiv 1 \bmod p\}$ and $P_{(a,d)}(x) = \#\{p \in \mathcal{P}_{(a,d)} : p \leq x\}$. Note that $\mathcal{P}_{(a,d)}$ is the set of rational primes which split completely in the field $L = \mathbb{Q}[\sqrt[d]{a}, \sqrt[d]{1}]$, or the set of rational primes whose conjugacy class of Frobenius automorphisms of L over $\mathbb{Q}$ contains only the identity element in the Galois group $\mathrm{Gal}(L/\mathbb{Q})$.

Effective estimates of $P_{(a,d)}(x)$ are critical to the derivation of Artin's conjecture (see [4, 7]). The conjecture asserts that, if a is the integer mentioned above, then the number of prime moduli up to x, for which integer a is a primitive root, is proportional to the number of primes up x. Hooley proved this conjecture [4] under the assumption of the extended Riemann Hypothesis (ERH). Many other results (see [2, 3, 4, 11]) were achieved in favor of the conjecture. One may refer to Murty [10] for a survey of the conjecture. In his conditional proof of Artin's conjecture [4], Hooley obtained the following sharp estimate for $P_{(a,d)}(x)$ under ERH for Dedekind zeta functions over the Kummerian field L:

$$P_{(a,d)}(x) = \frac{\varepsilon(d)}{d\varphi(d)}\mathrm{li}(x) + O(x^{\frac{1}{2}}\log(dx)) \tag{1}$$

if a is squarefree (see [4] for other cases). To describe $\varepsilon(d)$, let us write $a = a_1 \cdot a_2^2$ where a_1 is squarefree. Then $\varepsilon(d) = 2$ if $a_1 \equiv 1 \bmod 4$ and $2a_1$ divides d, and $\varepsilon(d) = 1$ otherwise. It is suggested in [7] that a uniform upper bound of $P_{(a,d)}(x)$, such as $P_{(a,d)}(x) \ll \mathrm{li}(x)/d\varphi(d)$ or $P_{(a,d)}(x) \ll \pi(x)/d\varphi(d)$, could lead to another proof of the conjecture.

What else can be said about $P_{(a,d)}(x)$? First, an unconditional estimate of $P_{(a,d)}(x)$ can be found in [6], but the range allowed for d would be much smaller and the estimate for the error terms would be less sharp than those in (1). Secondly, numerical verification of (1) with $x = 10^8$, $3 \le a \le 51$ and $3 \le d \le 97$ (where both a and d are squarefree) indicates that the absolute difference between $P_{(a,d)}(x)$ and $\frac{\varepsilon(d)}{d\varphi(d)}\pi(x)$ is on average less than 1.37% of $P_{(a,d)}(x)$ and the maximum difference is less than 9.24% of $P_{(a,d)}(x)$.

The goal of the article is to investigate $P_{(a,d)}(x)$ on average and some properties of primes in $\mathcal{P}_{(a,d)}$. First we will prove the following lemma.

Lemma 1 (see [5, 9]). *Let a be an integer not divisible by p and $p \equiv 1 \bmod d$. Then*

$$\frac{1}{d} \sum_{\substack{\chi \bmod p \\ \mathrm{ord}\,(\chi) \mid d}} \chi(a) = \begin{cases} 1, & \textit{if } a^{\frac{p-1}{d}} \equiv 1 \bmod p \\ 0, & \textit{otherwise} . \end{cases}$$

Thus we can write

$$\begin{aligned} P_{(a,d)}(x) &= \sum_{\substack{p \le x \\ p \equiv 1 \bmod d}} \frac{1}{d} \sum_{\substack{\chi \bmod p \\ ord(\chi) \mid d}} \chi(a) \\ &= \frac{1}{d}\,\pi(x;1,d) + \frac{1}{d} \sum_{\substack{p \le x \\ p \equiv 1 \bmod d}} \sum_{\substack{\chi \ne \chi_0 \bmod p \\ ord(\chi) \mid d}} \chi(a), \end{aligned} \tag{2}$$

