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ON BELLMAN SPHERES FOR LINEAR CONTROLLED OBJECTS OF SECOND ORDER

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Abstract. In this paper, a connection between Bellman's spheres for piecewise continuous and measurable controls is considered. The investigation is conducted for linear controlled objects of the second order with a rather complicated two–dimensional control region. A collection of examples is included to illustrate the obtained results.

1. Statements of classic results. Consider a linear controlled object

$$\dot{x} = Ax + Bu, \qquad u \in U, \tag{1}$$

where $x \in R^n$ is a contravariant vector (a *phase state*) with coordinates $x^1, ..., x^n$ and $u \in R^r$ is a contravariant vector (a *control*) with coordinates $u^1, ..., u^r$. The constant matrices $A = (a^i_j)$, $B = (b^i_k)$ (where $i, j = 1, ..., n$; $k = 1, ..., r$) define linear mappings $A : R^n \to R^n$, $B : R^r \to R^n$ by the formulas $(Ax)^i = a^i_j x^j$, $(Bu)^i = b^i_k u^k$ (here and in the sequel, summation over recurring indices is made). Finally, the *control region* $U \subset R^r$ is a compact, convex set containing the origin.

A measurable function $u(t)$, $t_0 \le t \le t_1$ with values in R^r is an *admissible control*, if $u(t) \in U$ for all $t \in [t_0, t_1]$.

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Let $u(t),\ t_0 \le t \le t_1$ be an admissible control and $x_0 \in R^n$. Then the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} (Ax(s) + Bu(s))\, ds$$

has an absolutely continuous solution $x(t),\ t_0 \le t \le t_1$ defined uniquelly. This solution complies with the condition $x(t_0) = x_0$ and its derivative $\dot{x}(t)$ (defined almost everywhere on $[t_0, t_1]$) satisfies the equation $\dot{x}(t) = Ax(t)+Bu(t)$. Therefore $x(t)$ is named the *phase trajectory* of the controlled object (1) corresponding to the control $u(t)$ and the initial point x_0. We say the control $u(t),\ t_0 \le t \le t_1$ *transits* the initial point x_0 to the origin if the aforesaid trajectory satisfies the terminal condition $x(t_1) = 0$.

The *linear problem in time-optimal control* requires to find an admissible control that transits a given initial point x_0 to the origin in the shortest time. The admissible control and the corresponding phase trajectory which solve the problem are said to be *time-optimal* (for brevity, *optimal*). In other words, a process $u(t), x(t),\ t_0 \le t \le t_1$ is optimal if $u(t)$ is an admissible control, $x(t)$ is the corresponding phase trajectory with $x_0 = x(t_0)$ and $x(t_1) = 0$, and in addition, the transit time is minimal, i.e., it is impossible to transit x_0 to the origin in a time lesser than $t_1 - t_0$.

Let $u(t), x(t),\ t_0 \le t \le t_1$ be an admissible process for the object (1) with boundary conditions $x(t_0) = x_0,\ x(t_1) = 0$. Then $\bar{u}(t) = u(t + t_0), \bar{x}(t) = x(t + t_0),\ 0 \le t \le t_1 - t_0$ also is a process with the same boundary conditions: $\bar{x}(0) = x_0,\ \bar{x}(t_1 - t_0) = 0$. Thus it is possible to limit ourselves by examination of processes with the initial moment $t_0 = 0$.

Bellman's sphere Σ_T^{mes} is the set of all initial points $x_0 \in R^n$ which can be transited to the origin in a time not greater than T. In other words, $x_0 \in \Sigma_T^{\mathrm{mes}}$ if there exists an admissible, measurable control $u(t),\ t_0 \le t \le t_1$ with $t_1 - t_0 \le T$ such that the corresponding phase trajectory $x(t)$ with the initial condition $x(t_0) = x_0$ satisfies the terminal condition $x(t_1) = 0$.

Similarly, Σ_T^{pwc} is the set of all initial points $x_0 \in R^n$ which can be transited to the origin in a time not greater than T by *piecewise continuous*, admissible controls.

We now formulate some well-known results (for proofs see [1, 2]).

Theorem 1. *For every $T > 0$, Bellman's sphere $\Sigma_T^{\mathrm{mes}} \subset R^n$ is a compact, convex set containing the origin. Besides, the inclusions*

$$\mathrm{ri}\ \Sigma_T^{\mathrm{mes}} \subset \bigcup_{t<T} \Sigma_t^{\mathrm{pwc}} \subset \Sigma_T^{\mathrm{pwc}} \subset \Sigma_T^{\mathrm{mes}}$$

hold, where ri *denotes the relative interior of a convex set.*

The first inclusion means every point $x_0 \in \mathrm{ri}\,\Sigma_T^{\mathrm{mes}}$ can be transited to the origin in a time *lesser* than T by an admissible, *piecewise continuous* control.

Theorem 2 (existence theorem). *If there is an admissible control transiting a given initial point $x_0 \in R^n$ to the origin, then there exists an optimal, measurable control that transits x_0 to the origin.*

Theorem 2 treats only *measurable*, optimal controls and Example 1 below illustrates this circumstance: for every initial point $x_0 \neq 0$, there is a *measurable* optimal control transiting x_0 to the origin, whereas there is no *piecewise continuous* optimal control. Generally, the aim of the paper is establishment of connection between Bellman's spheres Σ_T^{mes} and Σ_T^{pwc} for linear controlled objects of second order.

The linear controlled object (1) is *stable* if for every $\varepsilon > 0$, Bellman's sphere $\Sigma_\varepsilon^{\mathrm{mes}}$ contains the origin in its *interior*.

