PEG SOLITAIRE ON CATERPILLARS

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Received: 5/20/15, Revised: 5/26/16, Accepted: 1/31/17, Published: 2/13/17

Abstract

In a 2011 paper by Beeler and Hoilman, the traditional game of peg solitaire is generalized to graphs in the combinatorial sense. One of the important open problems was to classify solvable trees. In this paper, we extend this classification to several infinite classes of caterpillars. We also give the maximum number of pegs that can be left on such caterpillars under the restriction that we jump whenever possible.

1. Introduction

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (in other words, a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z. In [2], peg solitaire is generalized to graphs.

A graph, $G = (V, E)$, is a set of vertices, V, and a set of edges, E. If there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z, provided that $xy, yz \in E$. Such a jump will be denoted $x \cdot \overrightarrow{y} \cdot z$. All other notation and terminology will be consistent with Chartrand [10].

If S is a starting state of pegs, then a terminal state T is *associated* with S if T can be obtained from S via a sequence of peg solitaire moves. A graph G is

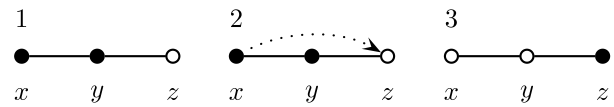

Figure 1: A typical jump in peg solitaire, $x\cdot\overrightarrow{y}\cdot z$

solvable if there exists some vertex s so that, starting with a hole in s, there exists an associated terminal state consisting of a single peg. A graph G is *freely solvable* if for all vertices s there exists a terminal state consisting of a single peg associated with starting state $\{s\}$. A graph G is *k-solvable* if there exists some vertex s so that, starting with a hole in s, there exists an associated minimum terminal state consisting of k nonadjacent pegs. In particular, a graph is *distance 2-solvable* if there exists some vertex s so that, starting with a hole in s, there exists an associated terminal state consisting of two pegs that are distance 2 apart [2]. The *dual* of a peg configuration S is obtained from S by reversing the roles of pegs and holes. For additional variations of peg solitaire on graphs, refer to [9, 11, 12]. For more information on traditional peg solitaire, refer to [1, 8].

In [2, 3, 7], the solvability of several families of graphs was determined. One of the more important open problems in [2] was to classify the solvability of trees. Progress in this area has been difficult, proceeding one family of trees at a time. So far, the solvability of stars [2], paths [2], double stars [3], and trees of diameter four [7] has been determined. It is likely that determining necessary and sufficient conditions for the solvability of all trees is a difficult problem. It is for this reason that results about specific classes of trees are interesting.

We are motivated by the above comments to consider the solvability of caterpillars. A caterpillar can be obtained from the path on n vertices by appending pendant vertices to the existing vertices of the path. The vertices of the original path, which are called the *spine* of the caterpillar, are labeled x_1,...,x_n in the natural way. We append a_i pendant vertices to x_i. The pendant vertices adjacent to x_i are denoted $x_{i,1}$,...,x_{i,a_i}. The caterpillar with parameters n, a_1,...,a_n will be denoted $P_n(a_1, ..., a_n)$ (see Figure 2). For convenience of exposition, the set of vertices $\{x_{i,1}, ..., x_{i,a_i}\}$ will be denoted X_i. Such a set is called a *cluster*. Without loss of generality, we may assume that $a_1 \geq 1$ and $a_n \geq 1$. When listing our conditions on solvability, we only consider one caterpillar from each isomorphism class. Our representative from each class will satisfy $a_1 \geq a_n$. If $a_1 = a_n$, then our representative will satisfy $a_2 \geq a_{n-1}$ and so on. At various points during our proofs, the resulting caterpillar will not satisfy this condition. However, in such cases we can reverse the order of parameters so that the resulting caterpillar does satisfy this condition. Further, we may assume that $n \geq 4$ as the solvability of caterpillars of shorter spine length was determined in [2, 3, 7].

When dealing with clusters, it is useful to generalize our jump notation for these clusters. In this case, $X_i\cdot\overrightarrow{x_i}\cdot x_j$ indicates a jump from a vertex in X_i over x_i into x_j. This notation is dependent on there being at least one peg in X_i, a peg in x_i,

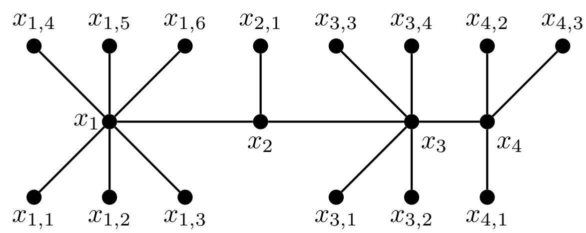

Figure 2: The caterpillar $P_4(6, 1, 4, 3)$

and a hole in x_j. The notations $x_j \cdot \overrightarrow{x_i} \cdot X_i$ and $X_i \cdot \overrightarrow{x_i} \cdot X_i$ are defined analogously.

We will also make heavy use of the *double star purge* from [7]. Using our notation, the double star can be represented as $P_2(a_1, a_2)$. Beginning with the hole in x_2, we can remove d pegs from both clusters by using the jumps $X_1 \cdot \overrightarrow{x_1} \cdot x_2$ and $X_2 \cdot \overrightarrow{x_2} \cdot x_1$ d times. This can be generalized between any two adjacent clusters on the spine. The notation $\mathcal{DS}(X_i, X_j, d)$ will be used to denote a double star purge where X_i is the cluster with a peg in x_i, X_j is the cluster with a hole in x_j, and d is the number of pegs that will be removed from each cluster in the process. Additional information on packages and purges can be found in [6]. Other useful results that we will use are given in the following proposition.

Proposition 1. *[2, 3, 5]*

(i) (Inheritance Principle) *If G is a k-solvable spanning subgraph of H, then H is (at worst) k-solvable.*

(ii) (Duality Principle) *Suppose that S is a starting state of G with associated terminal state T. These have associated duals S' and T', respectively. It follows that T' is a starting state of G with associated terminal state S'.*

(iii) The path on n vertices is freely solvable if and only if $n = 2$. The path of n vertices is solvable if and only if n is even or $n = 3$. The path on n vertices is distance 2-solvable when n is odd and $n \geq 5$.

