DETAILED STRUCTURE FOR FREIMAN'S $3k - 3$ THEOREM

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Abstract

Let A be a set of k integers. We study Freiman's inverse problem with small doublings and continue the work of G. A. Freiman, I. Bardaji and D. J. Grynkiewicz by characterizing the detailed structure of A in Theorem 2.2 below when the sumset $A+A$ contains exactly $3k-3$ integers. Besides some familiar structures, such a set A can have a configuration composed of "additively minimal triangles."

1. Introduction and Propositions

The letters A, B always represent finite sets of integers and $|A|$ means the cardinality of A. Let $A \pm B := \{a \pm b : a \in A,\, b \in B\}$ for any sets of integers A and B. Let $2A := A + A$. If a is an integer, then $A \pm a := A \pm \{a\}$ and $a \pm A := \{a\} \pm A$.

Freiman's inverse problem for small doubling constants seeks structural information of A or $2A$ when the size of $2A$ is small, say for example, less than $4|A|$.

A set B is called a *bi-arithmetic progression* if $B = I_0 \cup I_1$ where I_0 and I_1 are arithmetic progressions with a common difference such that $2I_0$, $I_0 + I_1$, $2I_1$ are pairwise disjoint[1]. The common difference of I_0 and I_1 is called the difference of B. The expression $B = I_0 \cup I_1$ gives a (bi-arithmetic progression) decomposition of B. For example, $B = \{0, 3, 5, 6, 8\}$ is a bi-arithmetic progression of difference 3 and has a decomposition $\{0, 3, 6\} \cup \{5, 8\}$.

Let G and G' be two abelian semi-groups, $A \subseteq G$ and $B \subseteq G'$. A bijection $\varphi : A \mapsto B$ is a Freiman isomorphism (of order 2) if

$$a + b = c + d \text{ if and only if } \varphi(a) + \varphi(b) = \varphi(c) + \varphi(d)$$

for any $a, b, c, d \in A$.

The following two classical theorems on Freiman's inverse problem with small doublings were proven more than fifty years ago (see [2, page 11, page 15] or [8]).

[1] B is a bi-arithmetic progression if and only if B is Freiman isomorphic to two parallel line segments of the integral lattice points on the plane.

Theorem 1.1 (G. A. Freiman). *Let A be a set of k integers with $k > 2$. If $|2A| = 2k-1+b < 3k-3$, then A is a subset of an arithmetic progression of length at most $k+b$.*

Theorem 1.2 (G. A. Freiman). *Let A be a set of k integers with $k \geqslant 2$. If $|2A| = 3k-3$, then one of the following is true.*

1. *A is a bi-arithmetic progression;*
2. *A is a subset of an arithmetic progression of length at most $2k-1$;*
3. *$k = 6$ and A is a Freiman isomorphism image of the set K_6 where*

$$K_6 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\} \subseteq \mathbb{Z}^2. \tag{1}$$

Notice that the part 2 above implies that $k > \frac{1}{2}|I|$, i.e. A is a large subset of the arithmetic progression I.

Part 1 and 2 in Theorem 1.2 show the regularity of the structure of A when $|2A| = 3k-3$. Part 3 is an exception. If A is the set $\{0,1,2,a,a+1,2a\}$ for any $a > 3$, then A is Freiman isomorphic to K_6. Clearly, this A can be made neither a subset of an arithmetic progression nor a subset of a bi-arithmetic progression of reasonable length.

We call each element in $V = \{(0,2),(2,0),(0,0)\}$ a vertex of K_6. Notice that each permutation of V can be extended to a Freiman isomorphism from K_6 to K_6. If $\varphi : K_6 \mapsto B$ is a Freiman isomorphism, we also call the elements in $\varphi(V)$ vertices of $\varphi(K_6)$.

Theorem 1.2 is much more difficult to prove than Theorem 1.1 is. There has been a few generalizations of Theorem 1.2. In [5] it is proved that the structure of A is the same as the structure of A characterized in Theorem 1.2 when $|A \pm A| = 3k-3$. In [6], a generalization of Theorem 1.2 is given, which characterizes the structure of A when k is sufficiently large and $|2A| = 3k-3+b$ for $0 \leqslant b \leqslant \epsilon k$, where ϵ is a small positive real number independent of k.

Recently, Freiman discovered in [3, 4] some interesting detailed structural information of A when $|2A| < 3k-3$. By saying "detailed structural information" we mean any structural information other than that of A being a large subset of an arithmetic progression. The term "detailed structure" first appeared in [4]. The main result in [3, 4] is the following.

Theorem 1.3 (A. G. Freiman, 2009). *Let A be a set of k integers. If $|2A| < 3k-3$, then $2A$ contains an arithmetic progression of length $2k-1$.*

For the sum of two distinct sets, the following theorem in [1] adds extra detailed structural information to the structural information obtained in [7] and [9].

Theorem 1.4 (I. Bardaji and D. J. Grynkiewicz, 2010). *Let A and B be nonempty sets of k_1 and k_2 integers, respectively, with*

$$\max B - \min B \leqslant \max A - \min A \leqslant k_1 + k_2 - 3$$

$$\text{and } |A+B| \leqslant k_1 + 2k_2 - 3 - \delta(A,B).$$

Then $A+B$ contains an arithmetic progression of length k_1+k_2-1.

The number $\delta(A,B)$ in Theorem 1.4 is defined to be 1 if $A+t \subseteq B$ for some integer t and 0 otherwise. If checking the proof in [1] carefully, the reader can find that the condition $\max A - \min A \leqslant k_1+k_2-3$ in Theorem 1.4 can be weakened to $\max A - \min A \leqslant k_1+k_2-2$ when $\max B - \min B < \max A - \min A$.

In this paper we seek *detailed* structural information for A when $|2A| = 3k-3$. The most part of the structural information we have found is consistent with that in Theorem 1.3 and Theorem 1.4. But there are some significant exceptions involving a new concept of set configurations called triangles (see Definition 1.7 and Theorem 2.2).

Let a and b be integers. Throughout this paper we will write $[a,b]$ for the interval of *integers* between a and b including a and b. Notice that $[a,b] = \emptyset$ if $a > b$. For any set A of integers, we will use the following notation:

$$A(a,b) := |A \cap [a,b]|.$$

We now introduce a few propositions, which will be used in the proof of the main result.

Proposition 1.5. *If $A(x,y) > \frac{1}{2}(y-x+1)$, then $y+x \in 2A$.*

Proof. The conclusion is true because $A \cap (x+y-A) \cap [x,y] \neq \emptyset$. □

Proposition 1.6. *If $\varphi : K_6 \mapsto B \subseteq \mathbb{Z}$ is an Freiman isomorphism from K_6 in (1) to B, then*

1. *$\min B$ and $\max B$ are vertices of $\varphi(K_6)$.*
2. *If $x, y \in B$ are vertices, then $\frac{1}{2}(x+y) \in B$.*
3. *If $B \subseteq [a,b]$, then $b-a \geqslant 10$.*
4. *If $B \subseteq [0,10]$, then B is either $B_1 = \{0,1,2,5,6,10\}$, or $B_2 = \{0,2,4,5,7,10\}$, or $B_3 = 10 - B_1$, or $B_4 = 10 - B_2$.*

Proof. Parts 1 and 2 follow from the definition of Freiman isomorphism. For Part 3, suppose $\varphi(\{(0,0),(0,1),(0,2)\}) = \{a, a+d, a+2d\}$ where $a = \min B$. Then part 3 can be easily verified for $d = 1,2,3$, or $\geqslant 4$. For Part 4, suppose $\varphi(\{(0,0),(0,1),(0,2)\}) = \{0,d,2d\}$. Then part 4 can be easily verified for $10 = \varphi((0,2))$ or $10 = \varphi((2,0))$. □

We introduce new names of some set configurations in order to be efficient and informative in describing them in part 4 of Theorem 2.2.

Definition 1.7. Let $B \subseteq [u, v]$.

- B is **anti-symmetric** in $[u, v]$ if
$$B \cap (u + v - B) = \emptyset \text{ and } B \cup (u + v - B) = [u, v];$$
- B is **half dense** in $[u, v]$ if $B(u, v) = \frac{1}{2}(v - u + 1)$;
- A half dense set B in $[u, v]$ is a **forward triangle** in $[u, v]$ if $B(u, x) > \frac{1}{2}(x - u + 1)$ for any $x \in [u, v - 1]$. We denote $\mathcal{FT}[u, v]$ for the collection of all forward triangles in $[u, v]$;
- A half dense set B in $[u, v]$ is a **backward triangle** in $[u, v]$ if $B(x, v) > \frac{1}{2}(v - x + 1)$ for any $x \in [u + 1, v]$. We denote $\mathcal{BT}[u, v]$ for the collection of all backward triangles in $[u, v]$;
- $B \in \mathcal{FT}[u, v]$ is **additively minimal** if
$$|2(B \cup \{v + 1\})| = 3(|B| + 1) - 3;$$
- $B \in \mathcal{BT}[u, v]$ is **additively minimal** if
$$|2(B \cup \{u - 1\})| = 3(|B| + 1) - 3;$$
- Let $\mathcal{FT}_{am}[u, v]$ and $\mathcal{BT}_{am}[u, v]$ denote the collection of all additively minimal forward triangles and the collection of all additively minimal backward triangles, respectively.

We call the interval $[u, v]$ in Definition 1.7 the *host interval* of B because B is half dense in $[u, v]$ even though u or v may not be in B.

The following are some consequences of Definition 1.7. For simplicity, we sometimes list only the properties of forward triangles. For backward triangles, one can easily formulate symmetric properties.

Proposition 1.8. *Let $B \subseteq [u, v]$.*

1. *B is anti-symmetric in $[u, v]$ if and only if $B(u, v)$ is half dense in $[u, v]$ and $u + v \notin 2B$.*
2. *If $B \in \mathcal{FT}[u, v]$ and $v - u > 1$, then $u, u + 1 \in B$ and $v, v - 1 \notin B$.*
3. *If $B \in \mathcal{FT}[u, v]$ and $b > v$, then*
$$2(B \cup \{b\}) \supseteq [2u, u + v - 1] \cup (b + (B \cup \{b\})),$$
which implies that $|2(B \cup \{b\})| \geqslant 3|B \cup \{b\}| - 3$.