For the object (1), we consider the *conjugate equation*

$$\dot{\psi} = -\psi A. \tag{2}$$

Here $\psi = (\psi_1, ..., \psi_n)$ is an auxiliary *covariant* vector (i.e., a row vector). The equation (2) has coordinate form $\dot{\psi}_j = -\psi_i a^i_j$, $j = 1, ..., n$. We remark for any $u \in U$ and any covariant vector ψ, the scalar product $\langle \psi, Bu \rangle = \psi_i b^i_k u^k$ is defined.

Let $u(t)$, $t_0 \le t \le t_1$ be an admissible control and $\psi(t)$ be a solution of (2). We say $u(t)$ satisfies the *maximum condition* with respect to $\psi(t)$ if

$$\langle \psi(t),\, Bu(t) \rangle = \max_{u \in U} \langle \psi(t),\, Bu \rangle \quad \text{almost everywhere on} \quad [t_0,\, t_1]. \tag{3}$$

Theorem 3 (the maximum principle). *For stable linear controlled objects,* (3) *is a necessary and sufficient condition of optimality. More detailed, an admissible process $u(t)$, $x(t)$, $t_0 \le t \le t_1$ transiting a point x_0 to the origin is optimal if and only if $u(t)$ satisfies the maximum condition with respect to a nontrivial solution $\psi(t)$ of the conjugate equation.*

The auxiliary proposition stated below illuminates geometrical sense of the function $\psi(t)$ in the maximum principle.

Lemma 1. *Let $u(t)$, $t_0 \le t \le t_0 + T$ be an admissible, measurable control optimally transiting x_0 to the origin in the time T and $\psi(t)$ be a nontrivial solution of* (2). *The control $u(t)$ satisfies the maximum condition with respect to $\psi(t)$ if and only if $x_0 \in L \cap \Sigma_T^{\mathrm{mes}}$, where L is the supporting hyperplane of Σ_T^{mes} with outward normal $-\psi(t_0)$.*

In the following theorem, we consider controlled objects with *strictly convex* Bellman's spheres (i.e., $\Sigma_T^{\rm mes}$ has only one commom point with any its supporting hyperplane).

THEOREM 4 (UNIQUENESS THEOREM). *Let* (1) *be a stable linear controlled object with strictly convex Bellman's spheres. If* $x_0 \in \Sigma_T^{\rm mes}$, *then the corresponding optimal trajectory* (*along which* x_0 *can be transited to the origin*) *is defined uniquelly.*

We remark the inverse assertion for Theorem 4 is correct as well: *If for a linear object* (1), *Bellman's sphere* $\Sigma_T^{\rm mes}$ *is not strictly convex, then for a point* $x_0 \in {\rm bd}\, \Sigma_T^{\rm mes}$, *there are infinitely many optimal trajectories going from* x_0 *to the origin.* More exactly, for a point $x_0 \in {\rm bd}\ \Sigma_T^{\rm mes}$, an optimal trajectory along which x_0 can be transited to the origin is defined uniquely if and only if x_0 is an *extremal* boundary point of $\Sigma_T^{\rm mes}$, i.e., there are no points $y,\, z \in \Sigma_T^{\rm mes}$ distinct from x_0 such that $x_0 \in [y,\, z]$.

2. Preliminare examples. Here and in the sequel, we consider a linear controlled object

$$\dot{x} = Ax + u \qquad (4)$$

with $x \in R^2$ and $u \in U \subset R^2$. We assume the control region U is a two-dimensional, compact, convex set in R^2 containing the origin in its interior. Thus in comparison with (1), $B : R^2 \to R^2$ is the identity mapping.

For every two-dimensional, compact, convex set $M \subset R^2$ and every nonzero covariant vector ψ, we denote by $M_{(\psi)}$ the intersection $M \cap L_\psi$, where L_ψ is the supporting line of M with the outvard normal ψ. Thus for the object (4), an admissible control $u(t)$, $t_0 \le t \le t_1$ satisfies the maximum condition with respect to a nontrivial solution $\psi(t)$ of (2) if and only if $u(t) \in U_{(\psi(t))}$ almost everywhere on $[t_0,\, t_1]$. Also, the condition $x_0 \in L \cap \Sigma_T^{\rm mes}$ in Lemma 1 can be reformulated in the form $x_0 \in {\Sigma_T^{\rm mes}}_{(-\psi(t_0))}$.

Consider several examples which clarify connection between $\Sigma_T^{\rm mes}$ and $\Sigma_T^{\rm pwc}$.

EXAMPLE 1. Let $\Phi = \{\varphi_1, \varphi_2, ...\}$ be a countable set dense in the segment $[0,\, 2\pi]$. For every $k = 1, 2, ...$, denote by $U_k \subset R^2$ the segment that is centered at the origin, forms with x^1-axis the angle φ_k, and has the length $\frac{1}{2^k}$. By U designate the set of all points $u_1 + u_2 + ...$, where $u_k \in U_k$, $k = 1, 2, ...$ Then U is a compact, convex, centrally symmetric set in R^2. Intuitively, U is a "polygon" with infinitely many sides such that the directions of its sides form a set everywhere dense in $[0,\, 2\pi]$. If $\psi \ne 0$ is a covariant vector orthogonal to U_k for an index k, then $U_{(\psi)}$ is a segment of nonzero length.

Consider the linear controlled object

$$\dot{x}^1 = x^2 + u^1, \quad \dot{x}^2 = -x^1 + u^2,$$

where the control $u \in R^2$ with coordinates u^1, u^2 runs over the control region U described above. The conjugate system $\dot{\psi}_1 = \psi_2$, $\dot{\psi}_2 = -\psi_1$ has general solution

$$\psi_1(t) = a\sin(t+\theta), \quad \psi_2(t) = a\cos(t+\theta),$$

where a, θ are constant parameters. For any nontrivial solution $\psi(t) = (\psi_1(t), \psi_2(t))$, $t_0 \le t \le t_1$ of the conjugate system, the vector $\psi(t)$ rotates clockwise.

