ACCURATE ASYMPTOTIC ESTIMATES OF CATALAN'S SEQUENCE

Vito Lampret
University of Ljubljana, Ljubljana, Slovenia, EU
vito.lampret@guest.arnes.si

Received: 4/11/16, Accepted: 1/1/17, Published: 2/13/17

Abstract

An asymptotic approximation of Catalan's sequence $n \mapsto c_n := \frac{1}{n+1}\binom{2n}{n}$ is given as

$$c_n = \frac{4^n}{(n+1)\sqrt{\pi\, n}} \exp\big(-\widetilde{s}_r(n) + \delta_r(n)\big),$$

where

$$\widetilde{s}_r(n) = \sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)n^{2i-1}} \quad (B_k \text{ are the Bernoulli coefficients}),$$

and

$$|\delta_r(n)| < \frac{2\exp\left(\frac{1}{24r}\right)}{\pi n(1-2^{-(2r+1)})}\sqrt{\frac{r}{\pi}}\left(\frac{r}{e\pi n}\right)^{2r}, \qquad \text{for integers } n, r \geq 1\,.$$

Parameter r controls the error factor $\exp\big(\delta_r(n)\big)$.

1. Introduction

Catalan's sequence $\big(c_n\big)_{n\geq 0}$, defined as

$$c_n := \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n}\binom{2n}{n-1} \quad (n \geq 0), \tag{1}$$

was discovered around 1730 by a Chinese–Mongolian mathematician Antu Ming [3] while he was investigating how to express series expansions of $\sin(m\alpha)$ in terms of powers of $\sin(\alpha)$ for $m \in \{2,3,4,5,10,100,1000,10000\}$, for example $\sin(2\alpha) = 2\big(\sin(\alpha) - \sum_{k=1}^{\infty} 2^{1-2k} c_{k-1} \sin^{2k+1}(\alpha)\big)$. Later, around 1751 Leonhard Euler came across the numbers c_n while studying the triangulations of convex polygons (Euler's triangulation problem 1751). These delightful numbers are named after Eugene

Charles Catalan[1] who, during the study of well-formed parentheses, came across these objects (Catalan's parenthesization problem 1838). Up to this time, over 400 articles and problems have appeared on Catalan's numbers [3]. Catalan's sequence is the most frequently encountered sequence present everywhere in combinatorics. Even more, Catalan's numbers form probably the most ubiquitous sequence of numbers in mathematics due to their ubiquitous nature [4]. The book [4] contains a comprehensive collection of their properties and applications in combinatorics, algebra, analysis, number theory, probability theory, geometry, topology, and other areas of pure and applied mathematics. All these and similar facts have attracted mathematicians to study Catalan's sequence for a long time.

According to (1) we have $c_n \in \mathbb{R}^+$ and

$$\frac{c_{n+1}}{c_n} = 2\frac{2n+1}{n+2} > 1, \quad \text{for } n \geq 1. \tag{2}$$

Catalan's sequence can also be given recursively as $c_0 = 1$ and $c_{n+1} = \sum_{i=1}^{n} c_i c_{n-i}$, for $n \geq 0$.

Due to (2), Catalan's sequence $\big(1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796\dots\big)$ is strictly increasing from the index $n = 1$ onward. All its terms are positive integers due to the identity $c_n = \binom{2n+1}{n+1} - 2\binom{2n}{n+1}$, for $n \in \mathbb{N}$. Referring to (2), $c_{n+1}/c_n < 4$ for $n \geq 0$ and, for big n, Catalan's sequence grows approximately like a geometric sequence with common ratio 4. Consequently, $c_n < 4^n$ and $c_n \approx 4^n$, for big n. Of course, this is a very rough observation. Therefore, our intention is to find more accurate approximations of Catalan's sequence. The solution of this problem provides Stirling's approximations to factorials appearing in c_n. Indeed, according to (1), we have