where $\pi(x;1,d)$ is the number of primes up to x which are congruent to 1 modulo d. If $a_1 \equiv 1 \bmod 4$ and $2a_1 \mid d$ then there is another character χ within the inner sum of (2) for which $\chi(a) = 1$. Indeed, this $\chi(a)$ is the Legendre symbol $\left(\frac{a}{p}\right)$, which is equal to

$$\left(\frac{a_1}{p}\right) = \left(\frac{p}{a_1}\right)$$

by the law of quadratic reciprocity and the condition $a_1 \equiv 1 \bmod 4$. $\left(\frac{p}{a_1}\right) = 1$ since $a_1 \mid p-1$. Thus (2) can be written as

$$P_{(a,d)}(x) = \frac{2}{d}\,\pi(x;1,d) + \frac{1}{d} \sum_{\substack{p \le x \\ p \equiv 1 \bmod d}} \sum_{\substack{\chi \ne \chi_0, \chi_1 \bmod p \\ ord(\chi) \mid d}} \chi(a), \tag{3}$$

where χ_1 is the Legendre symbol for modulus p. The first terms of (2) and (3) are the same as the major term of (1). Thus the rest sums in (2) and (3) are $O(x^{\frac{1}{2}}\log(dx)) + O(\mathrm{li}(x) - \pi(x))$ according to (1). In the next section we obtain a heuristic argument of why such an upper bound for the error terms should be within expectation. It would be desirable to prove that the sums in (2) and (3) are $o\left(\frac{\pi(x)}{d\varphi(d)}\right)$ unconditionally. Instead we can prove a similar result like this on average, namely the following theorem.

Theorem 2. *Let d be a positive squarefree integer such that $d \le x^{\frac{1}{2}}$. Let $E(x,y) = y^{-1/21}$ if $y \le x^{2/3}$ and $E(x,y) = x^{-1/6}\log x$ if $y > x^{2/3}$. Then, for sufficiently large x, we have*

$$\frac{1}{y}\sum_{2\le a\le y} P_{(a,d)}(x) = \frac{\pi(x;1,d)}{d} + O\left(\frac{x\cdot E(x,y)}{\varphi(d)\log x}\right)$$

provided that $y \ge \exp(3(\log x\log\log x)^{\frac{1}{2}})$ if $y \le x^{2/3}$. The involved constant in the error term above is independent of d.

The result that can be achieved in the theorem is due to the counting function by the character sum in Lemma 1, which makes it possible to apply Stephens' method first used in [11] on average results of Artin's conjecture. The exponent $-\frac{1}{21}$ in $E(x,y) = y^{-\frac{1}{21}}$ when $y \le x^{2/3}$ can be reduced to any number bigger than $-\frac{3}{16}$ if we require $y > x^{\delta}$ for some positive number δ. The similar method can also be used to obtain a non-trivial estimate for $\sum_{a\le y}\left(P_{(a,d)}(x) - \pi(x;1,d)/d\right)^2$.

From the above results it seems reasonable to conjecture that each individual $P_{(a,d)}(x)$ would follow the estimate of its average in Theorem 2, or a much weaker form $P_{(a,d)}(x) \ll \frac{\pi(x)}{d\varphi(d)}$ (such as $P_{(a,d)}(x) \le \frac{3\pi(x)}{d\varphi(d)}$ uniformly for every a and d). The weaker form may be helpful to deduce Artin's conjecture unconditionally (see [7]).

The author is grateful to the referee for valuable comments, and suggestions for sharper error term in Theorem 2.