(iv) The double star $P_2(a_1, a_2)$ is freely solvable if and only if $a_1 = a_2$ and $a_1 \neq 1$. The double star $P_2(a_1, a_2)$ is solvable if and only if $a_1 \leq a_2 + 1$. The double star $P_2(a_1, a_2)$ is distance 2-solvable if and only if $a_1 = a_2 + 2$. The double star $P_2(a_1, a_2)$ is $(a_1 - a_2)$-solvable in all other cases.

In particular, the even path is solvable with the initial hole in the second vertex. In this case, the final peg is in the next to the last vertex [2].

2. The Solvability of Certain Caterpillars

In this section, we determine the solvability of certain infinite classes of caterpillars. The first class of caterpillars that we consider will be caterpillars in which only the end vertices of the spine have pendants.

Theorem 1. *(i) The graph $G = P_n(a_1, \dots, a_n)$, where $n \geq 4$, $a_2 = a_3 = \cdots = a_{n-1} = 0$, and $a_n = 1$ is solvable if and only if $a_1 \leq 2$ and n is an even number or $a_1 \leq 3$ and $n = 4$.*

(ii) Graphs of the form $P_n(a_1, 0, ..., 0, a_n)$ are solvable if and only if $a_1 = 2$, $a_n \geq 1$, and n is even.

Proof. *(i)* First we will show that $P_{2n}(2, 0, ..., 0, 1)$ is a solvable graph. Let the initial hole be in x_2. The jump $X_1 \cdot \overrightarrow{x_1} \cdot x_2$ will result in an even path with a hole in the second vertex. Therefore, this graph is solvable by [2]. For the graph $P_4(3, 0, 0, 1)$, begin with the initial hole in x_2. The required jumps to solve the graph are $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, $x_3 \cdot \overrightarrow{x_2} \cdot x_1$, $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, $X_4 \cdot \overrightarrow{x_4} \cdot x_3$, $x_3 \cdot \overrightarrow{x_2} \cdot x_1$, and $X_1 \cdot \overrightarrow{x_1} \cdot x_2$. Notice that these graphs are not freely solvable for the same reasons as the path.

We claim that the graph is not solvable in all other cases. First, consider $G = P_{2n+1}(2, 0, ..., 0, 1)$. Any attempt to clear the cluster first will result in an odd path on the remaining vertices. Hence, it is not solvable. Thus, we need to consider the case where we remove the pegs from the spine first. Since the spine forms a path, we must solve it as if it were a path. For this reason, it is optimal to begin with the initial hole in x_n and proceed to solve the graph as an even path on the vertices x_1,...,x_n, $x_{n,1}$ using the methods outlined in [2]. This will result in one peg remaining in x_2 and both pegs remaining in X_1. Therefore, G is not solvable. Finally, we will consider the case where $n \geq 5$ and $a_1 \geq 3$. In order to solve the graph, we must remove all pegs from X_1 and from the path (including $x_{n,1}$). A peg in X_1 can be removed only if there is first a peg in x_1. However, to place a peg in x_1, we must jump $x_3 \cdot \overrightarrow{x_2} \cdot x_1$. Likewise, this requires a peg in x_3. In order to place a peg in x_3, we must jump $x_5 \cdot \overrightarrow{x_4} \cdot x_3$. This results in holes three adjacent vertices on the spine, namely x_3, x_4, and x_5. Hence, it is impossible to reach X_1 and x_2. Including an initial jump out of X_1, this means that we can remove at most three pegs from X_1. This is accomplished by having the initial hole in x_2. The required jumps are $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, $x_3 \cdot \overrightarrow{x_2} \cdot x_1$, $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, $x_5 \cdot \overrightarrow{x_4} \cdot x_3$, $x_3 \cdot \overrightarrow{x_2} \cdot x_1$, and $X_1 \cdot \overrightarrow{x_1} \cdot x_2$. This leaves $a_1 - 3$ pegs in X_1 and a peg in x_2. It follows that $a_1 \leq 3$ is necessary for solvability. Furthermore, this shows that if $n \geq 5$, our attempts to remove pegs from X_1 will make it impossible to remove all pegs on the spine.

(ii) Consider the graph $G = P_n(a_1, 0, ..., 0, a_n)$ where $a_1 = a_n = 2$ and n is even. By (i), the graph obtained by removing the vertex $x_{n,2}$ is solvable when the initial hole is in x_2. In this solution, the final peg is in x_n. Thus, we can solve G with the initial hole in x_2 leaving pegs in x_n and $x_{n,2}$. We can then make the jump $x_{n,2} \cdot \overrightarrow{x_n} \cdot x_{n-1}$ to solve the graph. The unsolvable cases follow for the same reasons as in (i). □

Making statements regarding the solvability of more general caterpillars may be difficult. One such statement is given in the following theorem.

Theorem 2. *(i) The graph $P_n(a_1, ..., a_n)$ is not solvable if $a_1 \geq a_2 + n$ or $a_n \geq a_{n-1} + n$.*

(ii) For $2 \leq i \leq n-1$*, the graph* $P_n(a_1, ..., a_n)$ *is not solvable if* $a_i \geq a_{i-1} + a_{i+1} + n - 2$*.*

(iii) The graph $P_n(a_1, ..., a_n)$ *is not freely solvable if* $a_1 \geq a_2 + n - 1$ *or* $a_n \geq a_{n-1} + n - 1$*.*

(iv) For $2 \leq i \leq n-1$*, the graph* $P_n(a_1, ..., a_n)$ *is not freely solvable if* $a_i \geq a_{i-1} + a_{i+1} + n - 3$*.*

Proof. (i) For the first claim, it suffices to show that we can remove at most a_2+n-1 pegs from X_1. Using a double star purge with X_2, it is possible to remove at most a_2 pegs from X_1. Consequently, the removal of any additional peg from X_1 will require the removal of a peg from the spine. If the initial hole is in one of the clusters, say X_i, then the first jump must be $x_{i\pm1} \cdot \overrightarrow{x_i} \cdot X_i$. Thus, we have $n-2$ pegs on the spine when the initial hole is on the spine and $n-1$ pegs on the spine when the hole is on the spine. Hence, we can remove at most $a_2 + n - 1$ pegs from X_1, leaving a peg on the spine. Thus the graph is not solvable when $a_1 \geq a_2 + n$. To achieve this bound, begin with the initial hole in x_2. We perform a sequence of jumps $n-2$ times. The jth iteration ($j = 1, ..., n-2$) begins with the moves $X_1 \cdot \overrightarrow{x_1} \cdot x_2$ and $x_3 \cdot \overrightarrow{x_2} \cdot x_1$. The next $n-j+1$ jumps are $X_\ell \cdot \overrightarrow{x_\ell} \cdot x_{\ell-1}$ for $\ell = 4, ..., n-j+1$. Notice that this leaves holes in x_2 and x_{n-j+1}. After completing this sequence of jumps, we will have removed a total of $a_2 + n - 2$ pegs from X_1, and $n - j + 2$ pegs from X_j for $i = 4, ..., n$. After this series of jumps, the only peg left on the spine will be in x_1. This will allow a final jump $X_1 \cdot \overrightarrow{x_1} \cdot X_1$. We now have $a_1 - a_2 - n + 2$ pegs in X_1 and no further moves are possible. The case where $a_n \geq a_{n-1} + n$ is analogous.