If at a moment τ the vector $\psi(\tau)$ is orthogonal to U_k for an index k, then $I = U_{(\psi(\tau))}$ is a segment of nonzero length. For $t < \tau$, the set $U_{(\psi(t))}$ is situated on one side of the segment I whereas for $t > \tau$ the set $U_{(\psi(\tau))}$ is situated on the other said of I. This means the control $u(t)$ satisfying (3) with respect to $\psi(t)$ is *discontinuous* at $t = \tau$. Hence the maximum condition defines a measurable control $u(t)$, $t_0 \le t \le t_1$ whose discontinuity points are situated *densely* in $[t_0, t_1]$. Consequently no optimal control is piecewise continuous. Thus Σ_T^{pwc} coincides with the *interior* of Σ_T^{mes}.

Example 2. Consider the linear controlled object

$$\dot{x}^1 = x^2 + u^1, \quad \dot{x}^2 = u^2, \qquad |u^1| \le 1,\ |u^2| \le 1. \tag{5}$$

This object is stable, since U (i.e., the square $|x^1| \le 1$, $|x^2| \le 1$) contains the origin $0 \in R^2$ in its interior. Hence the maximum principle gives a necessary and sufficient condition for optimality.

The maximum condition has the form

$$u^1 = \text{sign}\ \psi_1(t) \quad \text{as} \quad \psi_1 \ne 0; \qquad u^2 = \text{sign}\ \psi_2(t) \quad \text{as} \quad \psi_2 \ne 0,$$

where $\psi(t) = (\psi_1(t), \psi_2(t))$ is a solution of the conjugate system $\dot{\psi}_1 = 0$, $\dot{\psi}_2 = -\psi_1$. The general solution of this system has the form

$$\psi_1 = c_1, \quad \psi_2 = -c_1 t + c_2,$$

c_1, c_2 being constants. For $c_1 \ne 0$, denoting by θ the switching moment, i.e., $\theta = \frac{c_2}{c_1}$, we obtain all controls satisfying the maximum condition:

$$u^1 = 1, \quad u^2 = -\ \text{sign}\,(t-\theta) \quad \text{as} \quad c_1 > 0; \tag{6}$$

$$u^1 = -1, \quad u^2 = \text{sign}\,(t-\theta) \quad \text{as} \quad c_1 < 0; \tag{7}$$

$$-1 \le u^1 \le 1, \quad u^2 = \text{const} = \pm 1 \qquad \text{as} \quad c_1 = 0,\ c_2 \ne 0. \tag{8}$$

To obtain the synthesis of optimal trajectories, first find the trajectories corresponding to constant controls $u^1 = \pm 1,\ u^2 = \pm 1$. For $u^1 = 1,\ u^2 = 1$, we obtain from (5)

$$\frac{dx^1}{dx^2} = x^2 + 1.$$

This gives the phase trajectories

$$x^1 = \frac{1}{2}\left(x^2 + 1\right)^2 + \text{const} .$$

Similarly we find the trajectories for other constant values $u^1 = \pm 1,\ u^2 = \pm 1$.

Thus there are four parabolic arcs which correspond to constant controls $u^1 = \pm 1,\ u^2 = \pm 1$ and arrive to the origin. They divide R^2 into four "curved quadrants" I, II, III, IV, where I and III contain positive and negative x^1-semiaxes respectively. The optimal trajectories corresponding to the controls (6), (7) (and arriving to the origin) cover the "quadrants" I and III and for every point x_0 in I or III there is a *unique* optimal trajectory transiting x_0 to the origin. Every boundary point x_0 of Bellman's sphere $\Sigma_T^{\text{mes}} = \Sigma_T^{\text{pwc}}$ situated in "quadrant" I or III is an *extremal point* of this Bellman sphere. Hence there is a *unique* optimal trajectory along which x_0 can be transited to the origin.

We show the optimal trajectories corresponding to controls (8) cover the "quadrants" II, IV and this completes the optimal synthesis. First investigate the controls (8) for $u^2 \equiv +1$. Let $u^1(t),\ t_0 \le t \le t_1$ be a function with $-1 \le u^1(t) \le 1$. Denote by $x(t)$ the trajectory corresponding to the control $u^1 = u^1(t),\ u^2 \equiv 1$ and arriving to the origin at $t = t_1$. From (5) we obtain $\dot{x}^1 \le (x^2 + 1)\,\dot{x}^2$, since $u^1(t) \le 1,\ u^2(t) \equiv 1$. Integrating this inequality from t to t_1 and taking into account $x^1(t_1) = x^2(t_2) = 0$, we find

$$-x^1(t) \le -\frac{1}{2}\left(x^2(t)\right)^2 - x^2(t).$$

This means the trajectory $x(t)$ is situated *on the right* of the parabola $x^1 = \frac{1}{2}(x^2)^2 + x^2$, i.e., on the right of the common boundary of the "quadrants" III and IV.

Similarly (since $u^1(t) \ge -1$), the trajectory $x(t)$ is situated *on the left* of the parabola $x^1 = \frac{1}{2}(x^2)^2 - x^2$ i.e., of the common boundary of the "quadrants" IV and I. In other words, $x(t)$ is situated in "quadrant" IV. Analogously, for $u^2 \equiv -1$, the controls (8) define trajectories lying in the "quadrant" II.

