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UNCHAINED *R*-SEQUENCES AND A GENERALIZED CASSINI FORMULA

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Abstract

This paper posits a new method to analyze r-sequences without limitation as to type of number or the value of the seeds. That method is then used to derive identities applicable to all r-sequences, including a counterpart to Cassini's Formula. The paper also shows that any element in an r-sequence can be expressed as a linear function of r randomly chosen consecutive elements of the r-sequence using coefficients which are identical for all sequences of the same order.

1. Introduction

There have been many articles concerning sequences which satisfy the recurrence equation

$$G_{i+1}^{(r)} = \sum_{j=0}^{r-1} G_{i-j}^{(r)} \tag{1}$$

where r is an integer greater than 1, the term (r) is a superscript and not an exponent, and i and j are any integers (including negative integers in the case of i). If a subscript i is negative, the value of $G_i^{(r)}$ is determined by

$$G_i^{(r)} = G_{i+r}^{(r)} - (\sum_{j=1}^{r-1} G_{i+j}^{(r)}). \tag{2}$$

The seeds (the values of $G_i^{(r)}$ for $0 \le i < r$) are often set at $G_0^{(r)} = 0, G_1^{(r)} = 1$ and $G_i^{(r)} = \sum_{j=0}^{i-1} G_j^{(r)}$ for $1 < i < r$. For instance, if $r = 4$, then $G_{i+1}^{(4)} = G_i^{(4)} + G_{i-1}^{(4)} + G_{i-2}^{(4)} + G_{i-3}^{(4)}$ for $i > 3$ and frequently $G_0^{(4)} = 0, G_1^{(4)} = 1, G_2^{(4)} = 1$ and $G_3^{(4)} = 2$, yielding the sequence $0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, ...$ *However, much of the analysis which follows is independent of the value of the seeds.* For example, the analysis applies equally to the 4-sequence $2, 5, 7, 14, 28, 54, 103, 199, ...$

and to the 4-sequence with seeds $\sqrt{13}, -2.9, \pi$ and $tan^{-1}(2)$. The analysis also applies regardless of whether the numbers in the sequence are positive or negative, rational or irrational, real or imaginary. As suggested in [6], it may be worthwhile to be spreading our nets more widely. The present article is the outgrowth of several prior articles covering broad results for uniform power identities [2, 3].

For $r = 3$ and 4, these sequences have been referred to as tribonacci and tetranacci sequences. Instead, we use the language of [5] and refer to sequences which fulfill the condition in (1) as *r-sequences* or *sequences of order r*, notwithstanding that our definition of the term is significantly broader than that used in [5].

2. A Preliminary Model: $r = 5$

Solely for the purpose of introducing certain concepts and terminology, we let $r = 5$, pick n randomly and look for certain patterns in sequences of order 5. For greater ease in discovering patterns, we set $v = G^{(5)}_{n+4}, w = G^{(5)}_{n+3}, x = G^{(5)}_{n+2}, y = G^{(5)}_{n+1}$ and $z = G^{(5)}_n$ and then expand for nearby values of $G^{(5)}_{n+s}$. As s increases, we calculate $G^{(5)}_{n+s}$ by repeated use of equation (1). For values of $G^{(5)}_j$ where $j < n$, we calculate $G^{(5)}_j = G^{(5)}_{j+5} - G^{(5)}_{j+4} - G^{(5)}_{j+3} - G^{(5)}_{j+2} - G^{(5)}_{j+1}$. Putting the results in tabular form, we obtain Table 1. We also tentatively define a related r-sequence $L^{(5)}_m = G^{(5)}_m + G^{(5)}_{m-1}$

s	$G^{(5)}_{n+s}$	$G^{(5)}_{n-s}$
0	z	$-$
1	y	$v - w - x - y - z$
2	x	$-v + 2w$
3	w	$-w + 2x$
4	v	$-x + 2y$
5	$v + w + x + y + z$	$-y + 2z$
6	$2v + 2w + 2x + 2y + z$	$2v - 2w - 2x - 2y - 3z$
7	$4v + 4w + 4x + 3y + 2z$	$-3v + 5w + x + y + z$
8	$8v + 8w + 7x + 6y + 4z$	$v - 4w + 4x$
9	$16v + 15w + 14x + 12y + 8z$	$w - 4x + 4y$
10	$31v + 30w + 28x + 24y + 16z$	$x - 4y + 4z$
11	$61v + 59w + 55x + 47y + 31z$	$4v - 4w - 4x - 3y - 8z$
12	$120v + 116w + 108x + 92y + 61z$	$-8v + 12w + 4x + 4y + 5z$

Table 1: r = 5

as the Lucasian counterpart to $G^{(5)}_m$. A similar table can be easily prepared for $L^{(5)}_m$. But for now we will be focusing on identities in which only $L^{(5)}_{n+1}$ appears from the sequence L. In the case of a 5-sequence, $L^{(5)}_{n+1} = y + z$.