(ii) For $2 \leq i \leq n-1$, there are only three ways to remove pegs from X_i. Namely, a_{i-1} pegs in X_i can be removed using a double star purge with X_{i-1}, a_{i+1} pegs can be removed using a double star purge with X_{i+1}, or pegs can be removed using those in the spine. As in (i), we can remove at most $n-i-1$ pegs from X_i using the pegs in x_i, x_{i+1},...,x_n. Similarly, we can remove at most $i-1$ pegs from X_i using the pegs in x_1,...,x_{i-1}. However, to do this we must reposition the hole to make the required jumps. To do so, we must jump out of one the adjacent clusters or along the spine. In either case, we can remove at most $a_{i-1} + a_{i+1} + n - 3$ pegs from X_i, leaving a peg on the spine. Hence, this graph is not solvable if $a_i \geq a_{i-1} + a_{i+1} + n - 2$. To achieve this bound, begin with the initial hole in x_{i+1} and perform $\mathcal{DS}(X_i, X_{i+1}, a_{i+1})$. Then we use a series of jumps along the spine as in the first part of the theorem. This removes an additional $n-i-1$ pegs from X_i. We then perform $X_{i-1} \cdot \overrightarrow{x_{i-1}} \cdot x_i$ and $\mathcal{DS}(X_i, X_{i-1}, a_{i-1} - 1)$. Apply a series of jumps along the spine as in the first part of the theorem. This removes an additional $i-1$ pegs from X_i.

(iii) Note that the methods given above are optimal for removing pegs from a large cluster. In these methods, we place the initial hole on a spine vertex that is adjacent to the support vertex of the large cluster. This allows for an initial jump out of the large cluster. Hence, the placement of the initial hole allows us to remove one additional peg from the large cluster than we could otherwise. For this reason,

our methods given above are dependent on this placement of the initial hole. This observation will be crucial in determining that certain cases are not freely solvable.

To show that a graph is not freely solvable, it suffices to show that it can not be solved if the initial hole is in a specific vertex. Suppose that $a_1 \geq a_2 + n - 1$ and that the initial hole is in x_1. The first jump must be either $X_2 \cdot \overrightarrow{x_2} \cdot x_1$ or $x_3 \cdot \overrightarrow{x_2} \cdot x_1$. If the first jump is $X_2 \cdot \overrightarrow{x_2} \cdot x_1$, then we ignore the hole in X_2. Thus, we are trying to solve $P_n(a_1, a_2 - 1, a_3, ..., a_n)$ with a hole in x_2. Since $a_1 \geq a_2 + n - 1$, this is impossible by (i).

In the second case, the initial jump is $x_3 \cdot \overrightarrow{x_2} \cdot x_1$. At this point, we can assume without lose of generality that the double star purge $\mathcal{DS}(X_1, X_2, a_2)$ has been performed. Hence, this reduces to (i) after the first two moves and the double star purge have been completed. The remaining moves in (i) remove $n - 2$ additional pegs from X_1. Since $a_1 \geq a_2 + n - 1$, the graph is not solvable. The case where $a_n \geq a_{n-1} + n - 1$ is analogous.

(iv) Follows in a similar manner to (ii) and (iii). □

Note that if $a_1 = n - 1$, $a_2 = a_3 = 0$, and $a_j = n - j + 1$ for $j = 4, ..., n$, then the graph $P_n(a_1, ..., a_n)$ is solvable using the algorithm given above. Thus, the bound in Theorem 2 (i) is sharp. Likewise, if $a_i = n - 3$, $a_{i-2} = a_{i-1} = a_{i+1} = a_{i+2} = 0$, $a_j = j$ for $j = 1, ..., i - 3$, and $a_\ell = n - \ell + 1$ for $\ell = i + 3, ..., n$, then $P_n(a_1, ..., a_n)$ is solvable using the algorithm above. Therefore, the bound in (ii) is also sharp. To what extent the bounds given in (iii) and (iv) are sharp is unknown.

We end this section by providing two theorems which involve reducing a caterpillar to one of smaller order. These theorems can be used to obtain a solution of certain caterpillars. However, they do not answer the more difficult question of which caterpillars are unsolvable.

Theorem 3. *Suppose that $P_n(a_1, ..., a_n)$ is k-solvable. If there exists non-negative integers k_1,...,k_{n-1} such that $b_1 = a_1 + k_1$, $b_n = a_n + k_{n-1}$, and $b_i = a_i + k_{i-1} + k_i$ for $i = 2$,...,$n - 1$, then $P_n(b_1, ..., b_n)$ is (at worst) k-solvable.*

Proof. Suppose that we have a solution of $P_n(a_1, ..., a_n)$ that results in k nonadjacent pegs. We modify this solution for use on $P_n(b_1, ..., b_n)$ by including additional double star purges within the solution.