If x_0 is situated in the boundary of the "quadrant" IV, there is a *unique* optimal trajectory along which x_0 can be transited to the origin (since x_0

belongs also to I or III). But for any *interior* point x_0 of the "quadrant" IV, there are *infinitely many* optimal trajectories from x_0 to the origin. Indeed, we can take any piecewise continuous function $u^1(t)$ with $-1 \le u^1(t) \le 1$ and move under action of the control $u^1 = u^1(t)$, $u^2 \equiv 1$ untill arrival to the boundary of the "quadrant" IV; after that the trajectory goes along the boundary under $u^1 =$const$= \pm 1$, $u^2 = 1$. These trajectories are optimal (since they satisfy the maximum principle) and the transit time is equal to $|x_0^2|$ (since $\dot{x}^2 = 1$). Thus the boundary of Σ_T^{mes} contains a segment that is the intersection of the "quadrant" IV and the line $x^2 = -T$. For the "quadrant" II the situation is similar.

Example 3. Consider the linear controlled object

$$\dot{x}^1 = -x^1 + u^1, \quad \dot{x}^2 = -2x^2 + u^2, \qquad u \in U,$$

where U is the control region as in Example 1. The conjugate system $\dot{\psi}_1 = \psi_1$, $\dot{\psi}_2 = 2\psi_2$ has general solution

$$\psi_1(t) = c_1 e^t, \quad \psi_2(t) = c_2 e^{2t}. \tag{9}$$

In particular (for $c_1 = 1$, $c_2 = 0$), we have the solution

$$\psi(t) = (e^t, 0). \tag{10}$$

Denote by e_1, e_2 the unit vectors of the x^1- , x^2-axes correspondingly and by f^1, f^2 the covariant vectors of dual basis, i.e., $\langle f^i, e_j\rangle = \delta_l^i$ (Kronecker delta). For (10), i.e., for $\psi(t) = e^t f^1$, the maximum condition takes the form $u(t) \in U_{(f^1)}$. Let $U_{(f^1)} = [a, b]$ (it is possible $[a, b]$ degenerates, i.e., $a = b$). For every point $p \in [a, b]$, denote by $x_p(t)$, $0 \le t \le T$ the phase trajectory corresponding to the control $u_p(t) \equiv p$ with the terminal condition $x_p(T) = 0$. Then $u_p(t)$ transits optimally the point $x_p(0)$ to the origin in the time T, since $u_p(t)$ satisfies the maximum condition with respect to (10). Hence by Lemma 1, $x_p(0) \in \Sigma_T^{\mathrm{mes}}{}_{(-f^1)}$. Thus the point $x_p(0)$ belongs to the segment $[x_a(0), x_b(0)]$ and this segment coincides with $\Sigma_T^{\mathrm{mes}}{}_{(-f^1)}$. Since every point of this segment can be optimally transited to the origin in the time T by a constant (hence piecewise continuous) control, the face $\Sigma_T^{\mathrm{mes}}{}_{(-f^1)}$ is contained in Σ_T^{pwc}.

Similarly, for $c_1 = -1$, $c_2 = 0$, we obtain from (9) the solution $\psi(t) = (-e^t, 0) = -e^t f^1$ and conclude the face $\Sigma_T^{\mathrm{mes}}{}_{(f^1)}$ of Bellman's sphere Σ_T^{mes} is contained in Σ_T^{pwc}.

Furthermore, taking $c_1 = 0$, $c_2 = 1$, we obtain $\psi(t) = (0, e^{2t}) = e^{2t} f^2$. For this case, the maximum condition is $u(t) \in U_{(f^2)}$. As abowe, the face $\Sigma_T^{\mathrm{mes}}{}_{(-f^2)}$ of Bellman's sphere Σ_T^{mes} is contained in Σ_T^{pwc}. Finally, for $c_1 = 0$, $c_2 = -1$, we conclude the face $\Sigma_T^{\mathrm{mes}}{}_{(f^2)}$ of Bellman's sphere Σ_T^{mes} is contained in Σ_T^{pwc}. Thus we have four faces of Σ_T^{mes} which are contained in Σ_T^{pwc}.

Other points of $\mathrm{bd}\,\Sigma_T^{\mathrm{mes}}$ *do not belong* to Σ_T^{pwc}. Indeed, any solution $\psi(t)$ distinct from considered ones has the form (9) with $c_1 \neq 0$, $c_2 \neq 0$. Hence the vector $\psi(t)$ *rotates* and (as in Example 1) for any admissible control $u(t)$, $t_0 \le t \le t_1$ satisfying the maximum condition with respect to $\psi(t)$, its discontinuity points are situated *densely* in $[t_0, t_1]$, i.e., $u(t)$ is not piecewise continuous.

Thus

$$\Sigma_T^{\mathrm{pwc}} \cap \mathrm{bd}\,\Sigma_T^{\mathrm{mes}} = \Sigma_T^{\mathrm{mes}} \cap \mathrm{bd}\,P,$$

where P is the circumscribed parallelogram for Bellman's sphere Σ_T^{mes} with the sides parallel to e_1, e_2.

3. Main results. Consider the oblect (4) for $n = 2$, where $U \subset R^2$ is a *two-dimensional*, convex, compact set containing the origin in its interior. Any *one-dimensional* face of U (if exists) has the form $L \cap \mathrm{bd}\,U$, where L is a supporting line of U, having with U more than one common point. The set of all one-dimensional faces of U is no more than countable. We say a contravariant vector $\psi \neq 0$ is *exceptional* if it is the outward normal for an one-dimensional face of U, i.e., $U_{(\psi)}$ is a segment of nonzero length.