4. *If $B \in \mathcal{FT}[u,v]$, then B is additively minimal if and only if B is anti-symmetric in $[u,v]$ and*

$$(2B) \cap (v+1+[u,v]) \subseteq v+1+B.$$

5. *If $B \in \mathcal{FT}[u,v]$, then either $B = \left[u, \frac{1}{2}(u+v-1)\right]$ or $|2(B \cup \{b\})| > 3|B \cup \{b\}| - 3$ for any $b > v+1$.*

6. *If $C \in \mathcal{BT}_{am}[1,u]$ and $D \in \mathcal{FT}_{am}[u+2, n-1]$ for some $4 \leqslant u \leqslant n-6$ and $A = \{0\} \cup C \cup D \cup \{n\}$, then $|2A| = 3|A| - 3$.*

7. *If $P = \{0, 2, \ldots, 2(m-1)\}$ and $B \in \mathcal{FT}_{am}[2m, n-1]$ for $m \in \left[0, \frac{1}{2}n - 2\right]$, and $A = P \cup B \cup \{n\}$, then $|2A| = 3|A| - 3$.*

Proof. Parts 1, 2, and 3 are easy.

Part 4: Let $b = v+1$ in part 3. Then B is additively minimal if and only if two sides of the displayed expression in part 3 are the same set, which is true if and only if $u+v \notin 2B$ and $(2B) \cap (v+1+[u,v]) \subseteq v+1+B$. Now B is anti-symmetric if and only if $u+v \notin 2B$ by part 1 above. Notice that $|[2u, u+v-1]| = v-u = 2|B \cup \{v+1\}| - 3$.

Part 5: For convenience we can assume, without loss of generality, that $u = 0$. Suppose that B is not an interval.

Let $a = \max B$. Then $a > \frac{1}{2}(v-1)$, which implies $a \geqslant v-a$. If $v-a \notin B$, then $a > v-a$ because $a \in B$. Let $b > v+1$. It suffices to show that either $v \in 2B$ or $v+1 \in 2B$ by part 3. Suppose that $v \notin 2B$. Then B is anti-symmetric in $[0,v]$ and $v-a \notin B$. Hence

$$\frac{1}{2}(v+1) = B(0,v) = B(0, v-a-1) + B(v-a+1, a) \leqslant v-a + B(v-a+1, a),$$

which implies that $B(v-a+1, a) \geqslant \frac{1}{2}(2a-v+1) > \frac{1}{2}(2a-v)$. Hence

$$v+1 = a + (v-a+1) \in 2(B \cap [v-a+1, a]) \subseteq 2B$$

by Proposition 1.5.

Part 6: Let $A = \{0\} \cup C \cup D \cup \{n\}$. Then $|A| = |C| + |D| + 2 = \frac{1}{2}u + \frac{1}{2}(n-u-2) + 2 = \frac{1}{2}(n+2)$. By the additive minimality of the triangles C and D we have

$$\begin{aligned} |2A| &= |0 + (\{0\} \cup C)| + |[u+2, u+2+n-2]| + |n + (D \cup \{n\})| \\ &= |A| + n - 1 = 3|A| - 3. \end{aligned}$$

Part 7: Let $A = P \cup B \cup \{n\}$. Then $|A| = |P| + |B| + 1 = m + \frac{1}{2}(n-2m) + 1 = \frac{1}{2}n + 1$. By the additive minimality of the triangle B and the fact that $2m, 2m+1 \in B$, we have that

$$\begin{aligned} |2A| &= |P| + |[2m, 2m+n-2]| + |B| + 1 \\ &= m + n - 1 + \frac{1}{2}(n-2m) + 1 = \frac{3}{2}n = 3|A| - 3. \end{aligned}$$

□

Remark 1.9. Part 4 of Proposition 1.8 gives the structure of an additively minimal forward triangle. A symmetric structure can be described for an additively minimal backward triangle.

Part 5 of Proposition 1.8 justifies the definition of a forward triangle being additively minimal by looking at the cardinality of $2(B \cup \{v+1\})$ instead of the cardinality of $2(B \cup \{b\})$ for any $b > v+1$. Since $v+1$ is implicitly determined by the definition of additive minimality, we can call B additively minimal in its host interval $[u, v]$ without mentioning the element $v+1$.

Blank Assumption. After normalization, we can always assume, throughout this paper, that the set A satisfies (2) below with letters n and k reserved throughout this paper.

$$0 = \min A,\ \gcd(A) = 1, n = \max A, \text{ and } k = |A|. \tag{2}$$

Proposition 1.10. *Suppose that* $0 < a < b < n$ *and* $A \cap [a, b] = \{a, b\}$.

1. *Clearly,*
$$(2(A \cap [0, b])) \cap (2(A \cap [a, n])) = \{2a, a+b, 2b\}. \tag{3}$$

2. *If* $|2A| = 3k - 3$,
$$|2(A \cap [0, b])| \geqslant 3A(0, b) - 3, \quad \textit{and} \quad |2(A \cap [a, n])| \geqslant 3A(a, n) - 3,$$
$$\textit{then} \ |2(A \cap [0, b])| = 3A(0, b) - 3,\ |2(A \cap [a, n])| = 3A(a, n) - 3, \tag{4}$$
$$\textit{and} \ (2A) \setminus ((2(A \cap [0, b])) \cup (2(A \cap [a, n]))) = \emptyset. \tag{5}$$

3. *Let* $B \subseteq [u, v]$, $u, v \in B$, *and* $\gcd(B - u) = 1$. *If* $|B| \leqslant \frac{1}{2}(v - u + 3)$, *then* $|2B| \geqslant 3|B| - 3$. *If* $|2B| = 3|B| - 3$ *and* $|B| \leqslant \frac{1}{2}(v - u + 1)$, *then* B *is either a bi-arithmetic progression or a Freiman isomorphism image of* K_6 *defined in (1).*

Proof. Part 1 is trivial. Part 2 follows from the inequalities

$$\begin{aligned}
3k - 3 &= |2A| \\
&\geqslant |2(A \cap [0, b])| + |2(A \cap [a, n])| \\
&\quad -|\{2a, a+b, 2b\}| \\
&\geqslant 3A(0, b) - 3 + 3A(a, n) - 3 - 3 \\
&= 3A(0, a-1) + 3 + 3A(a, n) - 6 = 3k - 3,
\end{aligned}$$

which imply (4) and $|2A| = |2(A \cap [0, b])| + |2(A \cap [a, n])| - 3$.

Part 3 follows from Theorem 1.1 and Theorem 1.2. □

2. Main Theorem

Definition 2.1. For any $m \in \left[0, \frac{1}{2}n - 2\right]$ let

$$\mathcal{TPI}_{m,n} := \{\{0, 2, \ldots, 2(m-1)\} \cup B \cup \{n\} : B \in \mathcal{FT}_{am}[2m, n-1]\}. \tag{6}$$

For any $u \in [4, n-6]$, let

$$\begin{aligned} \mathcal{TPII}_{u,n} := \{&\{0\} \cup C \cup D \cup \{n\} : \\ &C \in \mathcal{BT}_{am}[1, u] \text{ and } D \in \mathcal{FT}_{am}[u+2, n-1]\}. \end{aligned} \tag{7}$$

A set in $\mathcal{TPI}_{m,n}$ is said to have a type 1 structure and a set in $\mathcal{TPII}_{u,n}$ is said to have a type 2 structure. If A has a type 1 structure or a type 2 structure, then $|2A| = 3k - 3$ by Proposition 1.8.

The following is the main theorem.

Theorem 2.2. *If $|A| = k \geqslant 2$ and*

$$|2A| = 3k - 3, \tag{8}$$

then one of the following must be true.

1. *A is a bi-arithmetic progression;*
2. *$2A$ contains an interval of length $2k - 1$;*
3. *$k = 6$ and A is a Freiman isomorphism image of K_6 defined in (1).*
4. *$k = \frac{1}{2}n + 1$ and either A or $n - A$ is in $\mathcal{TPI}_{m,n}$ defined in (6) for some $m \in \left[0, \frac{1}{2}n - 2\right]$ or A is in $\mathcal{TPII}_{u,n}$ defined in (7) for some $u \in [4, n-6]$.*

Remark 2.3. Notice that there are generally more than one set in each of $\mathcal{FT}_{am}[u, v]$ and $\mathcal{BT}_{am}[u, v]$. For example, $\{0, 1, 2, 3, 4\}$, $\{0, 1, 3, 4, 7\}$, $\{0, 1, 2, 5, 6\}$, and $\{0, 1, 2, 4, 6\}$ are all in $\mathcal{FT}_{am}[0, 9]$.

Notice also that $2A$ contains an interval of length $2k - 3$ when $A \in \mathcal{TPI}_{m,n}$ or $A \in \mathcal{TPII}_{u,n}$.

Proof. of Theorem 2.2. If $n+1 > 2k-1$, then A is either a bi-arithmetic progression or a Freiman isomorphism image of K_6 by Theorem 1.2. Hence we can assume that $n + 1 \leqslant 2k - 1$ or equivalently, $k \geqslant \frac{1}{2}n + 1$.

Let $H = [0, n] \setminus A$ and $h = |H|$. The elements in H are called the holes of A. Thus h counts the number of holes in H. A non-empty interval $[x, y] \subseteq H$ is called a gap of A if $x - 1, y + 1 \in A$. We now divide the proof into two parts and devote one subsection for each of them.

2.1. Proof of Theorem 2.2 when $k > \frac{1}{2}n + 1$

For each $x \in [0, n]$, $k > \frac{1}{2}(n+2)$ implies that

$$\text{either } A(0,x) > \frac{1}{2}(x+1) \text{ or } A(x,n) > \frac{1}{2}(n-x+1). \tag{9}$$

So for any $x \in [0, n]$, either $x \in 2A$ or $x + n \in 2A$ by Proposition 1.5. Let

- $H_1 = \{x \in H : x \notin 2A \text{ and } x+n \in 2A\}$ and $h_1 = |H_1|$,
- $H_2 = \{x \in H : x \in 2A \text{ and } x+n \notin 2A\}$ and $h_2 = |H_2|$,
- $H_3 = \{x \in H : x \in 2A \text{ and } x+n \in 2A\}$ and $h_3 = |H_3|$.