Looking at Table 1, one can see certain identities without the need to solve a set of simultaneous linear equations. We note that $G^{(5)}_{n-5} = 2z - y$. Hence we have the *shift formula* catalogued as Theorem 1 in [1]:

$$G^{(5)}_{n+1} = 2G^{(5)}_{n} - G^{(5)}_{n-5}. \tag{3}$$

One also notes that $3y^2 + 6z^2 - (2z - y)^2 = 2y^2 + 4yz + 2z^2 = 2(y + z)^2$ or

$$3(G^{(5)}_{n+1})^2 + 6(G^{(5)}_{n})^2 - (G^{(5)}_{n-5})^2 = 2(L^{(5)}_{n+1})^2. \tag{4}$$

Likewise taking $G^{(5)}_{n+5}$ and $G^{(5)}_{n-1}$ and setting $A = w + x + y + z$, one sees that

$$\begin{aligned} {G^{(5)}_{n+5}}^2 - {G^{(5)}_{n-1}}^2 &= (v + A)^2 - (v - A)^2 \\ &= 4Av \\ &= 4Av + 4v^2 - 4v^2 \\ &= 4[v^2 + vw + vx + vy + vz] - 4v^2 \\ &= 4v[(v + w + x + y + z) - v] \\ &= 4G^{(5)}_{n+4}(G^{(5)}_{n+5} - G^{(5)}_{n+4}). \end{aligned} \tag{5}$$

Similarly, taking $G^{(5)}_{n+6}$ and $G^{(5)}_{n-6}$, one obtains the identity

$${G^{(5)}_{n+6}}^2 - {G^{(5)}_{n-6}}^2 = 8(G^{(5)}_{n+5} - G^{(5)}_{n+4})(2G^{(5)}_{n+4} - G^{(5)}_{n}). \tag{6}$$

Not only does the above method dispense with solving simultaneous equations in deriving identities, but it also provides a more transparent view of the relationship among numbers in a sequence.

3. All $r > 1$

We now generalize for all integers $r > 1$. For any integer $r > 1$, we set r variables to equal $G^{(r)}_n$ and the numbers in the sequence immediately after $G^{(r)}_n$. We label the variables v_0 to v_{r-1}. We then let $v_i = G^{(r)}_{n+i}$ and construct Table 2 by letting $A = G^{(r)}_{i+1} = \sum_{j=0}^{r-1} G^{(r)}_{i-j}$. Then $G^{(r)}_{i+2} = G^{(r)}_{i+1} + A - G^{(r)}_{i-r} = 2A - G^{(r)}_{i-r}$, etc. For negative i, we apply equation (2) by iteration.

Table 2 can be extended indefinitely. Using the same techniques used in the prior section, one can readily derive the following identities which hold true for any

s	$G^{(r)}_{n+s}$	$G^{(r)}_{n-s}$
0	v_0	$-$
1	v_1	$v_{r-1} - \sum_{j=0}^{r-2} v_j$
2	v_2	$2v_{r-2} - v_{r-1}$
3	v_3	$2v_{r-3} - v_{r-2}$
4	v_4	$2v_{r-4} - v_{r-3}$
. . . .	....	.....
r-1	v_{r-1}	$2v_1 - v_2$
r	$\sum_{j=0}^{r-1} v_j$	$2v_0 - v_1$
r+1	$2(\sum_{j=1}^{r-1} v_j) + v_0$	$2v_{r-1} - \sum_{j=1}^{r-2} v_j - 3v_0$
r+2	$4(\sum_{j=2)}^{k} v_j) + 3v_1 + 2v_0$	$-3v_{r-1} + 5v_{r-2} + \sum_{j=0}^{r-3} v_j$
r+3	$8(\sum_{j=3)}^{k} v_j) + 7v_2 + 6v_1 + 4v_0$	$v_{r-1} - 4v_{r-2} + 4v_{r-3}$
r+4	$16(\sum_{j=4)}^{k} v_j) + 15v_2 + 14v_2 + 12v_1 + 8v_0$	$v_{r-2} - 4v_{r-3} + 4v_{r-4}$
r+4	add prior r entries above	$v_{r-3} - 4v_{r-4} + 4v_{r-5}$
r+5	add prior r entries above	$v_{r-4} - 4v_{r-5} + 4v_{r-6}$
r+6	add prior r entries above	$v_{r-5} - 4v_{r-6} + 4v_{r-7}$