Lemma 2. *Let $u(t)$, $t_0 \le t \le t_1$ be an admissible control satisfying the maximum condition with respect to a nontrivial solution $\psi(t)$ of* (2)*. Assume $\psi(t)$ does not have the same direction for all t, i.e.,* $\frac{\psi(t)}{|\psi(t)|} \neq \mathrm{const}$. *Under this condition, the control $u(t)$ is piecewise continuous if and only if there are only finitely many moments $t \in [t_0, t_1]$ at which $\psi(t)$ is an exceptional vector for U.*

Proof. The reasoning in Example 1 shows for $t_0 < t < t_1$, the moment t is a discontinuity point of $u(t)$ if and only if $U_{(\psi(t))}$ is a segment of nonzero length, i.e., $\psi(t)$ is an exceptional vector for U. Lemma 1 follows immediately from this assertion. □

Theorem 5. *Let* (4) *be a linear controlled object of the second order with a two-dimensional, compact, convex control region $U \subset R^2$ containing the origin in its interior. If U has only finite number of one-dimensional faces, then for every $T > 0$,*

$$\Sigma_T^{\mathrm{pwc}} = \Sigma_T^{\mathrm{mes}}.$$

Proof. Let $x_0 \in \Sigma_T^{\mathrm{mes}}$ and $u(t)$, $t_0 \le t \le t_1$ be an admissible, measurable, optimal control transiting x_0 to the origin. Then $u(t)$ satisfies (3) with respect to a nontrivial solution $\psi(t)$ of (2). If $\psi(t)$ has a constant direction, then for all t, the control $u(t)$ belongs to the same face $U_{(\bar\psi)}$ of U, where $\bar\psi = \frac{\psi(t)}{|\psi(t)|} = \mathrm{const}$. In this case, as in Example 3, there is a *constant*, optimal control $\bar u(t)$, $t_0 \le t \le t_1$ transiting x_0 to the origin, i.e., $x_0 \in \Sigma_T^{\mathrm{pwc}}$.

If however the direction of $\psi(t)$ in not constant, then by Lemma 2, $u(t)$ is piecewise continuous (since there are only finitely many exceptional vectors) and also $x_0 \in \Sigma_T^{\mathrm{pwc}}$. □

Foregoing Example 2 illustrates Theorem 5. In that Example, the control region $U \subset R^2$, i.e., the square $|x^1| \le 1$, $|x^2| \le 1$, has four one-dimensional faces. As we have seen, for every initial point $x_0 \in R^2$, there is a piecewise continuous, admissible control transiting x_0 to the origin, i.e., $\Sigma_T^{\mathrm{pwc}} = \Sigma_T^{\mathrm{mes}}$.

In the sequel, we assume U has infinitely many one-dimensional faces. First we presuppose the eigenvalues of the matrix A are complex. Then $\psi(t)$ rotates in a fixed direction (counterclockwise or clockwise, depending on A) and for any nontrivial solutions $\psi^{(1)}(t)$, $\psi^{(2)}(t)$ of (2), the relation $\psi^{(2)}(t) = c\psi^{(1)}(t+\alpha)$ holds, c, α being real constants. This implies there is a mimimal positive period τ for directions of $\psi(t)$, i.e., $\psi(t+\tau) = q\,\psi(t)$, $q > 0$ for any solution $\psi(t)$ of (2) and any t.

Let $\psi(t)$ be a nontrivial solution of (2). Choose a convergent sequence $t_1, t_2, \ldots$ such that for any k the vector $\psi(t_k)$ is exceptional and t_k is distinct from $t' = \lim_{k\to\infty} t_k$. We put $\Psi(t) = \psi(t - t')$. Then $\Psi(t)$ is a nontrivial solution of (2) and for every interval I containing 0 in its interior, there is a moment $t \in I$ such that $t \ne 0$ and $\Psi(t)$ is an exceptional vector. We shall consider $\Psi(t)$ on the segment $[0, \tau]$.

Let now $u(t)$, $t_0 \le t \le t_1$ be a piecewise continuous, optimal control transiting an initial point x_0 to the origin. By a translation $t \to t + \mathrm{const}$, we can replace it by a piecewise continuous control $\bar u(t)$, $\bar t_0 \le t \le \bar t_1$ which transits x_0 to the origin in the same time and satisfies the maximum condition with respect to $\Psi(t)$. Moreover, since the directions of the vectors ψ are periodic with the period τ, we can assume $0 < \bar t_1 \le \tau$. Hence $\bar t_0 \ge 0$ (since by Lemma 2, the segment $[\bar t_0, \bar t_1]$ cannot contain 0 in its interior).

Thus we can limit ourselves by consideration of piecewise continuous, optimal controls $u(t)$, $t_0 \le t \le t_1$ with $[t_0, t_1] \subset [0, \tau]$. By Lemma 2, there are only finitely many moments $t \in [t_0, t_1]$ for which the vector $\Psi(t)$ is exceptional.

This leads us to the following definition. Let $V \subset [0, \tau]$ be an interval, a half-open interval or a segment. We say V is a *pwc*-interval if for any segment $[t_0,\, t_1] \subset V$ there are only finitely many moments $t \in [t_0,\, t_1]$ for which $\Psi(t)$ is exceptional. Later on, we consider maximal *pwc*-intervals (not contained in another one). Maximal *pwc*-intervals are pairwise disjoint and the set of all maximal *pwc*-intervals is no more than countable. Thus later on, any piecewise continuous, optimal control is defined on a segment contained in a maximal *pwc*-interval and satisfies the maximum condition with respect to $\Psi(t)$.

Let V be a maximal *pwc*-interval with endpoints $a,\, b,\; a < b$ (i.e., either $V = [a,\, b]$ or V is obtained from $[a,\, b]$ by removal of one or both endpoints). Then we put $\theta(V) = b - a$. Evidently, for every $T > 0$, there are only finitely many maximal *pwc*-intervals with $\theta(V) > T$.