In [3], the elements in H_1 are called left stable holes, the elements in H_2, right stable holes, and the elements in H_3, unstable holes. By (9) and Proposition 1.5, we have that $H = H_1 \cup H_2 \cup H_3$ and $h = h_1 + h_2 + h_3$.

Since $|A \cup (n+A)| = 2k-1$, the following is true:

$$k - 2 = |(2A) \smallsetminus (A \cup (n+A))|$$

by (8). It is easy to verify that three sets $B_1 = \{x+n : x \in H_1\}$, $B_2 = \{x : x \in H_2\}$, and $B_3 = \{x, x+n : x \in H_3\}$ are pairwise disjoint and $B_1 \cup B_2 \cup B_3 = (2A) \smallsetminus (A \cup (n+A))$. Hence $k-2 = h_1 + h_2 + 2h_3$, which implies that

$$k - 2 - h = k - 2 - h_1 - h_2 - h_3 = h_3. \tag{10}$$

We now prove the following lemma which implies that $2A$ contains $2k-1$ consecutive integers when A is not a bi-arithmetic progression of difference 1 or 4.

Lemma 2.4. *Let $l, r \in [0, n]$ be such that $A(0,l) \leqslant \frac{1}{2}(l+1)$ and $A(n-r, n) \leqslant \frac{1}{2}(r+1)$. If A is not a bi-arithmetic progression of difference 1 or 4, then $l < n - r$.*

Corollary 2.5. *Let A, l, and r be in Lemma 2.4. If A is not a bi-arithmetic progression of difference 1 or 4, then $2A$ contains $2k-1$ consecutive integers.*

Proof. Let l and r be maximal so that for any $x \in [l+1, n]$ we have $A(0,x) > \frac{1}{2}(x+1)$ and for any $x \in [0, n-r-1]$ we have that $A(x,n) > \frac{1}{2}(n-x+1)$. By Lemma 2.4 we have $l < n-r$. Let $x \in [l+1, 2n-r-1]$. If $x \leqslant n$, then $x \in 2A$ because $A(0,x) > \frac{1}{2}(x+1)$. If $x > n$, then $x \in 2A$ because $A(x-n, n) > \frac{1}{2}(2n-x+1)$. Hence $[l+1, 2n-r-1] \subseteq 2A$. Note that

$$\begin{aligned} k &= A(0,l) + A(l+1, n-r-1) + A(n-r, n) \\ &\leqslant \frac{1}{2}(l+1) + (n-r-l-1) + \frac{1}{2}(r+1) = n - \frac{1}{2}(l+r), \end{aligned}$$

which implies $2k - 1 \leqslant 2n - l - r - 1 = |[l+1, 2n-r-1]|$. □

Proof. of Lemma 2.4. Without loss of generality, we can assume that A is not a bi-arithmetic progression of difference 1 or 4. Assume to the contrary that $l \geqslant n-r$. Clearly, $l \neq n-r$ by (9). Hence we can assume that $l > n-r$. Let

$$r_0 = \min\left\{x \in [n-l, r\,] : A(n-x, n) \leqslant \frac{1}{2}(x+1)\right\}. \tag{11}$$

By (9) we have that $n - r_0 < l$. Let

$$l_0 = \min\left\{x \in [n-r_0, l] : A(0, x) \leqslant \frac{1}{2}(x+1)\right\}. \tag{12}$$

We have that $n-r_0 < l_0$ again by (9). By the minimality of l_0 and r_0, it is true that $A(0,x) > \frac{1}{2}(x+1)$ and $A(x,n) > \frac{1}{2}(n-x+1)$ for any $x \in H \cap [n-r_0+1, l_0-1]$. So every hole in $[n-r_0+1, l_0-1]$ is an unstable hole. Thus

$$H(n-r_0+1, l_0-1) \leqslant h_3. \tag{13}$$

Now we have that

$$\begin{aligned} h &\geqslant H(0,l_0) + H(n-r_0,n) - H(n-r_0,l_0) && (14)\\ &\geqslant \frac{1}{2}(l_0+1) + \frac{1}{2}(r_0+1) - H(n-r_0,l_0) && (15)\\ &\geqslant \frac{1}{2}(n+1) + \frac{1}{2}(l_0-(n-r_0)+1) - H(n-r_0,l_0) && (16)\\ &\geqslant \frac{1}{2}(k+h) - \frac{1}{2}H(n-r_0,l_0). && (17) \end{aligned}$$

By solving the inequality above, we get that $h \geqslant k - H(n-r_0, l_0)$, which implies that

$$0 \geqslant k-2-h-H(n-r_0,l_0)+2 \geqslant h_3 - H(n-r_0+1, l_0-1) \geqslant 0 \tag{18}$$

by (10) and (13). Thus all inequalities in (14)–(18) become equalities. In particular, it is true that

$$H(n-r_0, l_0) = l_0 - (n-r_0) + 1 = h_3 + 2. \tag{19}$$

Notice that (19) implies that $[n-r_0, l_0] \cap A = \emptyset$ and $H_3 = [n-r_0+1, l_0-1]$. Notice also that l_0 is a left stable hole and $n-r_0$ is a right stable hole. These facts are important for the rest of the proof.

All arguments above this line are due to Freiman in [3]. The remaining part of the proof is new. Notice that if $|2A| < 3k-3$, then (10) becomes $k-2-h > h_3$, which leads to a contradiction that $0 > 0$ in (18). So the rest of the proof is needed only because (18) does not lead to a contradiction so far. Notice also that $l_0 > n - r_0$ can happen when, for example, $A = [0,10] \cup [22,32]$ or $A = \{0,4,8,\ldots,40\} \cup$

$\{1, 5, 9, \ldots, 41\}$. Fortunately, in these two cases, A is a bi-arithmetic progression of difference 1 or 4, respectively.

It is easy to verify that

$$A(0, l_0) = \frac{1}{2}(l_0 + 1) \text{ and } A(n - r_0, n) = \frac{1}{2}(r_0 + 1). \tag{20}$$

Since (20), l_0 is left stable hole, and $n - r_0$ is a right stable hole, we have, by Proposition 1.8, that $A \cap [0, l_0]$ and and $A \cap [n - r_0, n]$ are anti-symmetric. Let

$$a = \max(A \cap [0, n - r_0]) \text{ and } b = \min(A \cap [l_0, n]). \tag{21}$$

Then $a < n - r_0$, $b > l_0$, and $b - a \geqslant 2 + l_0 - (n - r_0) \geqslant 3$. Since

$$A(0, a) = A(0, l_0) = \frac{1}{2}(l_0 + 1) \geqslant \frac{1}{2}(a + 3) \geqslant \frac{1}{2}(a + 1) + 1,$$

we have that $a > 0$ and

$$\gcd(A \cap [0, a]) = 1. \tag{22}$$

By the same reason, we have that $b < n$ and

$$\gcd(A \cap [b, n] - b) = 1. \tag{23}$$

By part 3 of Proposition 1.10, we can assume that $|2(A \cap [0, b])| \geqslant 3A(0, b) - 3$ and $|2(A \cap [a, n])| \geqslant 3A(a, n) - 3$. Hence (3), (4), and (5) are true by Proposition 1.10. We now use these facts to derive contradictions. Let

$$a' = \max A \cap [0, a - 1] \text{ and } b' = \min A \cap [b + 1, n].$$

A contradiction will be derived under each of the following conditions:

$b' - b < a - a'$,

$b' - b > a - a'$,

$b' - b = a - a' > b - a$,

$1 < b' - b = a - a' \leqslant b - a$, and

$b' - b = a - a' = 1$.

Assume that $b' - b < a - a'$. Then $a' + b' = 2a$ by (3) and (5), which implies that $2a \notin b + A \cap [0, b]$. The fact that $2a \notin b + A \cap [0, b]$ will be used in the next several paragraphs to show that $|2(A \cap [0, b])| > 3A(0, b) - 3$, which contradicts (4).

Let $z = \max\left\{x \in [-1, l_0 - 1] : A(0, x) \leqslant \frac{1}{2}(x + 1)\right\}$. Clearly, $z + 1 \in A$ and $A(0, z) = \frac{1}{2}(z + 1)$ by the maximality of z. Notice that $A \cap [z + 1, l_0] \in \mathcal{FT}[z + 1, l_0]$.

If $z = -1$, then $|2(A \cap [0, b])| \geqslant 3A(0, b) - 2 > 3A(0, b) - 3$ by part 3 and 4 of Proposition 1.8.

Suppose that $z > -1$. Then $z > 0$ because $A(0, 0) = 1 > \frac{1}{2}$.