$$c_n = \frac{1}{n+1} \cdot \frac{(2n)!}{(n!)^2} \quad (n \geq 0). \tag{3}$$

2. Stirling's factorial approximation formula

Using the Euler Gamma function, we have $n! = \Gamma(n+1) = n\Gamma(n)$. Consequently, considering Stirling's factorial formula of order $r \geq 0$ [2, sect. 9.5] we have, for any integer[2] $n \geq 1$,

$$n! = \left(\frac{n}{e}\right)^n \cdot \sqrt{2n\pi} \cdot \exp\big(s_r(n) + d_r(n)\big), \tag{4}$$

where, for all $n \in \mathbb{N}$,

$$s_0(n) = 0 \quad \textit{and} \quad s_r(n) = \sum_{i=1}^{r} \frac{B_{2i}}{(2i-1)(2i)n^{2i-1}} \quad \text{for } r \geq 1, \tag{5}$$

[1] Belgian mathematician, 1814–1894
[2] in fact for any $n \in \mathbb{R}^+$

and, for some $\vartheta_r(n) \in (0,1)$,

$$d_r(n) = \vartheta_r(n) \cdot \frac{B_{2r+2}}{(2r+1)\,(2r+2) \cdot n^{2r+1}}\,. \tag{6}$$

The symbols B_2, B_4, B_6, ... in (5)–(6) denote the Bernoulli coefficients, for example,

$$\begin{aligned}
B_2 &= \frac{1}{6},\ B_4 = B_8 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730},\\
B_{14} &= \frac{7}{6},\ B_{16} = -\frac{3617}{510},\ B_{18} = \frac{43867}{798},\ B_{20} = -\frac{174611}{330},\\
B_{22} &= \frac{854513}{138},\ B_{24} = -\frac{236364091}{2730} \quad \text{and} \quad B_{26} = \frac{8553103}{6}\,,
\end{aligned} \tag{7}$$

with the estimates $|B_{12}| < \frac{1}{3}$, $|B_{16}| < 7$, $B_{18} < 55$, $|B_{20}| < 530$, $B_{22} < 6200$, $|B_{24}| < 87000$, $B_{26} < 1.43 \cdot 10^6$.

3. Approximating c_n Accurately and Asymptotically

Using (3) and (4) we have

$$\begin{aligned}
c_n =& \frac{1}{n+1} \cdot \left(\frac{2n}{e}\right)^{2n} \sqrt{4n\pi}\, \exp\big(s_r(2n) + d_r(2n)\big)\\
&\cdot \left[\left(\frac{e}{n}\right)^{n} \frac{1}{\sqrt{2n\pi}} \cdot \exp\big(-s_r(n) - d_r(n)\big)\right]^2\\
=& \frac{4^n}{(n+1)\sqrt{n\pi}}\, \exp\big(s_r(2n) - 2s_r(n)\big) \cdot \exp\big(\delta_r(n)\big),
\end{aligned} \tag{8}$$

where, considering [1, 23.1.15, p. 805],

$$\begin{aligned}
\delta_r(n) &= \frac{\vartheta_r(n) \cdot B_{2r+2}}{(2r+1)(2r+2) \cdot (2n)^{2r+1}} - \frac{2\vartheta'_r(n) \cdot B_{2r+2}}{(2r+1)(2r+2) \cdot n^{2r+1}}\\
&= \frac{(-1)^r \big|B_{2r+2}\big|}{(2r+1)(2r+2)} \left(\frac{\vartheta_r(n)}{2^{2r+1}} - 2\vartheta'_r(n)\right) \frac{1}{n^{2r+1}}\,,
\end{aligned} \tag{9}$$

for some $\vartheta_r(n), \vartheta'_r(n) \in (0,1)$. Hence, considering [1, 23.1.15, p. 805], we obtain, for $r \geq 0$,

$$-\frac{2\big|B_{2r+2}\big|}{(2r+1)(2r+2)n^{2r+1}} < (-1)^r \delta_r(n) < \frac{\big|B_{2r+2}\big|}{(2r+1)(2r+2)(2n)^{2r+1}}\,. \tag{10}$$