Fix now a positive number T and denote by s the translation $t \to t - T$. Let V be a maximal *pwc*-interval. We put

$$\sigma_T(V) = \bigcup_{t \in V \cap s(V)} {\Sigma_T^{\mathrm{mes}}}_{(-\Psi(t))}\,. \tag{11}$$

Since the eigenvalues of A are complex, every Bellman's sphere Σ_T^{mes} is *strictly convex*. This means every set ${\Sigma_T^{\mathrm{mes}}}_{(-\Psi(t))}$ consists of only one point and for any maximal *pwc*-interval V, the set $\sigma_T(V) \subset \mathrm{bd}\; \Sigma_T^{\mathrm{mes}}$ is either an arc (containing or not its endpoints) or a point.

THEOREM 6. *Let* (4) *be a linear controlled object of the second order with complex eigenvalues of the matrix* A. *We assume the control region* $U \subset R^2$ *is a two-dimansional, compact, convex set containing the origin in its interior. For every* $T > 0$, *the relation*

$$\Sigma_T^{\mathrm{pwc}} = (\mathrm{int}\; \Sigma_T^{\mathrm{mes}}) \cup \left(\bigcup_V \sigma_T(V) \right) \tag{12}$$

holds, i.e., Σ_T^{pwc} *is the union of the interior of Bellman's sphere* Σ_T^{mes} *and finitely many sets* $\sigma_T(V)$ *each of which is either an arc* (*containing or not its endpoints*) *or a point.*

Proof. Let x_0 be a boundary point of Σ_T^{mes} belonging to Σ_T^{pwc}. Then there exists an admissible, piecewise continuous, optimal control $u(t)$, $t_0 \le t \le t_0 + T$ transiting x_0 to the origin. We can assume $u(t)$ satisfies the maximum condition with respect to $\Psi(t)$ and $[t_0,\, t_0 + T]$ is contained in a maximal *pwc*-interval V. Since $t_0,\, t_0 + T \in V$, the relation $t_0 \in V \cap s(V)$ holds. Moreover, by Lemma 1, $x_0 = {\Sigma_T^{\mathrm{mes}}}_{(-\Psi(t_0))}$, i.e., $x_0 \in \sigma_T(V)$. This means $\Sigma_T^{\mathrm{pwc}} \subset S$, where S is the right-hand side of (12).

To establish the inverse inclusion, it is sufficient to prove $\sigma_T(V) \subset \Sigma_T^{\mathrm{pwc}}$ for every maximal *pwc*-interval V. Let $x_0 \in \sigma_T(V)$, i.e., $x_0 = {\Sigma_T^{\mathrm{mes}}}_{(-\Psi(t_0))}$, where $t_0 \in V \cap s(V)$ (see (11)). Then $t_0,\, t_0 + T \in V$, i.e., $[t_0,\, t_0 + T] \subset V$. Hence there are only finitely many moments $t \in [t_0,\, t_0 + T]$ for which $\Psi(t)$ is exceptional. By Lemma 2, the admissible control $u(t),\, t_0 \le t \le t_0 + T$ satisfying (3) with respect to $\Psi(t)$, is piecewise continuous. Denote by $x(t),\, t_0 \le t \le t_0 + T$ the phase trajectory corresponding to this control with terminal condition $x(t_0 + T) = 0$. Then $u(t)$ optimally transits $x(t_0)$ to the origin in the time T. By lemma 1, $x(t_0) = \Sigma^{\mathrm{mes}}_{T\;(-\Psi(t_0))}$, i.e., $x(t_0)$ *coincides* with x_0. Thus x_0 can be transited to the origin in the time T by a *piecewise continuous* control, i.e., $x_0 \in \Sigma^{\mathrm{pwc}}_T$. This proves $S \subset \Sigma^{\mathrm{pwc}}_T$. $\square$

Consequence. *Under the conditions of Theorem* 6 *for* $T > 0$ *large enough, the intersection* $\Sigma^{\mathrm{pwc}}_T \cap \mathrm{bd}\, \Sigma^{\mathrm{mes}}_T$ *is empty, i.e.,* Σ^{pwc}_T *coincides with the interior of* Σ^{mes}_T.

Indeed, any maximal *pwc*-interval V is contained in $[0,\, \tau]$ and hence $\theta(V) \le \tau$. Consequently for $T > \tau$ every set $\sigma_T(V)$ is empty (since $V \cap s(V) = \emptyset$).

Example 1 considered abowe illustrates Theorem 6. In that Example, there is none *pws*-interval. Indeed, for any solution $\Psi(t)$ of (2) and any segment $[t_0,\, t_1]$, there are infinitely many moments $t \in [t_0,\, t_1]$ at which $\Psi(t)$ is exceptional. Consequently the right-hand side of (12) coincides with $\mathrm{int}\, \Sigma^{\mathrm{mes}}_T$, i.e., $\Sigma^{\mathrm{pwc}}_T = \mathrm{int}\, \Sigma^{\mathrm{mes}}_T$. And by an exchange of the set Φ in Example 1, it is possible to obtain linear controlled objects for which there are finitely many maximal *pwc*-intervals or infiniteli many ones placed arbitrarily. More detailed, let $\Delta = \{V_1,\, V_2,\, \ldots\}$ be a family of pairwise disjoint sets contained in $[0,\, 2\pi]$ such that (i) every V_i is either an interval, or a half-open interval, or a segment; (ii) at least one of the points $0,\, 2\pi$ is not contained in any V_i; (iii) if distinct sets $V_i,\, V_j$ have a common endpoint a, then $a \notin V_i$, $a \notin V_j$. Under these conditions, there exists a two-dimensional, compact, convex set $U \subset R^2$ containing the origin in its interior such that for the linear controlled object $\dot{x}^1 = x^2 + u^1,\, \dot{x}^2 = -x^1 + u^2$ with the control region U, the family of all maximal *pwc*-intervals coincides with Δ (and similarly for any linear controlled object (4) of second order with complex eigenvalues of A).