If $\gcd(A \cap [0, z+1]) = 1$, then $|2(A \cap [0, z+1])| \geqslant 3A(0, z+1) - 3$ by part 3 of Proposition 1.10. If $z \in A$, then $z + b \in 2A$ and

$$\begin{aligned} &|2(A \cap [0, b])| \\ &\quad \geqslant |2(A \cap [0, z+1])| - 1 \\ &\qquad + |[2z+2, z+l_0]| + |(b + A \cap [z, b]) \cup \{2a\}| \\ &\quad \geqslant 3A(0, z) + 3A(z+1, b) - 2 > 3A(0, b) - 3. \end{aligned}$$

So we can assume that $z \notin A$. Let

$$z' = \max A \cap [0, z-1].$$

Then $z' + z + 2 \in (2A) \setminus (2(A \cap [0, z+1]))$. Hence

$$\begin{aligned} &|2(A \cap [0, b])| \\ &\quad \geqslant |(2(A \cap [0, z+1])) \cup \{z' + z + 2\}| - 1 \\ &\qquad + |[2z+2, z+l_0]| + |(b + A \cap [z+1, b]) \cup \{2a\}| \\ &\quad \geqslant 3A(0, z) + 3A(z+1, b) - 2 > 3A(0, b) - 3. \end{aligned}$$

Thus we can assume that $\gcd(A \cap [0, z+1]) = d > 1$. Clearly, $d = 2$ and $A \cap [0, z+1]$ is an arithmetic progression of difference 2 by the fact that $A(0, z) = \frac{1}{2}(z+1)$. Hence

$$\begin{aligned} &|2(A \cap [0, b])| \\ &\quad \geqslant |A \cap [0, z-1] + A \cap [0, z+1]| + |(z+2) + A \cap [0, z-1]| \\ &\qquad + |[2z+2, z+l_0]| + |(b + A[z+1, b]) \cup \{2a\}| \\ &\quad \geqslant 3A(0, b) - 2 > 3A(0, b) - 3. \end{aligned}$$

Notice that $|2(A \cap [0, b])| > 3A(0, b) - 3$ is true if $2a$, which is not in $b + A \cap [0, b]$, is replaced by any element $c \in (2A \cap [b + z + 1, 2b]) \setminus (b + A \cap [z+1, b])$ in the argument above.

Assume that $a - a' < b' - b$. The proof is symmetric to the case for $a - a' > b' - b$.

Assume that $b' - b = a - a' = d'$.

If $d' > b - a$, then $2a \notin b + A \cap [0, b]$. Hence $|2(A \cap [0, b])| > 3A(0, b) - 3$ by the same argument as above, which contradicts (4). Thus, we can assume that $d' \leqslant b - a$.

Suppose that $1 < d' \leqslant b - a$. Let a'' be the greatest element in $A \cap [0, a']$ which is not congruent to a modulo d'. The number a'' exists by (22).

If $b' + a'' \in 2(A \cap [a, n])$, then $b' + a'' = 2a$ by (3), which implies $2a = b + (d' + a'') \notin b + A \cap [0, b]$ by the maximality of a''. Hence $|2(A \cap [0, b])| > 3A(0, b) - 3$. So by

(5), we can assume that $b' + a'' \in 2(A \cap [0, b])$. Notice that $b' + a'' \geqslant b + z + 1$. This is true because $b' \geqslant b + 2$ and $a'' \geqslant z - 1$ due to the facts that $d' > 1$, $a + 1 < l_0$, and $A \cap [z + 1, l_0] \in \mathcal{FT}[z + 1, l_0]$. Clearly, $b' + a'' \notin b + A \cap [0, b]$. Hence, again, $|2(A \cap [0, b])| > 3A(0, b) - 3$ by the same argument as above.

We can now assume that $d' = 1$, i.e.,

$$a' = a - 1 \in A \text{ and } b' = b + 1 \in A. \tag{24}$$

The derivation of a contradiction under this case is much harder that the previous cases. The reason for this is perhaps that A satisfies (24) when A is a bi-arithmetic progression of difference 1 or 4.

Since $A \cap [0, l_0]$ and $A \cap [n - r_0, n]$ are anti-symmetric and $[n - r_0, l_0] \cap A = \emptyset$, we have that $[0, l_0 - (n - r_0)] \subseteq A$ and $[2n - l_0 - r_0, n] \subseteq A$. In particular, we have that

$$0, 1, n - 1, n \in A. \tag{25}$$

Next we prove four claims for the existence of unstable holes if A has a certain configuration. These claims will be used to derive a contradiction.

Claim 1. *If $z \in A$, then $z - 1 \in A$ or $z + 1 \in A$.*

Proof. Suppose that Claim 1 is not true. Then $z \in [3, n - 3] \setminus [a - 2, b + 2]$ by (24) and (25). If $A(0, z - 1) > \frac{1}{2}z$, then $z - 1 \in H_3$ because $n + z - 1 = (n - 1) + z \in 2A$. Symmetrically, if $A(z + 1, n) > \frac{1}{2}(n - z)$, then $z + 1 \in H_3$. Both contradicts (19). However, $A(0, z - 1) \leqslant \frac{1}{2}z$ and $A(z + 1, n) \leqslant \frac{1}{2}(n - z)$ contradicts the assumption $k > \frac{1}{2}n + 1$. □

Claim 1 says that A does not contains any isolated points in A.

Claim 2. *If $z \in H$, then either $z - 1 \in H$ or $z + 1 \in H$.*

Proof. If $z - 1, z + 1 \in A$ and $z \in H$, then $z \notin [n - r_0, l_0]$ and $z = (z - 1) + 1, z + n = (z + 1) + (n - 1) \in 2A$, which contradict (19). □

Claim 2 says that there do not exist any isolated holes of A.

Claim 3. *(a) If $0 < x < y < z < n$ are such that $x, z, z + 1 \in H$, $y \in A$, and $A(0, z) = \frac{1}{2}(z + 1)$, then $z + 1$ is an unstable hole.*

(b) If $0 < x < y < z < n$ are such that $x - 1, x, z \in H$, $y \in A$, and $A(x, n) = \frac{1}{2}(n - x + 1)$, then $x - 1$ is an unstable hole.

Proof. We prove (a) only and (b) follows by symmetry. Without loss of generality, let $x = \max H \cap [0, y]$. Notice that $z \notin 2A$ because $z \notin H_3$. Now $A(0, z) = \frac{1}{2}(z + 1)$ implies that $c = z - x \in A$. Hence $z + 1 = c + (x + 1) \in 2A$. □

Claim 3 (a) implies that if $[0, a] \not\subseteq A$, then $b = l_0 + 1$ because $b > l_0 + 1$ implies that $l_0 + 1$ is an unstable hole, which contradicts (19). By symmetry, Claim 3 (b) implies that $a = n - r_0 - 1$ if $[b, n] \not\subseteq A$.

Claim 4. *If $[x, y] \subseteq H$ is a gap of A with $y - x \geqslant 2$, $H \cap [0, x-1] \neq \emptyset$, and $H \cap [y+1, n] \neq \emptyset$, then $[x, y]$ contains an unstable hole.*

Proof. If $A(0, x) \leqslant \frac{1}{2}(x+1)$, then $x \in H_3$. If $A(y, n) \leqslant \frac{1}{2}(n - y + 1)$, then $y \in H_3$. Otherwise, one can find a $t \in [x+1, y-1]$ such that $t \in H_3$ by Claim 3. □

Claim 4 says that if A has a gap $[x, y]$ of length at least 3, i.e., $y - x \geqslant 2$, then $[x, y]$ is either the first gap or the last gap or the middle gap $[a+1, b-1]$ of A.

We now continue the proof of Theorem 2.2 by deriving a contradiction under the assumption that $d' = 1$, i.e., $a-1, a, b, b+1 \in A$.

If $n - b < b - a$ and $a < b - a$, then A is a subset of the bi-arithmetic progression $[0, a] \cup [b, n]$ of difference 1. So $|2A| = 3k - 3$ implies that $A = [0, a] \cup [b, n]$ by Theorem 1.1. Thus we can now assume that either $n - b \geqslant b - a$ or $a \geqslant b - a$.

Without loss of generality, let $a \geqslant b - a$. So we have $H \cap [0, a] \neq \emptyset$. Let

$$z = \min\left\{x \in [0, l_0] : A(0, x) \leqslant \frac{1}{2}(x+1)\right\}. \tag{26}$$

Then $z \neq a$.

`Case 1` $z > a$. It is easy to see that $z > a$ implies $z = l_0$. Let $y = \min H \cap [0, a]$. Then $y \notin [n - r_0 + 1, l_0 - 1]$. Since $y \in 2A$ by Proposition 1.5 and the minimality of z, and $y + n = (y+1) + (n-1) \in 2A$, we have $y \in H_3$, which contradicts (19).

`Case 2` $z < a$. Notice that $z \notin A$ by the minimality of z and $z > 2$. By the same argument as in Case 1, we have that $A \cap [0, z] = \left[0, \frac{z-1}{2}\right]$. If $A(z-1, n) > \frac{1}{2}(n-z+2)$, then $z - 1$ is an unstable hole below a by Proposition 1.5. Hence we can assume that $A(z-1, n) \leqslant \frac{1}{2}(n - z + 2)$. Since

$$\begin{aligned} A(z-1, n) &= k - A(0, z-2) \\ &= k - A(0, z) \geqslant \frac{1}{2}(n+3) - \frac{1}{2}(z+1) = \frac{1}{2}(n - z + 2), \end{aligned}$$

we have that

$$A(z-1, n) = \frac{1}{2}(n - z + 2). \tag{27}$$

By Claim 3 (b), we can assume that $z - 2 \in A$ because otherwise $z - 2$ becomes an unstable hole below a. So $z - 2 = \frac{1}{2}(z-1)$, which implies that $z = 3$ and $A \cap [0, z] = [0, 1]$.

It is worth mentioning that (27) and $A(0, z) \leqslant \frac{1}{2}(z+1)$ imply

$$k = \frac{1}{2}(n+3) = \frac{1}{2}(k + h + 2), \tag{28}$$

which implies $k-2=h$ and

$$h_3 = k-2-h = 0. \tag{29}$$

So A has no unstable holes and $n-r_0 = l_0-1$.

Let

$$V = \{x \in [0,n] : x \equiv 0,1 \,(\mathrm{mod}\, 4)\}.$$

We can assume that $A \neq V$ because otherwise A is a bi-arithmetic progression of difference 4. Let

$$z' = \min\{x \in [0,n] : A \cap [0,x] \neq V \cap [0,x]\}.$$

Notice that $n \geqslant z' > z = 3$ and $A \cap [0, z'-1] = V \cap [0, z'-1]$ is the maximal bi-arithmetic progression of difference 4 inside A containing $0,1$. The rest of the proof is divided into four cases in terms of the value of z' modulo 4.

`Case 2.1` $z' \equiv 0 \,(\mathrm{mod}\, 4)$. Clearly, $z' \notin A$ because otherwise $A \cap [0,z'] = V \cap [0,z']$.