Any solution of the caterpillar involves a combination of moves of the form $x_i \cdot \overrightarrow{x_{i\pm1}} \cdot X_{i\pm1}$, $X_j \cdot \overrightarrow{x_j} \cdot x_{j\pm1}$, and $x_{i-1} \cdot \overrightarrow{x_i} \cdot x_{i+1}$. A jump of the form $x_{i-1} \cdot \overrightarrow{x_i} \cdot x_{i+1}$ will eventually be followed by $x_{i+2} \cdot \overrightarrow{x_{i+1}} \cdot x_i$ in order to either remove pegs from X_i or to solve the spine. For this reason, if x_i and x_{i+1} are two adjacent vertices on the spine, then there is an intermediate state of the solution in which exactly one of x_i and x_{i+1} has a peg. At this point, we perform either $\mathcal{DS}(X_i, X_{i+1}, k_i)$ or $\mathcal{DS}(X_{i+1}, X_i, k_i)$, depending on the location of the peg on the spine. After completing the original solution and the additional purges, we now have a solution of $P_n(b_1, ..., b_n)$ with k nonadjacent pegs. □

As an example of Theorem 3, consider $P_4(1, 1, 1, 1)$. This graph is freely solvable as confirmed by an exhaustive computer search [4]. We claim that if a, b, and c

are positive integers, then $P_4(a, a+b, b+c, c)$ is also freely solvable. Here, we let $k_1 = a-1$, $k_2 = b$, $k_3 = c-1$ and apply Theorem 3. Hence, the resulting graph is freely solvable.

Theorem 4. *Suppose that $P_n(a_1, ..., a_n)$ is k-solvable with the initial hole in x_2. If there exists a non-negative integer m such that $c_1 = m+1$, $c_2 = a_1+m$, $c_3 = a_2+1$, and $c_i = a_{i-1}$ for $i = 4, ..., n+1$, then $P_{n+1}(c_1, ..., c_{n+1})$ is (at worst) k-solvable with the initial hole in x_2.*

Proof. Consider $P_{n+1}(c_1, ..., c_{n+1})$ with the initial hole in x_2. Perform the double star purge $\mathcal{DS}(X_1, X_2, m)$. Now jump $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, $X_2 \cdot \overrightarrow{x_2} \cdot X_2$, and $X_3 \cdot \overrightarrow{x_3} \cdot x_2$. This removes $m+1$ pegs from X_1, m pegs from X_2, and one peg from X_3. On the spine, there are holes in x_1 and x_3 and pegs elsewhere. Ignoring the holes in X_1, x_1, X_2, and X_3, the resulting graph is $P_n(a_1, ..., a_n)$ with a hole in x_2. Hence, the graph is (at worst) k-solvable. □

3. Caterpillars With a Spine of Length Four

We now consider general caterpillars where the spine length is four. In order to facilitate showing that some of these caterpillars are not freely solvable, we present a lemma that considers the case where at least one of a_2 or a_3 is zero. To what extent this lemma can be generalized to caterpillars of longer spine length is unknown at this time.

Lemma 1. *If $a_2 = 0$ or $a_3 = 0$, then the graph $P_4(a_1, a_2, a_3, a_4)$ is not freely solvable.*

Proof. Without loss of generality, assume that $a_2 = 0$. If $a_3 \geq a_4 + 1$, then this graph is not freely solvable by Theorem 2. For this reason, we only consider the case where $a_3 \leq a_4$. It is sufficient to show that we cannot solve the graph given a specific initial hole. Suppose that the initial hole is in x_1. Since $a_2 = 0$, the only available move is $x_3 \cdot \overrightarrow{x_2} \cdot x_1$. In order to remove pegs from X_4, we must eventually make the jump $X_4 \cdot \overrightarrow{x_4} \cdot x_3$. So, we can assume that this jump has been made. If we jump $X_3 \cdot \overrightarrow{x_3} \cdot x_2$, then $x_1 \cdot \overrightarrow{x_2} \cdot x_3$ is forced. If there are pegs available in X_3 and X_4, then we may perform a double star purge between these clusters. However, this leaves at least one peg in X_1 and a peg in either X_3 or X_4. As no further moves are possible, the graph is not solvable.

For this reason, we must instead follow $X_4 \cdot \overrightarrow{x_4} \cdot x_3$ with the jump $X_1 \cdot \overrightarrow{x_1} \cdot x_2$. At this point, we can remove at most one additional peg from X_1 by using the jumps $x_3 \cdot \overrightarrow{x_2} \cdot x_1$ and $X_1 \cdot \overrightarrow{x_1} \cdot x_2$. For this reason, the graph will not be solvable from this state when $a_1 \geq 3$. Hence, we need only consider the case where $a_1 \in \{1, 2\}$.

If $a_1 = 1$, then we begin with the initial hole in x_4. In some order, we must perform a double star purge on X_3 and X_4 and jump $x_2 \cdot \overrightarrow{x_3} \cdot x_4$. In either case, it reduces to a double star. Hence, if $a_4 \geq a_3 + 3$, then the graph cannot be solved from this state. Thus the only remaining cases when $a_1 = 1$ are $a_4 = a_3 \geq 1$, $a_4 = a_3 + 1$, and $a_4 = a_3 + 2$.

If $a_1 = 2$, then begin with the initial hole in x_1. Without loss of generality, we assume that the first two jumps are $x_3 \cdot \overrightarrow{x_2} \cdot x_1$ and $X_1 \cdot \overrightarrow{x_1} \cdot x_2$. To remove the pegs in X_4, we must perform a double star purge on X_3 and X_4. So if $a_3 \neq a_4 - 1$, then the graph is not solvable from this state. Thus the only remaining case when $a_1 = 2$ is $a_4 = a_3 + 1 \geq 2$.

For the remaining cases, we reverse the order of the parameters. Hence, we need only consider caterpillars of the form $P_4(a_1, a_1 - 2, 0, 1)$, $P_4(a_1, a_1 - 1, 0, 2)$, $P_4(a_1, a_1 - 1, 0, 1)$, and $P_4(a_1, a_1, 0, 1)$.

First, consider the caterpillar $P_4(a_1, a_1 - 2, 0, 1)$, where $a_1 \geq 2$. Suppose that the initial hole is in X_4. Our first jump must be $x_3 \cdot \overrightarrow{x_4} \cdot X_4$. Similarly, our second jump must either be $x_1 \cdot \overrightarrow{x_2} \cdot x_3$ or $X_2 \cdot \overrightarrow{x_2} \cdot x_3$. If our second jump is $x_1 \cdot \overrightarrow{x_2} \cdot x_3$, then we have a_1 pegs in X_1, $a_1 - 2$ pegs in X_2, a peg in x_3, and a peg in X_4. Hence, no further moves are possible. If instead our second jump is $X_2 \cdot \overrightarrow{x_2} \cdot x_3$, then without loss of generality we may perform the double star purge between X_1 and X_2. This results in three pegs in X_1, a peg in x_3, and a peg in X_4. After the jump $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, we can either remove a peg in X_1 or the peg in X_4. In either case, there will be three pegs on the graph and no available moves. In either case, the graph cannot be solved with the initial hole in X_4.