We estimate roughly, for $r \geq 0$, referring to [1, 23.1.15, p. 805], as

$$\begin{aligned}
|\delta_r(n)| &< \frac{2|B_{2r+2}|}{(2r+1)(2r+2)n^{2r+1}} && (11)\\
&\overset{23.1.15}{<} \frac{4}{1-2^{1-(2r+2)}} \cdot \frac{(2r+2)!}{(2\pi)^{2r+2}} \cdot \frac{1}{(2r+1)(2r+2)n^{2r+1}}\\
&= \frac{2}{\pi(1-2^{-(2r+1)})} \cdot \frac{(2r)!}{(2\pi n)^{2r+1}}\,. && (12)
\end{aligned}$$

According to [1, 6.1.38, p. 257], we have, for $r \geq 1$,

$$(2r)! < 2\sqrt{\pi r}\left(\frac{2r}{e}\right)^{2r} \exp\left(\frac{1}{24r}\right).$$

Consequently, using (12), we obtain

$$|\delta_r(n)| < \widetilde{\delta}_r(n) := \frac{2\exp\left(\frac{1}{24r}\right)}{\pi n(1-2^{-(2r+1)})}\sqrt{\frac{r}{\pi}}m\left(\frac{r}{e\pi n}\right)^{2r}, \tag{13}$$

valid for $r \geq 1$.

Using formulas (5) and (8), we find very accurate approximations of Catalan's sequence given in the next theorem.

Theorem 1. *For integers $n, r \geq 1$ the equality*

$$c_n = \widetilde{c}_r(n) \cdot \exp\big(\delta_r(n)\big) \tag{14}$$

holds with

$$\widetilde{c}_r(n) = \frac{4^n}{(n+1)\sqrt{\pi\, n}} \exp\big(-\widetilde{s}_r(n)\big), \tag{15}$$

$$\widetilde{s}_r(n) = \sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)n^{2i-1}}, \tag{16}$$

and $\delta_r(n)$ estimated in (10)–(13).

Corollary 1 (asymptotic expansion). *For $n \in \mathbb{N}$,*

$$\ln\big(c_n\big) \sim \ln\left(\frac{4^n}{(n+1)\sqrt{\pi\, n}}\right) - \sum_{i=1}^{\infty} \frac{(1-4^{-i})B_{2i}}{i(2i-1)n^{2i-1}} \quad \text{as } n \to \infty.$$

Immediately from Theorem 1 we obtain, using the finite increment theorem, the next corollary.

Corollary 2. *The approximation $c_n \approx \widetilde{c}_r(n)$ has the relative error $\varepsilon_r(n) := \big(c_n - \widetilde{c}_r(n)\big)/\widetilde{c}_r(n)$ estimated, for any $n, r \in \mathbb{N}$, as follows (see* (13)*):*

$$\big|\varepsilon_r(n)\big| = \big|\exp\big(\delta_r(n)\big) - 1\big| \leq \widetilde{\varepsilon}_r(n) := \exp\big(\widetilde{\delta}_r(n)\big) \cdot \widetilde{\delta}_r(n).$$

For example, we have $\left|\varepsilon_1(n)\right| < 6\cdot 10^{-3}$ and $\left|\varepsilon_2(n)\right| < 2\cdot 10^{-3}$, both for $n \geq 1$, $\left|\varepsilon_3(n)\right| < 10^{-12}$, for $n \geq 20$, and $\left|\varepsilon_{10}(n)\right| < 3\cdot 10^{-30}$, for all $n \geq 30$.

Considering the estimates above and Corollary 2, we obtain, for $n \geq 20$, that $\widetilde{c}_3(n)(1-10^{-12}) < c_n < \widetilde{c}_3(n)(1+10^{-12})$, i.e., using (5) and (7),

$$\begin{aligned} c_n &> (1-10^{-12})\frac{4^n}{(n+1)\sqrt{\pi\, n}}\exp\left(-\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}\right)\\ c_n &< (1+10^{-12})\frac{4^n}{(n+1)\sqrt{\pi\, n}}\exp\left(-\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}\right). \end{aligned}$$

Setting $r=1$ in Theorem 1 and considering (5) and (7), yields Corollary 3.