Table 2: A neighborhood for general r

r-sequence where $r \geq 2$.

$$3{G^{(r)}_{n+1}}^2 + 6{G^{(r)}_n}^2 - {G^{(r)}_{n-r}}^2 = 2{L^{(r)}_{n+1}}^2, \tag{7}$$

$${G^{(r)}_{n+r}}^2 - {G^{(r)}_{n-1}}^2 = 4G^{(r)}_{n+r-1}(G^{(r)}_{n+r} - G^{(r)}_{n+r-1}), \tag{8}$$

$${G^{(r)}_{n+(r+1)}}^2 - {G^{(r)}_{n-(r+1)}}^2 = 8(G^{(r)}_{n+r} - G^{(r)}_{n+r-1})(2G^{(r)}_{n+r-1} - G^{(r)}_n). \tag{9}$$

One can readily derive other identities which hold true for all r-sequences.

4. Cassini's Formula

A review of Table 2 suggests a general counterpart to Cassini's formula.

Theorem 1. *For any r-sequence $G^(r)$ where r, m and n are integers and $n = m - (r + 1)$,*

$${G^{(r)}_m}^2 - G^{(r)}_{m+1}G^{(r)}_{m-1} = {G^{(r)}_n}^2 - G^{(r)}_{n+r}G^{(r)}_{n-r}. \tag{10}$$

Proof. If we let $A = \sum_{j=2}^{r-1} v_j$, then

$$
\begin{aligned}
{G_m^{(r)}}^2 - G_{m+1}^{(r)} G_{m-1}^{(r)} &= [2A + (2v_1 + v_0)]^2 - [A + (v_1 + v_0)][4A + (3v_1 + 2v_0)] \\
&= 4A^2 + 4A(2v_1 + v_0) + (2v_1 + v_0)^2 \\
&\quad - [4A^2 + A(7v_1 + 6v_0) + (v_1 + v_0)(3v_1 + 2v_0)] \\
&= A(v_1 - 2v_0) + [(v_1)^2 - (v_1)(v_0) - (v_0)^2] \\
&= A(v_1 - 2v_0) + [(v_1 + v_0)(v_1 - 2v_0)] + (v_0)^2 \\
&= (A + v_1 + v_0)(v_1 - 2v_0) + (v_0)^2 \\
&= ({v_0}^2) - [(2v_0 - v_1)(\sum_{j=0}^{r-1} v_j)] \\
&= (G_n^{(r)})^2 - G_{n-r}^{(r)} G_{n+r}^{(r)}. \qquad (11)
\end{aligned}
$$

□

This generalized version of Cassini's formula holds for all values of $r > 1$ without regard to the values of the seeds. For $r = 2$ and the traditional Fibonacci sequence, Theorem 1 gives the traditional Cassini Formulae ${F_m}^2 - F_{m+1}F_{m-1} = (-1)^{m+1}$ and ${F_n}^2 - F_{n+2}F_{n-2} = (-1)^n$. In addition, if $r = 2$ and the seeds of the traditional Fibonacci sequence are varied so that $F_0 = A$ and $F_{1.} = B$, then Theorem 1 yields ${G_n^{(2)}}^2 - G_{n+1}^{(2)} G_{n-1}^{(2)} = (B^2 - A^2 - AB)(-1)^{n+1}$.

It is stated in [7] that the identity ${L_n}^2 - 5{F_n}^2 = 4(-1)^n$ is a fundamental ingredient in many 2-sequence identities. But that identity is a mere reformulation of Cassini's formula.

$$
\begin{aligned}
4(-1)^n &= {L_n}^2 - 5{F_n}^2 \\
&= (F_{n+1} + F_{n-1})^2 - 4F_{n+1}F_{n-1} + 4F_{n+1}F_{n-1} - 5{F_n}^2 \\
&= [{F_{n+1}}^2 + 2F_{n+1}F_{n-1} + {F_{n-1}}^2 - 4F_{n+1}F_{n-1}] + 4F_{n+1}F_{n-1} - 5{F_n}^2 \\
&= (F_{n+1} - F_{n-1})^2 + 4F_{n+1}F_{n-1} - 5{F_n}^2 \\
&= {F_n}^2 + 4F_{n+1}F_{n-1} - 5{F_n}^2 \\
&= 4F_{n+1}F_{n-1} - 4{F_n}^2. \qquad (12)
\end{aligned}
$$

It may be that (10) plays a similar role when $r > 2$. See [4] for the author's use of Cassini's formula in non-uniform power identities.