Now, consider the caterpillar $P_4(a_1, a_1 - 1, 0, a_4)$, where $a_1, a_4 \in \{1, 2\}$. Suppose that the initial hole is in X_1. Our first jump must be either $x_2 \cdot \overrightarrow{x_1} \cdot X_1$ or $X_1 \cdot \overrightarrow{x_1} \cdot X_1$. If we jump $x_2 \cdot \overrightarrow{x_1} \cdot X_1$, then $x_4 \cdot \overrightarrow{x_3} \cdot x_2$ is forced. At this point, we only have one peg on the spine (in x_2). Thus, any jump into x_3 will result in no available moves. So, after performing a double star purge on X_1 and X_2, we will have a peg in X_1, a peg in x_2, and at least one peg in X_4. Hence, the graph is not solvable from this state.

If instead our initial jump is $X_1 \cdot \overrightarrow{x_1} \cdot X_1$, then our second jump must be either $X_2 \cdot \overrightarrow{x_2} \cdot x_1$ or $x_3 \cdot \overrightarrow{x_2} \cdot x_1$. In the first case, we must eventually jump $X_1 \cdot \overrightarrow{x_1} \cdot x_2$ in order to remove the remaining peg from X_1. Without loss of generality, assume that this jump has been made. The jumps $X_3 \cdot \overrightarrow{x_2} \cdot x_1$ and $X_4 \cdot \overrightarrow{x_4} \cdot x_3$ are then forced. At this point, we have two nonadjacent pegs on the spine and no further moves are possible. Hence, the graph is not solvable from this state. In the second case, we must eventually jump $X_4 \cdot \overrightarrow{x_4} \cdot x_3$. We are then forced to jump $X_1 \cdot \overrightarrow{x_1} \cdot x_2$. Regardless of the next move, we will have at least two nonadjacent pegs on the graph and no available moves. Hence, it is not solvable from this state.

Finally, consider the caterpillar $P_4(a_1, a_1, 0, 1)$. Suppose the initial hole is in x_4. Note that the initial jump $x_2 \cdot \overrightarrow{x_3} \cdot x_4$ is forced. The only way to remove pegs from X_2 is a double star purge with X_1. However, this leaves a peg in x_1. We now have one available jump, $X_4 \cdot \overrightarrow{x_4} \cdot x_3$. We now have pegs in x_1 and x_3 and holes elsewhere. Thus, the graph cannot be solved from this state. $\square$

With this lemma in mind, we are ready to proceed with our main result. Namely, we will find necessary and sufficient conditions for the solvability of $P_4(a_1, a_2, a_3, a_4)$. For ease of presentation, we do not include conditions for such graphs to be k-

solvable, distance 2-solvable, and freely solvable in the statement of Theorem 5. However, these conditions are made explicit throughout the proof.

Theorem 5. *(I) The caterpillar $P_4(a_1, a_2, a_3, a_4)$ with $a_1 \geq a_2 + 1$ is solvable if and only if one of the following is true:* (i) $a_1 = a_2 + 1$ *and either* $a_3 - a_4 \leq 1$ *or* $a_4 - a_3 \leq 3$*;* (ii) $a_1 = a_2 + 2$ *and* $a_4 - a_3 \leq 2$*;* (iii) $a_1 = a_2 + 2$*,* $a_2 \geq 1$*, and* $a_3 = a_4$*;* (iv) $a_1 = a_2 + 3$ *and* $a_3 = a_4 - 1$*.*

(II) The caterpillar $P_4(a_1, a_2, a_3, a_4)$ with $a_2 = a_1 + m$, where $m \geq 0$ is solvable if and only if one of the following is true: (i) $a_3 = a_4 + k$*, where* $k \geq 0$ *and* $-2 \leq m - k \leq 2$*;* (ii) $a_4 = a_3 + k$*, where* $k \geq 1$ *and* $m + k \leq 2$*.*

Proof. (I) Let $a_1 \geq a_2 + 1$. By Theorem 2, $a_1 \leq a_2 + 3$ is a necessary condition for solvability. Likewise, $a_1 \leq a_2 + 2$ is necessary for the graph to be freely solvable. For each of the cases where $a_2 + 1 \leq a_1 \leq a_2 + 3$, we determine the additional necessary conditions below. We also prove that these conditions are also sufficient.

(i) Suppose that $a_1 = a_2 + 1$ and $a_3 = a_4$. If $a_1 \geq 2$, then we apply Theorem 3 with $k_1 = a_1 - 2$, $k_2 = 0$, and $k_3 = a_3 - 1$. This reduces the graph to $P_4(2, 1, 1, 1)$. An exhaustive computer search [4] confirms that $P_4(2, 1, 1, 1)$ is freely solvable. Hence, the original graph is also freely solvable. If $a_1 = 1$, then we reverse the parameters to obtain a caterpillar of the form $P_4(a_3, a_3, 0, 1)$. This graph reduces to $P_4(1, 1, 0, 1)$, which is solvable, but not freely solvable. Note that graphs of the form $P_4(a_1, a_2, 0, a_4)$ are not freely solvable by Lemma 1.

Suppose that $a_1 = a_2 + 1$ and $a_3 = a_4 + k$, where k is a positive integer. In an optimal solution, the pegs in X_2 must be used to remove those in X_1. Using a similar argument as in the proof of Theorem 2, at most a_4 pegs can be removed from X_3. Hence, such graphs are at best k-solvable and cannot be freely solvable. This can be achieved by placing the initial hole in x_2. Perform $\mathcal{DS}(X_1, X_2, a_2)$, $X_3 \cdot \overrightarrow{x_3} \cdot x_2$, $X_4 \cdot \overrightarrow{x_4} \cdot x_3$, $\mathcal{DS}(X_3, X_4, a_4 - 1)$, $x_3 \cdot \overrightarrow{x_2} \cdot X_2$, $X_1 \cdot \overrightarrow{x_1} \cdot x_2$, $X_2 \cdot \overrightarrow{x_2} \cdot x_3$, and $X_3 \cdot \overrightarrow{x_3} \cdot x_4$. At this point no further moves are possible and a peg remains in x_4 while $k - 1$ pegs remain in X_3. Consequently, the graph is solvable if $k = 1$. The graph is k-solvable for $k \geq 2$. In particular, the graph is distance 2-solvable when $k = 2$.