Corollary 3. *For $n \in \mathbb{N}$,*

$$\frac{4^n}{(n+1)\sqrt{\pi\, n}}\exp\left(-\frac{1}{8n}-\frac{1}{2880n^3}\right) < c_n < \frac{4^n}{(n+1)\sqrt{\pi\, n}}\exp\left(-\frac{1}{8n}+\frac{1}{180n^3}\right).$$

Similarly, putting $r \in \{2,3,4\}$ in Theorem 1, we obtain the next corollary.

Corollary 4. *For $n \in \mathbb{N}$,*

$$\frac{4^n}{(n+1)\sqrt{\pi\, n}}\cdot\exp\big(a_r(n)\big) < c_n < \frac{4^n}{(n+1)\sqrt{\pi\, n}}\cdot\exp\big(b_r(n)\big), \tag{17}$$

where

$$a_2(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{630n^5} > -\frac{1}{8n}, \tag{17-2a}$$

$$b_2(n) = -\frac{1}{8n}+\frac{1}{192n^3}+\frac{1}{40320n^5} < -\frac{1}{8n}+\frac{1}{190n} < -\frac{1}{9n}, \tag{17-2b}$$

$$a_3(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}-\frac{1}{215040n^7}, \tag{17-3a}$$

$$b_3(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{1}{840n^7}, \tag{17-3b}$$

$$a_4(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{17}{14336n^7}-\frac{1}{594n^9}, \tag{17-4a}$$

$$b_4(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{17}{14336n^7}+\frac{1}{608256n^9}, \tag{17-4b}$$

$$a_5(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{17}{14336n^7}-\frac{31}{18432n^9}-\frac{1}{10^6n^{11}}, \tag{17-5a}$$

$$b_5(n) = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{17}{14336n^7}-\frac{31}{18432n^9}+\frac{1}{257n^{11}}. \tag{17-5b}$$

Corollary 5. *For $n \geq 1$,*

$$\frac{4^n}{(n+1)\sqrt{\pi\, n}}\left(1-\frac{1}{8n}\right) < c_n < \frac{4^n}{(n+1)\sqrt{\pi\, n}}\left(1-\frac{1}{9n}\right). \tag{18}$$

Proof. According to the well-known estimate $e^x > 1 + x$, true for $x \neq 0$, relations (17) and (17-2a) confirm the left inequality in (18). Furthermore, considering the right inequality in Corollary 3, we have $\exp\left(-\frac{1}{8n} + \frac{1}{180n^3}\right) < \exp(-y)$ with $y := -\frac{1}{8n} + \frac{1}{180n}$, where $0 < y < \frac{1}{8n}$. Therefore, for some $\vartheta \in (0,1)$, we have

$$e^{-y} = 1 - y + \frac{e^{-\vartheta y}}{2} y^2 < 1 - y + \frac{1}{2} y^2 < 1 - \left(\frac{1}{8n} - \frac{1}{180n}\right) + \frac{1}{2} \cdot \frac{1}{64n^2}$$
$$\leq 1 - \frac{1}{8n} + \frac{1}{180n} + \frac{1}{128n} < 1 - \frac{1}{9n}\,. \qquad \square$$

4. Bounding π Using Catalan's Sequence

From Theorem 1, considering (11) and (13), we extract the following theorem (see (16)).

Theorem 2. *For any $n, r \in \mathbb{N}$ we have*

$$\pi = \pi_r(n) \cdot \exp\big(2\delta_r(n)\big), \tag{19}$$

where

$$\pi_r(n) := \frac{1}{n}\left(\frac{4^n}{(n+1)c_n}\right)^2 \exp\big(-2\widetilde{s}_r(n)\big), \tag{20}$$

and

$$\big|2\delta_r(n)\big| < \frac{4|B_{2r+2}|}{(2r+1)(2r+2)n^{2r+1}} < \frac{4\exp\left(\frac{1}{24r}\right)}{\pi n(1 - 2^{-(2r+1)})} \cdot \sqrt{\frac{r}{\pi}} \cdot \left(\frac{r}{e\pi n}\right)^{2r}. \tag{21}$$

The next corollary follows directly from Theorem 2.