If $z' > 4$, then $A \cap [0,z'] = \{0,1,4,5,\ldots,z'-4,z'-3\}$ and z' is at least 8. Since $A(0,z'-1) = \frac{1}{2}z'$ by the definition of V, and since $3, z'-1, z' \in H$, and $4 \in A$, we have that z' is an unstable hole by Claim 3, which contradicts (29).

So we can now assume that $z' = 4$, which implies that $A \cap [0,4] = \{0,1\}$.

Let $c = \min A \cap [z', a]$.

Recall that l_0 is a left stable hole and $A \cap [0,l_0]$ is anti-symmetric in $[0,l_0]$ by Proposition 1.8. Since $0,1,c \in A$ and $[2,c-1] \subseteq H$, we have that $l_0, l_0-1, l_0-c \notin A$ and $[l_0-c+1, l_0-2] \subseteq A$. Consequently, $l_0-2 = a$ by (21). Clearly, $c \leqslant l_0-c+1$. Since $[2,c-1]$ is a gap of A with length at least 3, $[l_0-c+1, a]$ is an interval in A with length at most 3.

Suppose that $c < l_0-c+1$. Then $c < l_0-c$ and $t = l_0-c \in H$ because $A \cap [0,l_0]$ is anti-symmetric in $[0,l_0]$. Since $t+1 \in A$, we have that $t-1 \notin A$ by Claim 2. If $t-2 \in H$, then the gap I of A containing t has a length at least three. Hence I contains an unstable hole by Claim 4. But if $t-2 \in A$, then $t-1+n \in 2A$ because $A(t-1,n) = A(t-1,a) + A(n-r_0,n) > \frac{1}{2}(n-t+2)$, and $t-1 = (t-2)+1 \in 2A$. Hence $t-1$ is an unstable hole. Both contradicts (29).

We can now assume that $c = l_0-c+1$, which implies that $A \cap [0,b] = \{0,1\} \cup [c,a] \cup \{b\}$. So

$$\begin{aligned} 2(A \cap [0,b]) &\supseteq [0,2] \cup [c,a+1] \cup [2c,a+b] \cup \{2b\} \text{ and} \\ |2(A \cap [0,b])| &\geqslant 3 + a - c + 2 + a + b - 2c + 1 + 1 \\ &= 3a - 3c + 10 = 3A(0,b) - 12 + 10 = 3A(0,b) - 2. \end{aligned}$$

`Case 2.2` $z' \equiv 1 \,(\mathrm{mod}\, 4)$. We have that $z' \notin A$, $z'-1 \in A$, and $z'-2 \notin A$. Hence $z'-1$ is an isolated point of A, which contradicts Claim 1.

Case 2.3 $z' \equiv 2 \pmod 4$. We have that $z', z'-1, z'-2 \in A$ and $z'-3 \notin A$.

Let $c = \max\{x \geqslant z' : [z', x] \subseteq A\}$.

Notice that $[z'-2, c] \subseteq A$. The proof of Case 2.3 is divided into four subcases for $c = n$, $b < c < n$, $c = a$, or $c < a$. Notice that $c = b$ is impossible because $c-1 \in A$.

Case 2.3.1 $c = n$. Since $A = (V \cap [0, z'-3]) \cup [z'-2, n]$, we have that

$$\begin{aligned} k = A(0, z'-3) + A(z'-2, n) &= \frac{1}{2}(z'-2) + n - z' + 3 \\ &= \frac{1}{2}(n+1) + \frac{1}{2}(n - z' + 3) \geqslant \frac{1}{2}(n+1) + \frac{3}{2} = \frac{1}{2}(n+4), \end{aligned}$$

which contradicts (28).

Case 2.3.2 $b < c < n$. Clearly, $b \leqslant z'-2$. Recall that $a+3 = b$. Let $x = 2n-r_0-c$ and $y = 2n - r_0 - z' + 2$. Since $z'-3 \notin A$, $[z'-2, c] \subseteq A$, and $c+1 \notin A$, and since $n - r_0$ is a right stable hole and $A \cap [n-r_0, n]$ is anti-symmetric in $[n-r_0, n]$, we have that $[x, y]$ is a gap of A with length $y - x + 1 = c - z' + 3 \geqslant 3$. Notice also that $n-2 \notin A$ because $b = n - r_0 + 2 \in A$. Notice that $c < x$ because gaps of A below c are also gaps of V with length 2 while the length of $[x, y]$ is at least 3. By Claim 4 we can assume that $[y+1, n] \subseteq A$. Hence $y = n-2$.

Suppose that $c+1 < x$. Since $c \in A$ and $c+1 \notin A$, we have that $c+2 \notin A$ by Claim 2. If $c+3 \notin A$, then the gap I of A with $c+1 \in I$ contains an unstable hole by Claim 4. So we can assume that $c+3 \in A$. By the fact that the interval $[z'-2, c]$ contains at least three elements, we have that $A(0, c+2) > \frac{1}{2}(c+3)$, which implies that $c+2 \in 2A$ by Proposition 1.5. Also $c+2+n = (c+3)+(n-1) \in 2A$. Hence $c+2 \in H_3$, which contradicts (29).

Thus we can assume that $c+1 = x$. Now we have that

$$A \cap [a, n] = \{a\} \cup [b, c] \cup \{n-1, n\}.$$

Hence

$$2(A \cap [a, n]) = \{2a\} \cup [a+b, 2c] \cup [n-1+b, n+c] \cup [2n-2, 2n] \text{ and}$$

$$\begin{aligned} |2(A \cap [a, n])| &= 1 + 2c - a - b + 1 + c - b + 2 + 3 \\ &= 3c - 3b + 10 = 3A(a, n) - 12 + 10 = 3A(a, n) - 2. \end{aligned}$$

Case 2.3.3 $c = a$. Since $A \cap [0, l_0]$ is anti-symmetric in $[0, l_0]$, we have that $[x, y] = l_0 - [z'-2, a] = [2, l_0 - z' + 2] \subseteq H$ is a gap of A below $z'-2$ with length at least 3, which is impossible because all gaps of A below $z'-2$ must be the gaps of V with length 2.

Case 2.3.4 $c < a$. Since $A \cap [0, l_0]$ is anti-symmetric in $[0, l_0]$, we have that $[x, y] = [l_0 - c, l_0 - z' + 2] \subseteq H$ is a gap of A with length at least 3. Since gaps of

A below $z'-2$ has length 2, we have that $c < x$. By Claim 4, $[x, y]$ contains an unstable hole, which contradicts (29).

`Case 2.4` $z' \equiv 3 \pmod 4$. By the definition of z' we have that $z', z'-2 \in A$ and $z'-1 \notin A$. Therefore, $z'-1$ is an isolated hole, which contradicts Claim 2.

This completes the proof of Lemma 2.4 as well as Theorem 2.2 when $k > \frac{1}{2}(n+2)$. □

Remark 2.6. Part 2 of Theorem 2.2 is a structural property for $2A$. So it is an indirect description of a structural property of A. Let l' and r' be the maximal l and r, respectively, as defined in Lemma 2.4. Then $l' < n - r'$, which is a direct description of a structural property of A. The conclusion that $l' < n - r'$ in Lemma 2.4 not only implies part 2 of Theorem 2.2 (the converse is not true), but also gives some geometric information for A. Roughly speaking, $l' < n - r'$ indicates that A is thin in $[0, l']$ and in $[n - r', n]$, and A is thick in $[l' + 1, n - r' - 1]$. Therefore, we can say that Lemma 2.4 is a more detailed description of the structural property of A than part 2 of Theorem 2.2 when $k \geqslant \frac{1}{2}(n+3)$.

2.2. Proof of Theorem 2.2 when $k = \frac{1}{2}n + 1$

Throughout this subsection we set that

$$k = \frac{1}{2}(n+2). \tag{30}$$

Notice that (30) cannot occur when n is an odd number.

Let x be a hole in A. We call x a *balanced hole* if $A(0, x) = \frac{1}{2}(x+1)$ and $A(x, n) = \frac{1}{2}(n - x + 1)$. Notice that if $A(0, y) = \frac{1}{2}(y+1)$ and $A(y, n) = \frac{1}{2}(n - y + 1)$ for some $y \in [0, n]$, then $y \notin A$ and if $A(0, x) = \frac{1}{2}(x+1)$ for some hole x in A, then x is a balanced hole by (30).

We want to show that $A \in \mathcal{TPI}_{m,n}$ or $n - A \in \mathcal{TPI}_{m,n}$ or $A \in \mathcal{TPII}_{u,n}$ where $\mathcal{TPI}_{m,n}$ and $\mathcal{TPII}_{u,n}$ are defined in Definition 2.1. It is worth mentioning that if $n = 10$ and B is a Freiman isomorphism image of K_6 in (1), then $|B| = \frac{1}{2}(n+2)$ and $B = B_i$ for $i = 1, 2, 3,$ or 4 where B_i's are defined in part 4 of Proposition 1.6. Notice that $B_1 \in \mathcal{TPI}_{0,10}$ and $B_2 \in \mathcal{TPI}_{2,10}$.

`Case 1` $0, 1 \in A$. In this case we want to show that $A \in \mathcal{TPI}_{0,n}$ or A is an arithmetic progression of difference 1 or 4. Let

$$z = \min\left\{x \in [0, n] : A(0, x) \leqslant \frac{1}{2}(x+1)\right\}. \tag{31}$$

Then $z \geqslant 3$ and $z - 1, z \notin A$, $A(0, z) = \frac{1}{2}(z+1)$, and $A \cap [0, z] \in \mathcal{FT}[0, z]$ by the minimality of z. Notice that z is a balanced hole.

If $z = 2k-3 = n-1$, then $2A \supseteq [0, n-2] \cup (n+A)$. Hence $|2A| = 3k-3$ implies that $2A = [0, z-1] \cup (n+A)$. So $A \in \mathcal{TPI}_{0,n}$. Therefore, we can now assume that $z < 2k-3 = n-1$.

We now want to show that either $|2A| > 3k-3$ or A is a bi-arithmetic progression of difference 1 or 4.