5. Primary r-Sequences

Returning to Table 1, one notes that the coefficients of v form a 5-sequence. The same holds for each of w, x, y and z. In tabular form, they are set forth in Table

3. For ease of reference, we will temporarily call these sequences $(V), (W), (X), (Y)$ and (Z) with subscripts to coincide with the value of s at the top of Table 3. For example, $x_8 = 7$.

s	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
V	2	0	0	0	−1	1	0	0	0	0	1	1	2	4	8	16	31
W	−2	0	0	−1	2	−1	0	0	0	1	0	1	2	4	8	15	30
X	−2	0	−1	2	0	−1	0	0	1	0	0	1	2	4	7	14	28
Y	−2	−1	2	0	0	−1	0	1	0	0	0	1	2	3	6	12	24
Z	−3	2	0	0	0	−1	1	0	0	0	0	1	1	2	4	8	16

Table 3: Primary Sequences for r = 5

Note that

(1) (V) and (Z) are the basic 5-sequence (as presented in [5]) shifted respectively one and two notches up;

(2) $y_n = z_n + z_{n-1}, x_n = z_n + y_{n-1}, w_n = z_n + x_{n-1}$ and $v_n = z_n + w_{n-1}$; and

(3) the relationships in (2) can be joined together to yield

$$
\begin{aligned}
y_n &= z_n + z_{n-1} \\
x_n &= z_n + z_{n-1} + z_{n-2} \\
w_n &= z_n + z_{n-1} + z_{n-2} + z_{n-3} \\
v_n &= z_n + z_{n-1} + z_{n-2} + z_{n-3} + z_{n-4} \\
&= z_{n+1}. \qquad (13)
\end{aligned}
$$

We dub these sequences as the *primary sequences of order 5* and then distinguish them from one another (in the order Z, Y, X, W and V) as the *first, second, etc* primary sequence of order 5. The sequences will be represented by $P_i^{(5)}$ for $i = 1$ to r. The individual numbers in the sequences will be symbolized by $p_{i,j}^{(5)}$ for the j-th number in sequence $P_i^{(5)}$.

This table can be expanded to any $r > 1$. Instead of arranging the primary sequences horizontally as in Table 3, we arrange them vertically on a new Table 4, with the first primary sequence $P_1^{(r)}$ on the right. To save space, we exclude the easy cases of $r < 5$. For ease of reference, we break the table into three segments. In the middle segment, we arbitrarially choose the seed $p_{i,0}^{(r)}$ to be that row in which $p_{1,0}^{(r)} = 1$ and $p_{i,0}^{(r)} = 0$ for $1 < i \leq r$.

$P^{(r)}_{i,j}$	$i = r$	$r-1$	$r-2$	...	...	6	5	4	3	2	1
j =											
$r+4$	16	16	16	16	16	16	16	15	14	12	8
$r+3$	8	8	8	8	8	8	8	8	7	6	4
$r+2$	4	4	4	4	4	4	4	4	4	3	2
$r+1$	2	2	2	2	2	2	2	2	2	2	1
r	1	1	1	1	1	1	1	1	1	1	1
$r-1$	1	0	0	0	0	0	0	0	0	0	0
$r-2$	0	1	0	0	0	0	0	0	0	0	0
$r-3$	0	0	1	0	0	0	0	0	0	0	0
...	...	...	...	...	...	...	...	...	...	...	...
4	0	0	0	0	0	0	1	0	0	0	0
3	0	0	0	0	0	0	0	1	0	0	0
2	0	0	0	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1
-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-2	-1	2	0	0	0	0	0	0	0	0	0
-3	0	0	0	-1	2	0	0	0	0	0	0
-4	0	0	-1	2	0	0	0	0	0	0	0
...	...	...	...	...	...	...	...	...	...	...	...
$-r+2$	0	0	0	0	0	0	0	-1	2	0	0
$-r+1$	0	0	0	0	0	0	0	0	-1	2	0
$-r$	0	0	0	0	0	0	0	0	0	-1	2
$-r-1$	2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-3
$-r-2$	-3	5	1	1	1	1	1	1	1	1	1
$-r-3$	1	-4	4	0	0	0	0	0	0	0	0
$-r-4$	0	1	-4	4	0	0	0	0	0	0	0