Suppose that $a_1 = a_2 + 1$ and $a_4 = a_3 + k$, where $k \geq 1$. If $a_1 \geq 2$, $a_3 \geq 1$, and $k = 1$, then we apply Theorem 3 with $k_1 = a_1 - 2$, $k_2 = 0$, and $k_3 = a_4 - 2$ to reduce the graph to $P_4(2, 1, 1, 2)$. An exhaustive computer search confirms that $P_4(2, 1, 1, 2)$ is freely solvable. If $a_3 = 0$ or $a_2 = 0$, then such graphs cannot be freely solvable by Lemma 1. If $k = 2$, $a_2 \geq 1$, and $a_3 \geq 1$, then we apply Theorem 3 with $k_1 = a_2 - 1$, $k_2 = 0$, and $k_3 = a_4 - 3$ to reduce the graph to $P_4(3, 1, 1, 2)$, which is freely solvable. For other cases when $k = 2$, after reversing the parameters the graph reduces to $P_4(2, 0, 0, 1)$ which is solvable, but not freely solvable. Similarly, if $k = 3$, then we apply Theorem 3 with $k_1 = a_2$, $k_2 = 0$, and $k_3 = a_4 - 3$ to reduce the graph to $P_4(3, 0, 0, 1)$, which is solvable. Note that if $k \geq 3$, then the graph cannot be freely solvable by Theorem 2. Using the same argument as was used to prove Theorem 2, we can remove at most $a_3 + 3$ pegs from X_4. Thus, such graphs are at best $(k-2)$-solvable. To achieve this, begin with the initial hole in x_3.

Perform $\mathcal{DS}(X_4, X_3, a_3)$, $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$, $x_2{\cdot}\overrightarrow{x_3}{\cdot}x_4$, $\mathcal{DS}(X_1, X_2, a_2)$, $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$, $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$, $x_2{\cdot}\overrightarrow{x_3}{\cdot}x_4$, and $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$. This leaves $k-3$ pegs in X_4 and a peg in x_3. Thus, the graph is $(k-2)$-solvable when $k \geq 4$. In particular, the graph is distance 2-solvable when $k = 4$.

(ii) Suppose that $a_1 = a_2+2$ and $a_4 = a_3+k$, where $k \geq 1$. If $a_1 \geq 3$, $a_3 \geq 1$, and $k = 1$, then we apply Theorem 3 with $k_1 = a_1-3$, $k_2 = 0$, and $k_3 = a_4-2$ to reduce the graph to $P_4(3, 1, 1, 2)$, which is freely solvable. Note that if $a_2 = 0$ or $a_3 = 0$, then the graph is not freely solvable by Lemma 1. Using a similar argument as in the proof of Theorem 2, at most $a_3 + 2$ can be removed from X_4, leaving a peg in the spine. Hence, this graph is at best $(k-1)$-solvable when $k \geq 2$. To accomplish this, begin with the hole in x_2. The solution is $\mathcal{DS}(X_1, X_2, a_2)$, $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$, $x_3{\cdot}\overrightarrow{x_2}{\cdot}x_1$, $\mathcal{DS}(X_4, X_3, a_3)$, $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$, $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$, and $x_2{\cdot}\overrightarrow{x_3}{\cdot}x_4$. If $k = 1$, then the graph is solved. If $k \geq 2$, then $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$. This results in a peg in x_3 and $k-2$ pegs in X_4. Hence, the graph is solved if $k = 2$. Likewise, if $k \geq 3$, then the graph is $(k-1)$-solvable. In particular, the graph is distance 2-solvable for $k = 3$.

(iii) We now consider the case where $a_1 = a_2+2$ and $a_3 = a_4+k$, where $k \geq 0$. If $a_2 = 0$ and $k = 0$, then after applying Theorem 3 the only admissible graph under our parametrization is $P_4(2, 0, 1, 1)$. This graph has been found to be distance 2-solvable using an exhaustive computer search [4]. Thus we assume that $k \geq 1$ and $a_2 = 0$. Note that we can remove at most a_4+1 pegs from X_3. Begin with the hole in x_2 and $X_3{\cdot}\overrightarrow{x_3}{\cdot}x_2$, $\mathcal{DS}(X_4, X_3, a_4-1)$, $x_1{\cdot}\overrightarrow{x_2}{\cdot}x_3$, $X_3{\cdot}\overrightarrow{x_3}{\cdot}x_2$, $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$, $x_3{\cdot}\overrightarrow{x_2}{\cdot}x_1$, and $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$. This leaves a peg in X_1, a peg in x_2, and $k-1$ pegs in X_3. Hence the graph is $(k+1)$-solvable when $k \geq 1$. In particular, the graph is distance 2-solvable when $k = 0$ or $k = 1$.

Assume that $a_1 = a_2 + 2$, $a_2 \geq 1$, and $a_3 = a_4 + k$, where $k \geq 0$. In an optimal solution, the pegs in X_2 will be used to remove those in X_1. Similarly, we can remove at most a_4 pegs from X_3. To do this begin with the hole in x_2. The solution is $\mathcal{DS}(X_1, X_2, a_2)$, $X_3{\cdot}\overrightarrow{x_3}{\cdot}x_2$, $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$, $\mathcal{DS}(X_3, X_4, a_4-1)$, $x_3{\cdot}\overrightarrow{x_2}{\cdot}X_2$, $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$, $X_2{\cdot}\overrightarrow{x_2}{\cdot}x_1$, and $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$. At this point no further jumps are possible but k pegs remain in X_3 and one peg is in x_2. Therefore, the graph is solvable only if $k = 0$. Otherwise, the graph is $(k+1)$-solvable for $k \geq 1$. In particular, $k = 1$ implies that the graph is distance 2-solvable. Using a similar argument, we can show that the remaining caterpillars of this type are not freely solvable.