2. Properties and Preliminaries

Proof of Lemma 1. Let g be a primitive root modulo p. Then $a \equiv g^r \bmod p$ for some integer r, and thus $a^{\frac{p-1}{d}} \equiv 1 \bmod p$ is equivalent to $d \mid r$. Let χ be a Dirichlet character modulo p whose order divides d. Then $\chi(g)$ is a d-th root of unity. We can write $\chi(g) = e^{\frac{2\pi i k}{d}}$ for $0 \le k \le d-1$. As k runs through all integers in the range, we get all the characters whose orders divide d. Therefore,

$$\sum_{\substack{\chi \bmod p \\ \mathrm{ord}\,(\chi)\,\mid\, d}} \chi(a) = \sum_{k=0}^{d-1} e^{\frac{2\pi i k r}{d}},$$

which is 0 if $d \nmid r$, and d if $d \,|\, r$. This concludes the proof. $\square$

We can look at the phenomenon in Lemma 1 from another perspective. Let $\mathbb{Z}_p^{(d)}$ be the set of all elements b^d where $b \in \mathbb{Z}_p^* = (\mathbb{Z}/p\mathbb{Z})^*$. Obviously $\mathbb{Z}_p^{(d)}$ is a subgroup of $\mathbb{Z}_p^*$. If $a \in \mathbb{Z}_p^{(d)}$, then $\chi(a) = 1$ for all characters $\chi \bmod p$ with $\operatorname{ord}(\chi)|d$. Conversely if the condition holds, then $d \,|\, r$ since we can take $k = 1$ where r and k are defined in the above proof. Thus $a \in \mathbb{Z}_p^{(d)}$. Therefore, we proved

Proposition 3. *Let $\mathbb{Z}_p^{(d)} = \{b^d : b \in (\mathbb{Z}/p\mathbb{Z})^*\}$. Then $a \in \mathbb{Z}_p^{(d)}$ if and only if $\chi(a) = 1$ for all the characters $\chi \bmod p$ with $\operatorname{ord}(\chi)|d$.*

Since $d \,|\, r$ is equivalent to $a^{\frac{p-1}{d}} \equiv 1 \bmod p$, one can deduce Lemma 1, using the above proposition, and that $\mathbb{Z}_p^{(d)} = \{a : a^{\frac{p-1}{d}} \equiv 1 \bmod p\}$. Thus the order of $\mathbb{Z}_p^{(d)}$ is $(p-1)/d$ if $d \,|\, p-1$. The chance that integer a, not divisible by p, is in $\mathbb{Z}_p^{(d)}$ is $((p-1)/d)/(p-1) = 1/d$ if $d \,|\, p-1$ and $\left(\frac{a}{p}\right) \neq 1$ (in the case where $2 \,|\, d$ and $\left(\frac{a}{p}\right) = 1$, the chance becomes $2/d$). Thus the chance that prime p satisfies $p \equiv 1 \bmod d$ and $a^{\frac{p-1}{d}} \equiv 1 \bmod p$ is $\varepsilon(d)/(d\phi(d))$ by distribution of primes in arithmetic progression. This presents a heuristic explanation of why $P_{(a,d)}(x) \sim \varepsilon(d)\pi(x)/(d\varphi(d))$ for large x.

Let us consider a homomorphism from $\mathbb{Z}_p^*$ to itself $\varphi : a \longmapsto a^{\frac{p-1}{d}}$. Obviously $\ker(\varphi) = \mathbb{Z}_p^{(d)}$, and thus $\mathbb{Z}_p^*/\mathbb{Z}_p^{(d)} \simeq \varphi(\mathbb{Z}_p^*)$, a cyclic subgroup of $\mathbb{Z}_p^*$ of order d. This proves the following fact.

Lemma 4. *Let p be a prime such that $d \,|\, p-1$. Then, for each $a \in \mathbb{Z}_p^*$, $a^{\frac{p-1}{d}}$ is a d-th root of unity modulo p and is uniquely determined by the coset of $\mathbb{Z}_p^{(d)}$ containing a.*