We now assume A has real eigenvalues $\lambda_1 \ne \lambda_2$. Let $e_1,\, e_2$ be relevant eigenvectors: $Ae_1 = \lambda_1 e_1,\, Ae_2 = \lambda_2 e_2$ and $f^1,\, f^2$ be the dual basis for $e_1,\, e_2$, i.e., $\langle f^i,\, e_j\rangle = \delta^i_j$.

The *first* ψ*-quadrant* is the set of all covariant vectors ψ satisfying the inequalities $\langle \psi,\, e_1\rangle > 0$, $\langle \psi,\, e_2\rangle > 0$. The *second* ψ*-quadrant* consists of all covariant vectors ψ with $\langle \psi,\, e_1\rangle < 0$, $\langle \psi,\, e_2\rangle > 0$. Finally, the *third* one is given by $\langle\psi,\, e_1\rangle < 0,\ \langle\psi,\, e_2\rangle < 0$ and the *fourth* one is defined by $\langle\psi,\, e_1\rangle > 0,\ \langle\psi,\, e_2\rangle < 0$.

For any solution $x(t)$ of the homogeneous equation $\dot{x} = Ax$ and any solution $\psi(t)$ of (2), we have $\langle\psi(t),\, x(t)\rangle = \text{const}$. Consequently (since $x(t) = e_i e^{\lambda_i t},\ i = 1, 2$ satisfies $\dot{x} = Ax$) for any solution $\psi(t)$ of (2), the scalar product $\langle\psi(t),\, e_i\rangle$ preserves its sign. Hence if $\psi(0)$ is situated in i-th ψ-quadrant, then $\psi(t)$ belongs to i-th ψ-quadrant for $-\infty < t < \infty$ and $\psi(t)$ rotates in a fixed direction (counterclockwise or clockwise, depending on parity of i). And for any $\psi^{(1)}(t),\ \psi^{(2)}(t)$ situated in i-th ψ-quadrant, the relation $\psi^{(2)}(t) = c\psi^{(1)}(t+\alpha),\ \infty < t < \infty$ holds, $c,\, \alpha$ being real constants.

For every $i = 1, 2, 3, 4,$ we fix a solution $\Psi^{(i)}(t),\ -\infty < t < \infty$ of (2) situated in i-th ψ-quadrant. Let V be an interval, a half-open interval or a segment. We say V is a *pwc*-interval with respect to $\Psi^{(i)}(t)$ if for any $[t_0,\, t_1] \subset V$ there are only finitely many moments $t \in [t_0,\, t_1]$ for which $\Psi^{(i)}(t)$ is exceptional. Later on, we consider maximal *pwc*-intervals. For every $i = 1, 2, 3, 4,$ the maximal *pwc*-intervals with respect to $\Psi^{(i)}(t)$ are pairwise disjoint and the set of all maximal *pwc*-intervals is finite or countable.

Let V be a maximal *pwc*-interval with respect to $\Psi^{(i)}(t)$ and $a < b$ be its endpoints. Then we put $\theta(V) = b - a$. Unlike the case of complex eigenvalues, now it is possible for any $T > 0$, there are *infinitely* many maximal *pwc*-intervals with $\theta(V) > T$ (since the maximal *pwc*-intervals are situated in the line $-\infty < t < \infty$ instead of $[0,\, \tau]$.

Let now $u(t),\ t_0 \le t \le t_1$ be a piecewise continuous, optimal control transiting a point x_0 to the origin and $\psi(t)$ be a solution of (2) such that $u(t)$ satisfies the maximum condition with respect to $\psi(t)$. If $\psi(t)$ does not have a constant direction (i.e., $\psi(t)$ is situated in i-th ψ-quadrant for an i), then with a translation $t \to t + \text{const}$, we can replace $u(t)$ by a control $\bar{u}(t),\ \bar{t}_0 \le t \le \bar{t}_1$ which transits x_0 to the origin in the same time and satisfies the maximum condition with respect to $\Psi^{(i)}(t)$. By Lemma 2, there are only finitely many moments $t \in [\bar{t}_0,\, \bar{t}_1]$ for which $\Psi^{(i)}(t)$ is exceptional, i.e., $[\bar{t}_0,\, \bar{t}_1]$ is contained in a maximal *pwc*-interval with respect to $\Psi^{(i)}(t)$. Thus for piecewise continuous, optimal controls, we can limit ourselves by two cases: (i) the control satisfies the maximum condition with respect to a solution $\psi(t)$ with constant direction (i.e., either $\langle\psi(t),\, e_1\rangle \equiv 0$ or $\langle\psi(t),\, e_2\rangle \equiv 0$), (ii) for an $i = 1, 2, 3, 4,$ the control is defined on a segment contained in a maximal *pwc*-interval and satisfies the maximum condition with respect to $\Psi^{(i)}(t)$.

Fix now a positive number T and denote by s the translation $t \to t - T$. Let V be a maximal *pwc*-interval with respect to $\Psi^{(i)}(t)$. We put

$$\sigma_T(V) = \bigcup_{t \in V \cap s(V)} \Sigma_T^{\text{mes}}{}_{(-\Psi^{(i)}(t))} \,.$$

We remark for any $i = 1, 2, 3, 4$ and any t, the supporting line of Bellman's sphere Σ_T^{mes} with the outward normal $-\Psi^{(i)}(t)$ has with this Bellman's sphere only one common point. This means every set $\Sigma_T^{\text{mes}}{}_{(-\Psi^{(i)}(t))}$ consists of only one point. Furthermore, if V is a maximal *pwc*-interval with respect to $\Psi^{(i)}(t)$, then the set $\sigma_T(V) \subset \text{bd}\ \Sigma_T^{\text{mes}}$ is either an arc (containing or not its endpoints) or a point.