Table 4: Primary Sequences for $r > 4$

In the first segment of the table, we set forth the coefficents of the identity $G^{(r)}_k = \sum_{j=0}^{r-1}(p^{(r)}_{j,k}G^{(r)}_j)$ where $k > r-1$, based on the following iterative process. Let $A = \sum_{j=0}^{r-1} G^{(r)}_{i-j}$. Then $G^{(r)}_r = A, G^{(r)}_{r+1} = G^{(r)}_r + A - G^{(r)}_0 = 2A - G^{(r)}_0$; $G^{(r)}_{r+2} = G^{(r)}_{r+1} + G^{(r)}_r + A - G^{(r)}_1 - G^{(r)}_0 = 4A - G^{(r)}_1 - 2G^{(r)}_0$; etc. The entries for $r+k$ where $k < r$ are either $2^k - 2^{k-i}$ or $2^k - 2^{k-i} - 1$. The third segment of the Table A displays the values of the sequences as the subscripts move into negative territory, using equation (2) by iteration.

One can now easily posit the following theorem:

Theorem 2. *For any numbers $G^{(r)}_m$ and $G^{(r)}_n$ which are part of an r-sequence $G^{(r)}_{i+1} = \sum_{j=0}^{r-1} G^{(r)}_{i-j}$ where r, m, n, q, j and i are integers and $r > 1$, there exists r*

primary r-sequences $p_{q,k}^{(r)}$ *(being the k-th number in the q-th primary r-sequence) such that* $G_m^{(r)} = \sum_{j=0}^{r-1} [(p_{j+1,m}^{(r)})(G_{n+j}^{(r)})]$.

Proof. The proof follows from the definition of the primary sequences. □

Other findings include:

(1) $p_{2,j}^{(r)} = [p_{1,j}^{(r)} + p_{1,j-1}^{(r)}]$ and, for $2 < i \le r$, $p_{i,j}^{(r)} = [p_{1,j}^{(r)} + p_{i-1,j-1}^{(r)}]$, and
(2) for $1 < i \le r$, $p_{i,j}^{(r)} = \sum_{j=0}^{i-1} [p_{1,i-j}^{(r)}$, and
(3) the values of $p_{i,j}^{(r)}$ for $1 \le i \le r$ and $j > r$ can be expressed as a linear function of powers of 2.

All of these results can be proved using simple induction. I leave to the reader the derivation of the function referred to in clause (3) of the immediately preceding paragraph.

Most importantly, the primary sequences for any r-sequence are identical to that of any other sequence of the same order, regardless of the seeds of the r-sequence and regardless of whether the numbers of the r-sequence are positive or negative, rational or irrational, real or imaginary. Hence, in many cases, analysis of an r-sequence can be reduced to an analysis of the primary r-sequences. For instance, if one extends Table 4 upward, by just looking at Table 4, the possibility of the identity

$$G_{n+2}^{(r)} - 4G_{n+1}^{(r)} + 4G_n^{(r)} = G_{n-2r}^{(r)} \tag{14}$$

pops out. As indicated in [7], the derivation of an identity is often more difficult than the proof of the identity. The structure set forth in this paper makes the derivation considerably easier for a large number of identities.

6. Topics for Future Study

As indicated in [5], much further work needs to be done. Some issues are suggested in [5]. Still additional issues are raised by this paper. For example, [5] expresses a view as to the counterpart to the Lucas sequence for r-sequences. The author of this article believes that there are better candidates, based in part on the results set forth above. The author hopes to address that issue in a separate paper.

In addition, the author solicits the input of readers on the following conjectures:

1. In order that an r-sequence as defined solely by (1) be *seed neutral* (that is, the sequence applies regardless of the *seeds*), the spread of the identity (the difference

between the highest and lowest subscripts in the identity) must be at least r, the order of the sequence.

2. For any integers r and s where $r > 2$ and $0 < s < r$

$$\sum_{j=0}^{s} 2^j(-1)^{(s-j)}\binom{s}{j}G_{s-j}^{(r)} = G_{n-sr}^{(r)}. \tag{15}$$

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