Let $p \equiv 1 \bmod d$ be a fixed prime. By Lemma 4, natural numbers a, for which $a^{\frac{p-1}{d}} \equiv 1 \bmod p$, are evenly distributed in the sense that there are always $(p-1)/d$ such integers in each interval of length p. Conversely, for a fixed integer a, the conjecture below Theorem 2 asserts that the primes p for which $p \equiv 1 \bmod d$ and $a^{\frac{p-1}{d}} \equiv 1 \bmod p$ are also distributed evenly in the sense that the ratio of $P_{(a,d)}(x)$ and $\pi(x)$ approaches $\varepsilon(d)/(d\varphi(d))$ as x goes to infinity. Here let us assume that $2 \nmid d$, which yields $\varepsilon(d) = 1$. The two phenomena on distribution are connected through the identity

$$\sum_{\substack{p \le x \\ p \equiv 1 \bmod d}} \sum_{a \le z} t_p(a) = \sum_{a \le z} \sum_{\substack{p \le x \\ p \equiv 1 \bmod d}} t_p(a)$$

where $t_p(a) = 1$ if $a^{\frac{p-1}{d}} \equiv 1 \bmod p$ and $t_p(a) = 0$ otherwise. Because, by the distribution phenomena, the left side becomes $\sum_{\substack{p \le x \\ p \equiv 1 \bmod d}} \frac{p-1}{d} \cdot \frac{z}{p} \sim \frac{z\pi(x)}{d\varphi(d)}$ if x is large and $z \ge x$, while the right side becomes $\sum_{a \le z} \frac{\pi(x)}{d\varphi(d)} \sim \frac{z\pi(x)}{d\varphi(d)}$. According to

Lemma 4, for a fixed prime p, the number of natural numbers a in each interval of length p, for which $a^{\frac{p-1}{d}}$ is a fixed d-th root of unity modulo p, is also $(p-1)/d$. It is not known whether or not a similar phenomenon on distribution holds for the primes $p \le x$, for which $a^{\frac{p-1}{d}} \bmod p$ is any fixed d-th root of unity modulo p.

Next let us consider integer r, for which $a \equiv g^r \bmod p$. Suppose that $\tilde{g}$ is another primitive root modulo p. Then $a \equiv \tilde{g}^{r'} \bmod p$ for another integer r' and $g \equiv \tilde{g}^s \bmod p$ where $\gcd(s, p-1) = 1$. Obviously $\gcd(r, p-1) = \gcd(r', p-1)$. What can we say about the GCD? Let set $S = \{r \in \mathbb{N} : a \equiv g^r \bmod p \text{ and } g \text{ is a primitive root } \bmod p\}$. Then S has a least element. Let us call it r_a although it also depends on p.

Theorem 5. *Let r_a be the integer defined above for a not divisible by p. Then $r_a \,|\, p-1$. Thus $r_a = \gcd(r, p-1)$ if $a \equiv g^r \bmod p$ and g is a primitive root $\bmod p$.*

Proof. Let a be a fixed integer not divisible by p. Since $\gcd(r, p-1)$ remains the same if g is a primitive root $\bmod p$ and $a \equiv g^r \bmod p$, we only need to show that $r_a \,|\, p-1$, which yields $r_a = \gcd(r, p-1)$.

Suppose that $a \equiv g^r \bmod p$ and g is a primitive root for p. If we can write $r = s \cdot r_1$ where $\gcd(s, p-1) = 1$, then $g_1 = g^s$ is another primitive modulo p and $a \equiv g^{s \cdot r_1} \equiv g_1^{r_1} \bmod p$.

If r_a has a nontrivial factor coprime to $p-1$, then the above process yields another primitive root for which the exponent for $a \bmod p$ would be smaller than r_a, a contradiction. Thus every prime factor of r_a is a factor of $p-1$. If $r_a \nmid p-1$, there must be a prime factor q of r_a such that the exponent of q in $p-1$ is smaller than the exponent of q in r_a. Let $q_1, q_2, \cdots, q_t$ be all the prime factors of r_a such that their exponents in r_a are the same as their exponents in $p-1$. Consider

$$r_a + q_1 \cdots q_t(p-1) = v.$$

Obviously each prime factor of r_a divides v, and each prime factor of $\gcd(v, p-1)$ divides r_a. In addition, each q_i has the same exponent in the prime factorizations of v and r_a. For each other prime factor l of r_a, the exponent of l in the prime factorization of v is the smaller exponent between the exponents of l in the prime factorizations of r_a and $p-1$. Thus we can write

$$v = s \cdot r'$$

where $\gcd(s, p-1) = 1$ and r' has no prime factors coprime to $p-1$.