(iv) Suppose $a_1 = a_2+m$ and $a_4 = a_3+k$, where $m \geq 3$ and $k \geq 1$. Using a similar argument as in the proof of Theorem 2, we can remove at most a_2+3 pegs from X_1 and at most a_3+1 pegs from X_4. As this method will result in a peg in x_2, at best the graph is $(m+k-3)$-solvable. Begin with the hole in x_2 and perform $\mathcal{DS}(X_1, X_2, a_2)$, $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$, $x_3{\cdot}\overrightarrow{x_2}{\cdot}x_1$, $\mathcal{DS}(X_4, X_3, a_3)$, $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$, $X_4{\cdot}\overrightarrow{x_4}{\cdot}x_3$, $x_3{\cdot}\overrightarrow{x_2}{\cdot}x_1$, and $X_1{\cdot}\overrightarrow{x_1}{\cdot}x_2$. This will leave $m-3$ pegs in X_1, $k-1$ pegs in X_4, and a peg in x_2. If $m = 3$ and $k = 1$, then the graph is solved. Otherwise, the graph is $(m+k-3)$-solvable. Using an exhaustive computer search [4], $P_4(3, 0, 0, 2)$ and $P_4(4, 0, 0, 1)$ were found to be distance 2-solvable. Any such caterpillar with $m+k = 5$ will reduce to either $P_4(3, 0, 0, 2)$ or $P_4(4, 0, 0, 1)$ via Theorem 3. Therefore, these caterpillars are also distance 2-solvable.

Suppose $a_1 = a_2 + m$ and $a_3 = a_4 + k$, where $m \geq 3$ and $k \geq 0$. Using a similar argument as that used in the proof of Theorem 2, we can remove at most a_2+3 pegs from X_1 and at most a_4-1 pegs from X_3. As this method will result in a peg in x_2, at best the graph is $(m+k-1)$-solvable. We solve the graph in the same method as in the previous case, however we replace $\mathcal{DS}(X_4, X_3, a_3)$ with $\mathcal{DS}(X_4, X_3, a_4 - 1)$. This will leave $m - 3$ pegs in X_1, $k + 1$ pegs in X_3, and a peg in x_2. Thus, the graph is $(m+k-1)$-solvable. In particular, if $m+k = 3$, then the graph is distance 2-solvable.

(II) *(i)* Suppose that $a_2 = a_1+m$ and $a_3 = a_4+k$, where $m \geq 0$ and $k \geq 0$. If $m \in \{k-1, k, k+1\}$, then we apply Theorem 3 with $k_1 = a_1-1$, $k_2 = \min(a_2-a_1, a_3-a_4)$, and $k_3 = a_4 - 1$ to reduce the graph to $P_4(1,1,1,1)$ or $P_4(1,2,1,1)$. An exhaustive computer search [4] confirms that $P_4(1,1,1,1)$ and $P_4(1,2,1,1)$ are freely solvable. Thus, the original graph is freely solvable.

Suppose that $a_2 = a_1 + m$ and $a_3 = a_4 + k$, where $m \geq k + 2$ and $k \geq 0$. By Theorem 2, we can remove at most $a_1 + a_3 + 1$ pegs from X_2. However, the pegs in X_3 must also be used to remove those in X_4. Hence, we can remove at most $a_1 + a_3 - a_4 + 2$ pegs from X_2, leaving one peg left on the spine. This can be achieved with the initial hole in x_1. Perform $\mathcal{DS}(X_2, X_1, a_1 - 1)$, $X_2 {\cdot} \overrightarrow{x_2} {\cdot} x_1$, $X_3 {\cdot} \overrightarrow{x_3} {\cdot} x_2$, $\mathcal{DS}(X_4, X_3, a_4-1)$, $\mathcal{DS}(X_2, X_3, a_3-a_4)$, $X_2 {\cdot} \overrightarrow{x_2} {\cdot} x_3$, $X_1 {\cdot} \overrightarrow{x_1} {\cdot} x_2$, $x_2 {\cdot} \overrightarrow{x_3} {\cdot} X_3$, $X_4 {\cdot} \overrightarrow{x_4} {\cdot} x_3$, $X_3 {\cdot} \overrightarrow{x_3} {\cdot} x_2$, and $X_2 {\cdot} \overrightarrow{x_2} {\cdot} x_3$. This leaves $m - k - 2 = a_2 - a_1 - a_3 + a_4 - 2$ pegs in X_2 and a peg in x_3. If $m = k + 2$, then the graph is solved. If $m \geq k + 3$, then the graph is $(m - k - 1)$-solvable. In particular, if $m = k + 3$, then the graph is distance 2-solvable. In the case where $m \leq k - 2$, we reverse the order of the parameters to show that the graph is $(k - m - 1)$-solvable. To show that this case is not freely solvable, note that we can commit at most k of the pegs from X_3 to remove those in X_2. Hence this case is not freely solvable for the same reasons as in Theorem 2.

(ii) Suppose that $a_2 = a_1 + m$ and $a_4 = a_3 + k$, where $m \geq 0$ and $k \geq 1$. Note that if $m = 0$, $k = 1$, and $a_3 \geq 1$, then the graph reduces to $P_4(2,1,1,1)$, which is freely solvable. Note that if $m \geq 1$, $k \geq 3$, or $a_3 = 0$, then the graph cannot be freely solvable by Theorem 2 and Lemma 1. If $m = 0$, $k = 2$, and $a_3 \geq 1$, then the graph is not freely solvable using a similar argument to Theorem 2. Hence we can assume that $m \geq 1$ for the remainder of the proof. Using a similar argument to above, we can remove at most a_1 pegs from X_2 using X_1 and at most a_3 pegs from X_2 and X_4 using those in X_3. We can remove at most two additional pegs from X_2 or X_4 using the vertices on the spine. As this will leave a peg on the spine, such a graph is at best $(m+k-1)$-solvable. To achieve this, we begin with the initial hole in x_1. Perform $\mathcal{DS}(X_2, X_1, a_1 - 1)$, $X_2 {\cdot} \overrightarrow{x_2} {\cdot} x_1$, $x_4 {\cdot} \overrightarrow{x_3} {\cdot} x_2$, $X_2 {\cdot} \overrightarrow{x_2} {\cdot} x_3$, $\mathcal{DS}(X_3, X_4, a_3)$, $X_1 {\cdot} \overrightarrow{x_1} {\cdot} x_2$, $x_2 {\cdot} \overrightarrow{x_3} {\cdot} x_4$, $X_4 {\cdot} \overrightarrow{x_4} {\cdot} x_3$. This leaves $m-1$ pegs in X_2, $k-1$ pegs in X_4, and a peg in x_3. Hence, if $m = k = 1$, then the graph is solved. Otherwise, the graph is $(m + k - 1)$-solvable. Note that if $m + k = 3$, then we can apply Theorem 3 to reduce the graph to either $P_4(1,2,0,2)$ or $P_4(1,3,0,1)$. An exhaustive computer search [4] confirms that both $P_4(1,2,0,2)$ and $P_4(1,3,0,1)$ are distance 2-solvable. Thus, the graph is distance 2-solvable when $m + k = 3$. □