Let $a = \max A \cap [0, z]$ and $b = \min A \cap [z, n]$. By part 3 of Proposition 1.10, we can assume that

$$|2(A \cap [0, b])| \geqslant 3A(0, b) - 3. \tag{32}$$

Since $A(z, n) = A(b, n) = \frac{1}{2}(n - z + 1) \geqslant \frac{1}{2}(z + 2 - z + 1) > 1$, the set $A \cap [a, n]$ contains at least three elements.

Suppose that $\gcd(A \cap [b, n] - b) > 1$. Then $A(z, n) = \frac{1}{2}(n - z + 1)$ implies that $b = z + 1$ and $A \cap [b, n]$ is an arithmetic progression of difference 2. Since $A \cap [b, n]$ is an arithmetic progression of difference 2, we have that

$$2A \supseteq [0, z-1] \cup (A \cap [b, n-2] + \{0, 1\}) \cup (n + A). \tag{33}$$

So $|2A| = 3k - 3$ implies that two sides in (33) are the same set. Let E_0 be the set of all even numbers and O_0 be the set of all odd numbers in $A \cap [0, a]$. If there is an $x > 0$ such that $x \notin E_0$ and $x + 2 \in E_0$, then $n + x = (n-2) + (x+2)$ is in $2A$ but not in the right side of (33). So E_0 is a set of consecutive even numbers. By the same reason we can assume that O_0 is a set of consecutive odd numbers.

If $a = z - 1 = b - 2$, then a is even and $A(0, z) = \frac{1}{2}(z + 1) = |E_0|$. So $O_0 = \emptyset$, which contradicts $1 \in A$. Hence we can assume that $a < b - 2$ and $b - 2 \notin A$. Now $b + (n-2)$ is in $2A$ but not in the right side (33), a contradiction to the fact that two sides of (33) are the same set.

Remark 2.7. Notice that in the proof above we have that $|2A| \geqslant 3k - 2$ by identifying an element in $((2A) \smallsetminus (n + A)) \cap [n, 2n]$. If in some case we can also show that $z \in 2A$, then $|2A| \geqslant 3k - 1$. This fact will be mentioned later.

We can now assume that $\gcd(A \cap [b, n] - b) = 1$.

By part 3 of Proposition 1.10, we have that $|2(A \cap [a, n])| \geqslant 3A(a, n) - 3$. Together with (32), we conclude that (3), (4), and (5) are true.

`Case 1.1` $H \cap [0, a] = \emptyset$. This case implies that $z = 2a+1$, $2a < b$, and $A \cap [0, b] = [0, a] \cup \{b\}$.

By applying Theorem 1.2 we have that one of the following is true: (i) $A \cap [a, n]$ is a bi-arithmetic progression, (ii) $n - a + 1 \leqslant 2A(a, n) - 1$, or (iii) $A \cap [a, n]$ is Freiman isomorphic to K_6 in (1).

Notice that $n - a + 1 \leqslant 2A(a, n) - 1 = (n - z + 1) + 1$ implies that $2a + 1 = z \leqslant a + 1$, which is absurd. So we can assume that $A \cap [a, n]$ is either a bi-arithmetic progression or Freiman isomorphic to K_6.

Case 1.1.1 $A\cap[a,n]$ is Freiman isomorphic to K_6 in (1). Let $\varphi: K_6 \mapsto A\cap[a,n]$ be the Freiman isomorphism. Notice that $A(b+1,n-1)=3$.

Suppose that $b+1\notin A$. Let $b'=\min A\cap[b+1,n]$.

If $a>1$, then there is an $x\in\{a-1,a-2\}$ such that $x+b'\notin\{2a,a+b,2b\}$. Hence $x+b'$ is in the set in (5), which contradicts that the set is empty.

If $a=1$, then $z=3$, $b\geqslant 4$, and $n=12$ because $k=7$. If $2b\neq b'$, then b' is in the set in (5). So we can assume that $b'=2b\geqslant 8$. Notice that $a=1$ is a vertex of $\varphi(K_6)$ and b is not a vertex by part 2 of Proposition 1.6. So $2b-a=2b-1$ is another vertex of $\varphi(K_6)$. This contradicts the minimality of b'.

We can now assume that $b+1\in A$. If $b+1$ is a vertex of $\varphi(K_6)$, then $c=\frac{1}{2}(a+b+1)\in A\cap[a+1,b-1]$, which contradicts $A\cap[a+1,b-1]=\emptyset$. So $b+1$ is not a vertex in $\varphi(K_6)$. Hence $2b+2-a$ is in A and is a vertex. So $A=[0,a]\cup\{b,b+1,2b-a,2b-a+1,2b-a+2\}$, which implies that $(a-1)+(2b-a)=2b-1$ is in the empty set in (5).

Case 1.1.2 $A\cap[a,n]$ is a bi-arithmetic progression of difference d. Let $A\cap[a,n]=I_0\cup I_1$ be the bi-arithmetic progression decomposition and $a\in I_0$.

If $d=1$, then $A\cap[a,n]=\{a\}\cup[b,n]$ such that $n-b>b-a$. Hence $A=[0,a]\cup[b,n]$ is a bi-arithmetic progression of difference 1.

If $d=2$, then $\gcd((A\cap[b,n])-b)=2$ because $b-a\geqslant 3$. But this contradicts the assumption that $\gcd((A\cap[b,n])-b)=1$.

If $d=3$, then $b\in I_0$ and $b-a=3$. Hence $z=a+2$, which implies that $a=1$ because $z=2a+1$. Let $c=\min I_1$. If $c=b+2$, then $a-1+c$ is in the set in (5). If $c=b+1$ or $c>b+3$, then $a-1+b+3$ is in the set in (5). But both contradict that the set in (5) is empty.

If $d=4$, then $A(b+1,b+3)\leqslant 1$. If $a\not\equiv b\,(\mathrm{mod}\,4)$, then $b=a+3$ because $b-a\geqslant 3$ and $a+4\in I_0$. But $b=a+3$ implies $a=1$. So A is a bi-arithmetic progression of difference 4. Hence we can assume that $a\equiv b\,(\mathrm{mod}\,4)$. If $A(b+1,b+3)=0$ or $b+1\in A$, let $x=b+4$. If $b+3\in A$, let $x=b+3$. Then $a-1+x$ is in the empty set in (5). Notice that $b+2\notin A$ because otherwise $\gcd(A\cap[b,n]-b)=2$.

If $d\geqslant 5$, then $\frac{1}{2}(n-z+1)=A(z,n)\leqslant\frac{2}{5}(n-z+3)$, which implies that $n-z\leqslant 7$. Now $A(z,n)=\frac{1}{2}(n-z+1)$, $z<b$, and $d=5$ imply that $n-z=7$, $d=5$, $b=z+1$, and $A\cap[b,n]=\{b,b+1,b+5,b+6\}$. If $a\equiv b\,(\mathrm{mod}\,5)$, then $a-1+b+5$ is in the empty set in (5). If $a\not\equiv b\,(\mathrm{mod}\,5)$, then $b-a=4$, which implies that $a=2$ because $a+3=z=2a+1$. Hence $b+5=(a-2)+(b+5)=2b-1$ is in the empty set in (5).

Case 1.2 $H\cap[0,a]\neq\emptyset$. Notice that $z\leqslant 2a$. If $b>z+1$, then $|2(A\cap[0,b])|>3|A\cap[0,b]|-3$ by part 5 of Proposition 1.8. Hence we can assume that $b=z+1$. By (4), $A\cap[0,z]$ is an additively minimal backward triangle. Hence $A\cap[0,z]$ is anti-symmetric.

Since $A\cap[0,z]$ is anti-symmetric and $1\in A$, we have that $z-1\notin A$. So $b-a\geqslant 3$.

If $n-a+1 \leqslant 2A(a,n)-1=(n-z+1)+1$, then $z-1 \leqslant a$. Hence we can assume, by Theorem 1.2, that $A \cap [a,n]$ is either a bi-arithmetic progression or a Freiman isomorphism image of K_6 in (1).

`Case 1.2.1` $A \cap [a,n]$ is Freiman isomorphic to K_6 in (1). Let $\varphi : K_6 \mapsto A \cap [a,n]$ be the Freiman isomorphism.

Since $A(z,n)=5=\frac{1}{2}(n-b+2)$, we have that $n-b=8$. Notice that a is a vertex, b is not a vertex, and $2b-a$ is a vertex of $\varphi(K_6)$. Let c be the third vertex in $\varphi(K_6)$. If $2b-a=n$, then $\frac{1}{2}(a+c) \in A \cap [a+1,b-1]=\emptyset$, which is absurd. So we can assume that $2b-a<c=n$.

Notice that $\frac{1}{2}(n+2b-a)$ is in $A \cap [b,n]$. Clearly, $n-(2b-a)$ is even and $\leqslant 5$ because $n-b=8$ and $b-a \geqslant 3$. If $n-(2b-a)=4$, then $A \cap [b,n]=\{b,b+2,b+4,b+6,b+8\}$, which contradicts $\gcd(A \cap [b,n]-b)=1$. If $n-(2b-a)=2$, then $A \cap [b,n] = \{b,b+1,b+6,b+7,b+8\}$ and $a=b-6$. If $a-1 \in A$, then $(a-1)+(b+6)$ is in the empty set in (5). So we can assume that $a-1 \notin A$. Let $a' = \max A \cap [0,a-1]$. If $a'+b+1 \neq 2a$, then $a'+b+1$ is in the empty set in (5). If $a'+b+1=2a$, then $a'+b+6$ is in the empty set in (5). Both are absurd. This completes the proof of Case 1.2.1.

`Case 1.2.2` $A \cap [a,n]$ is a bi-arithmetic progression of difference d. Let $A \cap [a,n] = I_0 \cup I_1$ be the bi-arithmetic progression decomposition and $a \in I_0$.

If $d=1$, then $A \cap [a,n]=\{a\} \cup [b,n]$ with $n-b<b-a$. Let $A'=n-A$, $z'=n-z$, $b'=n-a$, and $a'=n-b$. Then $A' \cap [0,z']=[0,a']$, $A' \cap [a',b']=[0,a'] \cup \{b'\}$, and z' is a balanced hole of A'. The same proof for Case 1.1 works for A'.