Note that each prime factor of r' divides r_a, and the exponent of the prime factor in r' does not exceed that of the prime factor in r_a. Thus $r' \,|\, r_a$. It can also be noted that $r' < r_a$ because the exponent of prime q in the prime factorization of $p-1$ is strictly smaller than the exponent of q in the prime factorization of r_a by the above assumption. On the other hand, since $a \equiv g^{r_a} \equiv g^v = g^{s \cdot r'} \bmod p$ for some primitive root g modulo p, the process in the second paragraph of the proof implies $r' \in S$, contradiction to the minimum of r_a in S. Therefore, we have proved that $r_a \mid p-1$, and the theorem. $\square$

Lemma 6 (see [11]). *Let r be any natural number and x, y be positive real numbers. Then*

$$\sum_{n\leq x}\sum_{\chi \bmod n}{}' \left|\sum_{a\leq y}\chi(a)\right|^{2r} \ll (x^2+y^r)y^r\left(\ln(ey^{r-1})\right)^{r^2-1}, \tag{4}$$

where $\sum'$ means that the summation is taken over primitive characters only, and the involved constant is independent of r.

3. Proof of Theorem 2

By (2),

$$\begin{aligned}\frac{1}{y}\sum_{a\leq y}P_{(a,d)}(x) &= \frac{1}{y}\sum_{\substack{p\leq x\\ p\equiv 1 \bmod d}}\sum_{a\leq y}\frac{1}{d}\sum_{\substack{\chi \bmod p\\ \mathrm{ord}\,(\chi)\,\mid\, d}}\chi(a)\\ &= \frac{1}{y}\sum_{\substack{p\leq x\\ p\equiv 1\,(d)}}\sum_{a\leq y}\frac{\chi_0(a)}{d}+\frac{1}{y}\sum_{\substack{p\leq x\\ p\equiv 1\,(d)}}\sum_{a\leq y}\sum_{\substack{\chi\neq\chi_0\,(p)\\ \mathrm{ord}\,(\chi)\,\mid\, d}}\frac{\chi(a)}{d},\end{aligned} \tag{5}$$

where (d) means $\bmod\, d$ and the same for (p). The inner sum of the first double sum of (5) is $([y]-[y/p])/d$, and thus the double sum becomes

$$\frac{\pi(x;1,d)}{d}+O\left(\frac{\log\log x}{d}\right)+O\left(\frac{\pi(x;1,d)}{d\cdot y}\right). \tag{6}$$

Next let us focus on the triple sum in (5), which can be written as

$$\sum_{\substack{p\leq x\\ p\equiv 1\,(d)}}\sum_{\substack{\chi\neq\chi_0\,(p)\\ \mathrm{ord}\,(\chi)\,\mid\, d}}\frac{1}{d}\sum_{a\leq y}\chi(a) \ll \sum_{\substack{p\leq x\\ p\equiv 1\,(d)}}\sum_{\substack{\chi\neq\chi_0\,(p)\\ \mathrm{ord}\,(\chi)\,\mid\, d}}\frac{1}{d}\left|\sum_{a\leq y}\chi(a)\right|. \tag{7}$$

Let us denote the last triple sum by S. By the Pólya-Vinogradov inequality [1], we have

$$S \leq \sum_{\substack{p\leq x\\ p\equiv 1\,(d)}}\frac{d-1}{d}\cdot p^{\frac{1}{2}}\log p \leq x^{\frac{1}{2}}\log x\cdot\pi(x;1,d).$$