Finally, let P be the circumscribed parallelogram for Σ_T^{mes} with sides parallel to e_1, e_2, i.e., $P \supset \Sigma_T^{\text{mes}}$ and the sides of P are contained in the supporting lines of Σ_T^{mes} with outward normals $\pm f^1$, $\pm f^2$. Every side of P has with Σ_T^{mes} a common segment (maybe degenerating into one point) and $\Sigma_T^{\text{mes}} \cap \text{bd}\ P$ is the union of these four segments.

THEOREM 7. *Let* (4) *be a linear controlled object of the second order with real, distinct eigenvalues of the matrix A. We assume the control region $U \subset R^2$ is a two-dimansional, compact, convex set containing the origin in its interior. For every $T > 0$,*

$$\Sigma_T^{\text{pwc}} = (\text{int}\ \Sigma_T^{\text{mes}}) \cup \left(\bigcup_{i=1,2,3,4} \left(\bigcup_V^{(i)} \sigma_T(V) \right) \right) \cup (\Sigma_T^{\text{mes}} \cap \text{bd}\ P) \,, \qquad (13)$$

where $\bigcup^{(i)}$ means the union over all maximal pws-intervals with respect to $\Psi^{(i)}(t)$. In other words, Σ_T^{pwc} is the union of the interior of Bellman's sphere Σ_T^{mes} and no more than countable family of sets situated in bd Σ_T^{mes} *each of which is either an arc* (*containing or not its endpoints*) *or a point.*

Proof. Let x_0 be a boundary point of Σ_T^{mes} belonging to Σ_T^{pwc}. Then there exists an admissible, piecewise continuous, optimal control $u(t)$, $t_0 \le t \le t_0 + T$ transiting x_0 to the origin. We can assume without loss of generality that one of the following cases is realized: (i) the control $u(t)$ satisfies the maximum condition with respect to $\Psi^{(i)}(t)$ for an index $i = 1, 2, 3, 4$; (ii) the control $u(t)$ satisfies the maximum condition with respect to a solution $\psi(t)$ of (2) that has a constant direction.

In the case (i), the inclusion $[t_0, t_0 + T] \subset V$ holds, where V is a maximal *pwc*-interval with respect to $\Psi^{(i)}(t)$. Since $t_0, t_0 + T \in V$, the relation $t_0 \in V \cap s(V)$ holds. Moreover, by Lemma 1, $x_0 = \Sigma_T^{\text{mes}}{}_{(-\Psi^{(i)}(t_0))}$.

Consequently $x_0 \in \sigma_T(V)$. This means $x_0 \in S$, where S is the right-hand side of (13).

In the case (ii), the control $u(t)$ satisfies the maximum condition with respect to $\psi(t)$, where either $\psi(t) = \pm f^1 e^{-\lambda_1 t}$ or $\psi(t) = \pm f^2 e^{-\lambda_2 t}$. Let $\psi(t) = f^1 e^{-\lambda_1 t}$ (other cases are similar). Then $u(t) \in U_{(f^1)}$ for all t. As in Example 3, $x_0 \in \Sigma_T^{\text{mes}}{}_{(-f^1)}$. Hence $x_0 \in \Sigma_T^{\text{mes}} \cap \text{bd}\ P \subset S$. Thus in both the cases, $x_0 \in S$, i.e., $\Sigma_T^{\text{pwc}} \subset S$.

To establish the inverse inclusion, it is sufficient to prove the inclusions

$$\sigma_T(V) \subset \Sigma_T^{\text{pwc}} \quad \text{for every maximal } pwc\text{-interval } V \text{ with respect to } \Psi^{(i)}(t),$$

$$\Sigma_T^{\text{mes}} \cap \text{bd}\ P \subset \Sigma_T^{\text{pwc}}.$$

The first one can be established as in the end of the proof of Theorem 6 and the second one as in Example 3. □

Theorem 7 can be illustrated by Example 3 abowe. In that Example, there is none *pwc*-interval in any i-th ψ-quadrant. Hence the relation (13) takes the form $\Sigma_T^{\text{pwc}} = (\text{int}\ \Sigma_T^{\text{mes}}) \cup (\Sigma_T^{\text{mes}} \cap \text{bd}\ P)$. The intersection $\Sigma_T^{\text{mes}} \cap$ bd P contains four faces of Σ_T^{mes} situated in the supporting lines of Σ_T^{mes} parallel to x^1- and x^2-axes. For every point $x_0 \in \Sigma_T^{\text{mes}} \cap \text{bd}\ P$ there exists a *constant* optimal control transiting x_0 to the origin.

Remark. For every $T > 0$, the set $\Sigma_T^{\text{mes}} \cap \text{bd}\ P \subset \Sigma_T^{\text{pwc}}$ is in Theorem 7 nonempty, i.e., Σ_T^{pwc} does not coincide with int Σ_T^{mes}. Moreover, by an exchange of the set Φ (cf. Example 1), it is possible to construct a linear controlled objects like one in Example 3 such that there are *infinitely many* nonempty arcs $\sigma_T(V)$ for every $T > 0$. This distinguishes the case of real eigenvalues from the case of complex ones.

We remark in addition, if A has coinciding eigenvalues with only one Jordan's box of second order, then Theorem 7 (with obvious modification) holds. And if A has coinciding eigenvalues with two one-dimensional Jordan's boxes, then (for any $T > 0$) the relation $\Sigma_T^{\text{pwc}} = \Sigma_T^{\text{mes}}$ holds. The same relation holds (for any matrix A) if $U \subset R^2$ is a segment containing the origin in its relative interior.

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