If $d=2$, then $\gcd((A \cap [b,n])-b)=2$ because $b-a \geqslant 3$, which contradicts the assumption that $\gcd((A \cap [b,n])-b)=1$.

If $d=3$, then $a,b \in I_0$, $b-a=3$, and $z=a+2$. Let $c=\min I_1$ and $a'=\max A \cap [0,a-1]$. If $a'=a-1$, then $A(0,a'-1)=\frac{1}{2}a'$, which contradicts the minimality of z. Thus we can assume that $a'<a-1$. If $a' \equiv a \pmod 3$, then $c+a'$ is in the empty set in (5). So we can assume that $a' \not\equiv a \pmod 3$. If $c=b+1$, then $c+a'$ is in the empty set in (5). So we can assume that $c>b+1$. If $b+3 \notin A$, then $c=b+2$ and $A \cap [z,n]=\{b,b+2\}$ by the fact that $A(z,n)=\frac{1}{2}(n-z+1)$, which contradicts the assumption that $\gcd(A \cap [b,n]-b)=1$. So we can assume that $b+3 \in A$. But now $a'+b+3$ is in the empty set in (5).

If $d=4$, then $b-a=4$ or $b-a=3$ because $\gcd(A \cap [b,n]-b)=1$. Let $a'=\max(A \cap [0,a-1])$ and $c=\min\{x \in A \cap [b+1,n] : x \not\equiv b \pmod 4\}$.

Suppose that $a'=a-1$. If $b-a=4$, then $c \neq b+2$ and $b+4 \in A$ by the fact that $A(b-1,n)=\frac{1}{2}(n-b)$. If $c \neq b+3$, then $a'+b+4$ is in the empty set in (5). If $c=b+3$, then $a'+c$ is in the empty set in (5). If $b-a=3$, then $z-a=2$ and $A(0,a'-1)=\frac{1}{2}a'$, which contradicts the minimality of z.

So we can now assume that $a'<a-1$. If $b+1 \in A$, then $a'+b+1$ is in the empty set in (5) unless $a'+b+1=2a$. If $a'+b+1=2a$, then $2a \notin (b+A \cap [0,b])$, which

leads to a contradiction to (4). So we can assume that $b+1 \notin A$, which implies that $b \neq a+3$. So we have that $b = a+4$. Since $A(z,n) = \frac{1}{2}(n-z+1)$, we have that $b+3 \in A$ and $n \geqslant b+4$. If $a' \equiv a \pmod 4$, then $a'+b+3$ is in the empty set in (5). If $a' \not\equiv a \pmod 4$, then $a'+b+4$ is in the empty set in (5).

If $d \geqslant 5$, then $\frac{1}{2}(n-z+1) = A(z,n) \leqslant \frac{2}{5}(n-z+3)$, which implies that $n-z=7$ and $A \cap [b,n] = \{b, b+1, b+5, b+6\}$. Notice that $b-a$ is 5 or 4. Let $a' = \max A \cap [0, a-1]$.

If $a' < a-1$, then $a'+b+1$ is in the empty set in (5) unless $a'+b+1 = 2a$. If $a'+b+1 = 2a$, then $2a \notin (b + A \cap [0,b])$, which leads to a contradiction to (4). So we can assume that $a' = a-1$. If $a-b=5$, then $a'+b+5$ is the empty set in (5). If $a-b=4$, then let $a'' = \max\{x \in A \cap [0,a-1] : x \not\equiv b, b+1\}$. Notice that a'' exists because otherwise A is a subset of a bi-arithmetic progression of difference 5, which contradicts the assumptions that $H \cap [0,a] \neq \emptyset$ and $A \cap [0,z]$ is a backward triangle in $[0,z]$. If $a'' \equiv b+2$ or $b+3 \pmod 5$, then $a''+b+1$ is in the empty set in (5). If $a'' \equiv b+4 \pmod 5$, then $a''+b+5$ is the empty set in (5).

This completes the proof of Case 1.2.2 as well as Case 1.

`Case 2` $1 \notin A$. If $n-1 \in A$, then by the proof of Case 1 for $n-A$, we can conclude that $n - A \in \mathcal{TPI}_{0,n}$ or A is an arithmetic progression of difference 1 or 4. Thus we can assume that $n-1 \notin A$.

We want to show that $A \in \mathcal{TPI}_{m,n}$ or $n-A \in \mathcal{TPI}_{m,n}$ for some $m \in \left[1, \frac{1}{2}n-2\right]$ or $A \in \mathcal{TPII}_{u,n}$ for some $u \in [4, n-6]$. Let E be the set of all even numbers and

$$a = \max\{x \in [0,n] : A \cap [0,x] = E \cap [0,x]\}.$$

Notice that $2 \leqslant a \leqslant n-2$. Notice also that $a \in A$ implies $a+1 \in A$ and if $a \notin A$ implies $a+1 \notin A$ by the maximality of a.

`Case 2.1` $a \in A$. In this case we show that $A \in \mathcal{TPI}_{m,n}$ for $m = a/2$.

Let $A' = A \cap [a,n]$. Then $a, a+1 \in A$. Notice that

$$\begin{aligned} 3k-3 &= |2A| \\ &\geqslant |2(A \cap [0,a])| + |a+1+A \cap [0,a-2]| + |2(A \cap [a,n])| - 1 \\ &= 3A(0,a-2) + |2(A \cap [a,n])|. \end{aligned}$$

Hence $|2(A \cap [a,n])| \leqslant 3A(a,n) - 3$. Notice also that $A(a,n) = \frac{1}{2}(n-a+2)$. So $|2(A \cap [a,n])| = 3A(a,n) - 3$ by part 3 of Proposition 1.10. Let $n' = n-a$. Now $A' = A \cap [a,n] - a$ in $[0,n']$ satisfies all conditions for Case 1. So either $A' \in \mathcal{TPI}_{0,n'}$ or A' is a bi-arithmetic progression of difference 1 or 4.

Suppose that A' is a bi-arithmetic progression of difference 1. Since $n-1 \notin A$, we have that $A \cap [a,n] = [a,x] \cup \{n\}$. Since $|A \cap [a,n]| = \frac{1}{2}(n-a+1)$, we have that $A' \in \mathcal{TPI}_{0,n'}$. Hence $A \in \mathcal{TPI}_{m,n}$ for $m = a/2$.

Suppose that A' is a bi-arithmetic progression of difference 4. If $a+5 \notin A$, then $A' = \{0,1,4\}$ by the fact that $|A'| = \frac{1}{2}(n'+2)$. Since $\{0,1,4\} \in \mathcal{TPI}_{0,4}$, we have again that $A \in \mathcal{TPI}_{m,n}$ for $m = a/2$. If $a + 5 \in A$, then

$$\begin{aligned}3k-3 = |2A| &\geqslant |2(A \cap [0,a])| + |a+1+A \cap [0,a-2]| \\ &+|\{(a+5)+(a-2)\}| + |2(A \cap [a,n])| - 1 \\ &\geqslant 3A(0,a-2)+1+3A(a,n)-3 = 3k-2,\end{aligned}$$

which is absurd.

`Case 2.2` $a \notin A$. In this case we show that $a > 1$ implies $|2A| > 3k-3$ and $a = 1$ implies that either $n - A \in \mathcal{TPI}_{m,n}$ or $A \in \mathcal{TPII}_{u,n}$.

Recall that $a+1 \notin A$ and $a-1 \in A$. Notice that $A(0,a) = \frac{1}{2}(a+1)$. Thus a is a balanced hole. Notice also that $A(a,n-2) = \frac{1}{2}(n-a-1)$ because $n \in A$ and $n-1 \notin A$. Let

$$u = \min\left\{x \in [a,n-2] : A(a,x) \geqslant \frac{1}{2}(x-a+1)\right\}. \tag{34}$$

Notice that $u > a+2$ because $a, a+1 \notin A$. Notice also that $u, u-1 \in A$, $A(a,u) = \frac{1}{2}(u-a+1)$, and $A(x,u) > \frac{1}{2}(u-x+1)$ for every $x \in [a+1,u]$ by the minimality of u. So $A \cap [a,u]$ is a backward triangle in $[a,u]$. Notice that $A(u,n) = \frac{1}{2}(n-u+2)$.

`Case 2.2.1` $a > 1$. In this case we derive a contradiction by showing $|2A| > 3k-3$.

Since $0, 2 \in A$ and $1 \notin A$, A as well as $A \cap [0,u]$ can be neither a bi-arithmetic progression of difference 1 nor a bi-arithmetic progression of difference 4. Notice that since $0, 2 \in A$ and $1 \notin A$, $u-(A\cap[0,u])$ cannot be in $\mathcal{TPI}_{0,u}$. Hence by applying the proof of Case 1 to $u-(A \cap [0,u])$, we have that $|2(A \cap [0,u])| > 3A(0,u)-3$.

If $\gcd(A \cap [u,n] - u) > 1$, then

$$|2A| \geqslant |2(A \cap [0,u])| + |2(A \cap [u,n])| - 1 + |u-1+A \cap [u+2,n]| > 3k-3.$$

So we can assume that $\gcd(A \cap [u,n] - u) = 1$. Hence $|2(A \cap [u,n])| \geqslant 3A(u,n)-3$. Let $v = \min A \cap [u+1,n]$.

If $v > u+1$, then $u-1+v$ is in

$$(2A) \setminus ((2(A \cap [0,u])) \cup (2(A \cap [u,n]))). \tag{35}$$

Hence

$$|2A| \geqslant |2(A \cap [0,u])| + |2(A \cap [u,n])| > 3k-3.$$

Thus we can assume that $v = u+1$.