If $y > x^{\frac{2}{3}}$, by the Brun-Titchmarsh inequality $\pi(x;1,d) \ll \dfrac{x}{\varphi(d)\log x/d}$ (see [8]) and $d \leq x^{\frac{1}{2}}$, we have

$$\frac{S}{y} = O\left(\frac{x^{5/6}}{\varphi(d)}\right). \tag{8}$$

If $y \le x^{\frac{2}{3}}$, we apply Hölder's inequality and obtain

$$S^{2r} \le \left(\sum_{\substack{p \le x \\ p \equiv 1\,(d)}} \sum_{\substack{\chi \ne \chi_0\,(p) \\ \mathrm{ord}\,(\chi)\,|\,d}} \left(\frac{1}{d}\right)^{\frac{2r}{2r-1}} \right)^{2r-1} \cdot \sum_{\substack{p \le x \\ p \equiv 1\,(d)}} \sum_{\substack{\chi \ne \chi_0\,(p) \\ \mathrm{ord}\,(\chi)\,|\,d}} \left| \sum_{a \le y} \chi(a) \right|^{2r}$$

for any integer $r \ge 1$. Let s_1 and s_2 be the two factor on the right side. Then

$$s_1 = \left(\sum_{\substack{p \le x \\ p \equiv 1\,(d)}} \frac{d-1}{d} \cdot \frac{1}{d^{1/(2r-1)}} \right)^{2r-1} \le \frac{\pi(x;1,d)^{2r-1}}{d}.$$

Since each non-principal character modulo p is a primitive character,

$$s_2 \le \sum_{\substack{p \le x \\ p \equiv 1\,(d)}} \sum_{\chi \ne \chi_0\,(p)} \left| \sum_{a \le y} \chi(a) \right|^{2r} \ll (x^2 + y^r) y^r \left(\log(e y^{r-1})\right)^{r^2-1}$$

by Lemma 6. Choose $r = \left[\frac{2\log x}{\log y}\right] + 1$. Then $y^{r-1} \le x^2 < y^r$. Combining it with s_1 and s_2, we have

$$\begin{aligned} \frac{S}{y} &\ll \frac{1}{d^{\frac{1}{2r}}} \pi(x;1,d)^{1-\frac{1}{2r}} \left(\log(ex^2)\right)^{\frac{r^2-1}{2r}} \\ &\ll \frac{1}{\varphi(d)} \cdot \left(\frac{x}{\log x}\right)^{1-\frac{1}{2r}} \left(\log(ex^2)\right)^{\frac{r^2-1}{2r}}. \end{aligned} \tag{9}$$

Here we use the Brun-Titchmarsh inequality $\pi(x;1,d) \ll \dfrac{x}{\varphi(d)\log x/d}$ and $d \le x^{\frac{1}{2}}$. The involved constants in (9) are independent of r and d.

Since $y \le x^{2/3}$ we have $r \ge 4$ and $\frac{r^2-1}{2r} \le \frac{5}{8}(r-1) \le \frac{5\log x}{4\log y}$, which yields

$$\begin{aligned} \frac{S}{y} &\le \frac{1}{d^{(1/2r)}} \pi(x;1,d)^{1-\frac{1}{2r}} \left(\log(ex^2)\right)^{(5\log x)/(4\log y)} \\ &\ll \frac{1}{\varphi(d)} \cdot \left(\frac{x}{\log x}\right)^{1-\frac{1}{2r}} \left(\log(ex^2)\right)^{(5\log x)/(4\log y)}. \end{aligned} \tag{10}$$

But $r \le \frac{8\log x}{3\log y}$. Thus if we require $\log y > 3(\log x \log\log x)^{1/2}$, then

$$\begin{aligned} & -\frac{1}{2r}(\log x - \log\log x) + \frac{5\log x}{4\log y}\log\log(ex^2) \\ \le \quad & -\frac{3}{16}\log y + \frac{1}{8}\log\log x + \frac{5\log x}{4\log y}\log\log(ex^2) \le -\frac{1}{21}\log y \end{aligned}$$

when x is large enough. By (10), we have $S/y = O\left(\frac{x \cdot y^{-1/21}}{\varphi(d)\log x}\right)$. Combining this result with (5), (6), (7) and (8), we have proved Theorem 2.

It should be noted that if we require that $y > x^{\delta}$ for some positive number δ, then both positive terms in the second line of the above display are less than a constant multiple of $\log\log x$. Thus exponent $-\frac{1}{21}$ of y in the estimate for S/y can be reduced to any number bigger than $-\frac{3}{16}$.

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