Corollary 6. *The approximation $\pi \approx \pi_r(n)$ has the relative error $\rho_r(n) := \big(\pi - \pi_r(n)\big)/\pi$ bounded, for any $n \in \mathbb{N}$, as follows (see (13) and Corollary 2):*

$$\big|\rho_r(n)\big| = \big|1 - \exp\big(-2\delta_r(n)\big)\big| \leq 2\widetilde{\varepsilon}_r(n) = 2\exp\big(2\widetilde{\delta}_r(n)\big) \cdot \widetilde{\delta}_r(n).$$

For example, referring to Corollary 6 and example on page 5, we have $\big|\rho_1(n)\big| < 2 \cdot 10^{-2}$ and $\big|\rho_2(n)\big| < 4 \cdot 10^{-3}$, for all $n \geq 1$, $\big|\rho_3(n)\big| < 2 \cdot 10^{-12}$, for $n \geq 20$, and $\big|\rho_{10}(n)\big| < 6 \cdot 10^{-30}$, for $n \geq 30$.

From Corollary 5 we read the following consequence.

Corollary 7. *For $n \geq 1$,*

$$\pi_n^* := \frac{1}{n}\left[\frac{4^n}{(n+1)c_n}\left(1 - \frac{1}{8n}\right)\right]^2 < \pi < \frac{1}{n}\left[\frac{4^n}{(n+1)c_n}\left(1 - \frac{1}{9n}\right)\right]^2 =: \pi_n^{**}.$$

Consequently,

$$\pi = \lim_{n\to\infty}\left[\frac{1}{n}\left(\frac{4^n}{n\,c_n}\right)^2\right].$$

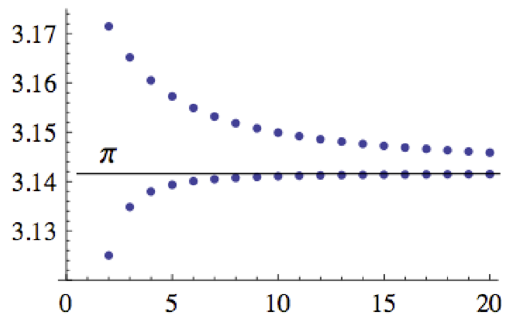

Figure 1: The graphs of the sequences π_n^* and π_n^{**} from Corollary 7.

Figure 1, where the graphs of the sequences $n \mapsto \pi_n^*$ and $n \mapsto \pi_n^{**}$ are plotted, illustrates the roughness of the estimates in Corollary 7. However, directly from Corollary 4 more accurate estimates are obtained, given in the following corollary.

Corollary 8. *For every* $n \in \mathbb{N}$,

$$\pi > \frac{1}{n}\left[\frac{4^n}{(n+1)c_n}\right]^2 \exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7} - \frac{31}{9216n^9} - \frac{2}{10^6 n^{11}}\right)$$

and

$$\pi < \frac{1}{n}\left[\frac{4^n}{(n+1)c_n}\right]^2 \exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7} - \frac{31}{9216n^9} + \frac{2}{257n^{11}}\right).$$

The inequalities in Corollary 8 are quite sharp. Indeed, setting $n = 100$ in Corollary 8, for example, we have the following bounds

$$3.141\,592\,653\,589\,793\,238\,462\,640\ldots < \pi < 3.141\,592\,653\,589\,793\,238\,462\,643\ldots.$$

Consequently, the value of π is calculated to 23 decimal places:

$$\pi = 3.141\,592\,653\,589\,793\,238\,462\,64\ldots.$$

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 9th edn, Dover Publications, New York, (1974).

[2] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1994.

[3] T. Koshy, *Catalan Numbers With Applications*, Oxford University Press, 2009, Oxford, NY.

[4] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[5] S. Wolfram, *Mathematica*, version 9.0, Wolfram Research, Inc., 1988–2013.