Recall that in the argument in the proof of Case 1 before case 1.1, we showed that if $b < n$, $\gcd(A \cap [b,n] - b) > 1$, and $z = b-1 \in 2A$ then $|2A| \geqslant 3k-1$ (see Remark 2.7). So we can apply the same argument to $A' = u - (A \cap [0,u])$ with $z' = u-a$ and $b' = u-a+1$ to show that

$$|2(A \cap [0,u]) \cup \{u+a\}| = |(2A') \cup \{z'\}| \geqslant 3|A'| - 1 = 3A(0,u) - 1.$$

Hence

$$|2A| \geqslant |2(A\cap[0,u])\cup\{a-1+v\}| + |2(A\cap[u,n])| - 1$$
$$\geqslant 3A(0,u) - 1 + 3A(u,n) - 4 = 3k-2.$$

Case 2.2.2 $a=1$. In this case we show that either $n-A \in \mathcal{TPI}_{m,n}$ for some $m>0$ or $A\in \mathcal{TPII}_{u,n}$ for some $u\in[4,n-6]$.

Notice that $A\cap[1,u]$ is a backward triangle in $[1,u]$ and $|2(A\cap[0,u])| \geqslant 3A(0,u)-3$ by part 3 of Proposition 1.10.

Case 2.2.2.1 $u+1\in A$. If $|2(A\cap[0,u])| > 3A(0,u)-3$, then

$$3k-3 = |2A| \geqslant |2(A\cap[0,u])| + |2(A\cap[u,n])| - 1$$
$$\geqslant 3A(0,u) - 2 + 3A(u,n) - 3 - 1 = 3k-3.$$

Hence $|2(A\cap[u,n])| = 3A(u,n)-3$. By applying the proof of Case 1 to the set $A' = A\cap[u,n]-u$ and $n'=n-u$, we can conclude that either $A'\in\mathcal{TPI}_{0,n'}$ or A' is a bi-arithmetic progression of difference 1 or 4. We now want to show that $|2A| > 3k-3$ by identifying one element in the set in (35), which implies that $|2A| > 3A(0,u)-3+3A(u,n)-3-1+1 = 3k-3$.

If $A'\in\mathcal{TPI}_{0,n'}$, then $u-1+n$ is in the set in (35). If A' is a bi-arithmetic progression of 1, then $A\cap[u,n]=[u,x]\cup\{n\}$. So again $A'\in\mathcal{TPI}_{0,n'}$. If A' is a bi-arithmetic progression of difference 4, then $u-1+u+4$ is in the set in (35).

Thus we can assume that $|2(A\cap[0,u])| = 3A(0,u)-3$. So $A' = u-(A\cap[0,u]) \in \mathcal{TPI}_{0,n'}$ for $n'=u$ or A' is a bi-arithmetic progression of difference 1 or 4. Notice that if A' is a bi-arithmetic progression of difference 1, then $A'\in\mathcal{TPI}_{0,n'}$ because $1\notin A$. And if A' is a bi-arithmetic progression of difference 4, then $A'=\{0,1,4\}\in\mathcal{TPI}_{0,4}$ because A' is an forward triangle in $[0,u-1]$ and $A'(0,3)=\frac{1}{2}(n'+1)$. As a consequence we have that $u+1$ is in the set in (35).

If $|2(A\cap[u,n])| > 3A(u,n)-3$, then $|2A| > 3A(0,u)-3+3A(u,n)-3 = 3k-3$. Hence we can now assume that $|2(A\cap[u,n])| = 3A(u,n)-3$.

Recall that we have assumed that $|2(A\cap[0,u])| = 3A(0,u)-3$, $|2(A\cap[u,n])| = 3A(u,n)-3$, $\{u-1,u,u+1\}\subseteq A$, and $u-(A\cap[0,u])\in\mathcal{TPI}_{0,u}$. By applying the proof of Case 1 to $A' = (A\cap[u,n])-u$, we have that either $(A\cap[u,n])-u\in\mathcal{TPI}_{0,n-u}$ or $(A\cap[u,n])-u$ a bi-arithmetic progression of difference 4. Notice that if $A\cap[u,n]$ is a bi-arithmetic progression of difference 1, then $(A\cap[u,n])-u\in\mathcal{TPI}_{0,n-u}$. We now want to show that $|2A|>3k-3$ by identifying two elements in the set in (35), which implies that $|2A|\geqslant 3k-2$.

If $(A\cap[u,n])-u\in\mathcal{TPI}_{0,n-u}$, then $u+1, u-1+n$ are in the set in (35). If $A\cap[u,n]$ is a bi-arithmetic progression of difference 4, then $0+u+1, u-1+u+4$ are in the set in (35).

Case 2.2.2.2 $u+1\notin A$. Let $v=\min A\cap[u+1,n]$. Notice that $v+u-1$ is in the set in (35). If $|2(A\cap[0,u])| > 3A(0,u)-3$, then $|2A|>3k-3$. Hence we can assume that $|2(A \cap [0,u])| = 3A(0,u) - 3$. By applying the proof of Case 1, we have that $u - (A \cap [0,u]) \in \mathcal{TPI}_{0,u}$. If $\gcd((A \cap [u,n]) - u) = d > 1$, then $d = 2$ and $A \cap [u,n]$ is an arithmetic progression of difference 2. So $n - A \in \mathcal{TPI}_{m,n}$ for $m = (n-u)/2$. Hence we can assume that $\gcd(A \cap [u,n] - u) = 1$. If $|2(A \cap [u,n])| > 3A(u,n) - 3$, then $|2A| > 3k - 3$. Hence we can assume that $|2(A \cap [u,n])| = 3A(u,n) - 3$.

Suppose that $v = u + 2$. We want to show that $A \in \mathcal{TPII}_{u,n}$.

Let $a' = \max\{x \in [u,n] : A \cap [u,x] = (u + E) \cap [u,x]\}$. The number a' is well-defined because $\gcd((A \cap [u,n]) - u) = 1$. Notice that $a' \geqslant u + 2$.

If $a' \notin A$, then $a' > u+2$. By the proof of Case 2.2.1 we have that $|2(A \cap [u,n])| > 3A(u,n) - 3$. So we can assume that $a' \in A$, which implies that $a' + 1 \in A$ by the maximality of a'. By applying the proof of Case 2.1 to the set $A' = (A \cap [u,n]) - u$, we have that $A' \in \mathcal{TPI}_{m,n-u}$. Hence

$$A = \{0\} \cup C \cup \{u, u+2, \ldots, u+2m\} \cup D \cup \{n\}$$

where $C \in \mathcal{BT}_{am}[1,u]$, and $D \in \mathcal{FT}_{am}[u+2m, n-1]$. Thus

$$\begin{aligned}
|2A| &= |2(A \cap [0,u])| + |[2u+1, 2u+4m-1]| + |2(A \cap [u+2m, n])| \\
&= 3A(0,u) - 3 + 4m - 1 + 3A(u+2m,n) - 3 \\
&= 3A(0,u) - 3 + 4A(u+2, u+2m-2) + 3 + 3A(u+2m,n) - 3 \\
&= 3k - 3 + A(u+2, u+2m-2) > 3k - 3
\end{aligned}$$

unless $m = 1$. If $m = 1$, then $A \in \mathcal{TPII}_{u,n}$.

Now we assume that $v > u + 2$. We show that $|2A| > 3k - 3$ by identifying two elements in the set in (35).

If $u - 2, u - 3 \notin A$, then $u = 4$ and $A \cap [0,u] = \{0,3,4\}$ by the minimality of u. If A is not a bi-arithmetic progression of difference 4, we can define

$$\begin{aligned}
c = \min\{x \in [u,n] : &\ A \cap [0,x] \\
&\text{is not a subset of a bi-arithmetic progression of difference } 4\}.
\end{aligned}$$

Since $3, 4 \in A$, we have that c is either congruent to 5 or congruent to 6 modulo 4.

Suppose that $c = v$. Recall that $v > 6$. So we can assume that $v \geqslant 9$. Then $v + 3, v + 0$ are in the set in (35).

Suppose that $c > v$. Then v is congruent to 3 or 4 modulo 4. If $c \equiv 5 \pmod 4$, then $c + 0, v + 3$ are in the set in (35). If $c \equiv 6 \pmod 4$, then $c + 3, v + 3$ are in the set in (35).

So we can assume that $A(u-3,u) \geqslant 3$. If $u - 2 \in A$, then $v + u - 1, v + u - 2$ are in the set in (35). So we can assume that $u - 2 \notin A$ and $u - 3 \in A$.

If $v \geqslant u + 4$, then $v + u - 1, v + u - 3$ are in the set in (35). So we can assume that $v = u + 3$. Let $v' = \min A \cap [v+1, n]$. If $v' = v + 1$, then $v + u - 1, v' + u - 3$ are in the set in (35). If $v' > v + 1$, then $v + u - 1, v' + u - 1$ are in the set in (35) unless $v' + u - 1 = 2v$. But if $v' + u - 1 = 2v$, then $v + u - 1, v' + u - 3$ are in the set in (35).

This completes the proof of Theorem 2.2. □

References

[1] I. Bardaji and D. J. Grynkiewicz, *Long arithmetic progressions in small sumsets*, Integers, 10 (2010)

[2] G. A. Freiman, *Foundations of a Structural Theory of Set Addition*. Translated from the Russian. Translations of Mathematical Monographs, Vol 37. American Mathematical Society, Providence, R. I., 1973

[3] G. A. Freiman, *Inverse additive number theory, XI. long arithmetic progressions in sets with small sumsets*, Acta Arith. 137.4 (2009), 325–331

[4] G. A. Freiman, *On the detailed structure of sets with small additive property*, Combinatorial Number Theory and Additive Group Theory, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser Vergal AG, Basel-Boston-Berlin, (2009), 233–239

[5] Y. O. Hamidoune and A. Plagne, *A generalization of Freiman's* $3k - 3$ *theorem*, Acta Arith. 103 (2002), No. 2, 147–156.

[6] R. Jin, *Freiman's Inverse Problem with Small Doubling Property*, Adv. Math. 216 (2007), No. 2, 711–752.

[7] V. Lev and P. Y. Smeliansky, *On addition of two distinct sets of integers*, Acta Arith. 70 (1995), No. 1, 85–91.

[8] M. B. Nathanson, *Additive Number Theory–Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[9] Y. V. Stanchescu, *On addition of two distinct sets of integers*, Acta Arith. 75 (1996), no. 2, 191–194.