4. Fool's Solitaire

So far we have dealt with solving the graph and leaving the minimum number of pegs. Now, we determine the maximum number of pegs we leave if we follow the stipulation that we jump whenever possible. This is the *fool's solitaire* problem. The *fool's solitaire number* of a graph G, denoted $Fs(G)$, is the cardinality of the largest terminal state associated with a starting state with only one hole. Note that the independence number of G gives a sharp upper bound on $Fs(G)$. For more information on the fool's solitaire problem, refer to [5, 13]. In this section, we find the fool's solitaire number for caterpillars of the form $P_4(a_1, a_2, a_3, a_4)$ and $P_n(a_1, 0, ..., 0, a_n)$.

Theorem 6. *Let $G = P_4(a_1, a_2, a_3, a_4)$.* (i) *If $a_2 \geq 1$ and $a_3 \geq 1$, then $Fs(G) = a_1 + a_2 + a_3 + a_4$.* (ii) *If $a_2 = 0$ or $a_3 = 0$, then $Fs(G) = a_1 + a_2 + a_3 + a_4 + 1$.*

Proof. (i) If $a_2 \geq 1$ and $a_3 \geq 1$, then the maximum independent set is $X_1 \cup X_2 \cup X_3 \cup X_4$. The dual of this configuration has pegs in x_1, x_2, x_3, and x_4 and holes elsewhere. We can solve this configuration by making the jumps $x_1 \cdot \overrightarrow{x_2} \cdot X_2$, $x_4 \cdot \overrightarrow{x_3} \cdot x_2$, and $X_2 \cdot \overrightarrow{x_2} \cdot x_1$. It follows from the Duality Principle that $Fs(P_4(a_1, a_2, a_3, a_4)) = a_1 + a_2 + a_3 + a_4$.

(ii) Without loss of generality, suppose that $a_2 = 0$. A maximum independent set is $X_1 \cup \{x_2\} \cup X_3 \cup X_4$. The dual of this configuration has pegs in x_1, x_3, and x_4 and holes elsewhere. We can solve this configuration with the jumps $x_4 \cdot \overrightarrow{x_3} \cdot x_2$ and $x_1 \cdot \overrightarrow{x_2} \cdot x_3$. It follows from the Duality Principle that $Fs(P_4(a_1, a_2, a_3, a_4)) = a_1 + a_2 + a_3 + a_4 + 1$. □

We now consider caterpillars with longer spine length.

Theorem 7. *Let $G = P_n(a_1, 0, ..., 0, 1)$, where $a_1 \geq 2$. The fool's solitaire number of G is $Fs(G) = a_1 + \lfloor n/2 \rfloor$.*

Proof. Suppose that n is even, say $n = 2t$. A maximum independent set is $X_1 \cup \{x_2, x_4, ...x_{2t-2}, x_{2t,1}\}$. The dual of this set is $\{x_1, x_3, ..., x_{2t-1}, x_{2t}\}$. This can be solved by jumping $x_{2t-i+1} \cdot \overrightarrow{x_{2t-i}} \cdot x_{2t-i-1}$ for $i = 1, ..., t-1$ followed by $x_1 \cdot \overrightarrow{x_2} \cdot x_3$. By the Duality Principle, $Fs(G) = a_1 + t$.

Suppose that n is odd, say $n = 2t + 1$. The maximum independent set is $X_1 \cup \{x_2, x_4, ..., x_{2t}, x_{2t+1,1}\}$. The associated dual is $\{x_1, x_3, , ..., x_{2t+1}\}$. This is not solvable. Thus, at least one peg must be added to the dual. We add $x_{1,1}$ to the dual. This configuration can be solved by jumping $x_{1,1} \cdot \overrightarrow{x_1} \cdot x_2$ and then $x_{2i} \cdot \overrightarrow{x_{2i+1}} \cdot x_{2i+2}$ for $i = 1, ..., t-1$. Finally, we jump $x_{2t+1} \cdot \overrightarrow{x_{2t}} \cdot x_{2t-1}$. Hence, by the Duality Principle, $Fs(G) = a_1 + t$ when $n = 2t + 1$. □

Theorem 8. *If $G = P_n(a_1, 0, ..., 0, a_n)$ and $a_1 \geq a_n \geq 2$, then $Fs(G) = a_1 + a_n + \lfloor n/2 \rfloor - 1$.*

Proof. Suppose that n is even, say $n = 2t$. The maximum independent set is $X_1 \cup X_{2t} \cup \{x_2, x_4, ..., x_{2t-2}\}$. The associated dual, $\{x_1, x_3, ..., x_{2t-1}, x_{2t}\}$, is solvable by the same argument as in the proof of Theorem 7. So $Fs(G) = a_1 + a_n + t - 1$.

Suppose that n is odd, say $n = 2t + 1$. The maximum independent set is $X_1 \cup X_{2t+1} \cup \{x_2, x_4, ..., x_{2t}\}$. The associated dual is $\{x_1, x_3, ..., x_{2t+1}\}$. This is not solvable. So we must add at least one peg to the dual. We choose to add $x_{1,1}$ to the dual. The resulting configuration is solvable by the same argument as the proof of Theorem 7. So $Fs(G) = a_1 + a_n + t - 1$. □

5. Open Problems

Naturally, an important problem remains the classification of solvable trees. Thus, a logical continuation of the work of this paper would be to finish the classification of all solvable caterpillars. While we were not able to accomplish this, we have reason to believe that the results (and techniques involved) will be similar to those presented in this paper. As evidence, `faculty.etsu.edu/beelerr/cat-catalog.pdf` lists the solvability of all caterpillars that have twelve vertices or less and a spine length of at least five.

Another possible avenue of research would be to determine necessary and sufficient conditions for the solvability of all trees of diameter five. Note that the caterpillars in Section 3 are a special case of this type of tree. This would also serve to extend the work in [7].

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