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AN ADAPTIVE PARALLEL PROJECTION METHOD FOR SOLVING CONVEX FEASIBILITY PROBLEMS

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Abstract. We present an adaptation of the (by now classical) parallel projection method for finding a point in the nonempty intersection of a finite number of closed convex sets in a Hilbert space. The adaptation consists of controlling at each iteration step whether or not some condition is fulfilled; if not, the adapted next iteration point is determined such that its position with respect to the intersection is better than the usual next iteration point. This may improve the speed of convergence.

1. Introduction

Many problems in applied mathematics may be reduced to the following standard form: given a finite number of closed convex sets $\{C_t\}_{t=1}^m$ with nonempty intersection $C^* \equiv \cap_{t=1}^m C_t$ in a real Hilbert space H with inner product $\langle\ ,\ \rangle$ and norm $||\quad||$ derived from $\langle\ ,\ \rangle$, find in an iterative manner a point in C^*. For an overview of such problems in fields as diverse as control theory, image processing, statistics and solving systems of linear inequalities we refer to [6], [12], [13], [18]. Mainly theoretical results may be found in

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[3], [14], [15] and [17]. For generalized methods and problems in related areas we refer to [4], [5] and [10]. An extensive bibliography is given in [1].

The possibility of doing the iteration by using a parallel computer with r processors ($1 \le r \le m$) focuses the interest of the researchers on constructing parallel algorithms where at each step in the iteration r sets among the m sets $\{C_t\}_{t=1}^m$ are involved. Denoting for short the set $\{1, 2, \ldots, m\}$ by J, and by J_{k+1} the subset of J consisting of the r indexes $\{i_1, \ldots, i_r\}$ that determine the sets $C_{i_1}, \ldots, C_{i_r}$ that are used at step $k+1$ in the iteration, the typical parallel iteration step used to solve the problem starting from some point x_o in H has the following form ([2], [7], [9]):

$$x_{k+1} = x_k + \lambda_{k+1} \sum_{j \in J_{k+1}} \mu_{k+1}(j)(P_j x_k - x_k), \tag{1}$$

with a positive relaxation coefficient λ_{k+1}, a set of r nonnegative weights $\mu_{k+1}(j)$ (i.e., $\mu_{k+1}(j) \ge 0$ for each $j \in J_{k+1}$, and $\sum_{j \in J_{k+1}} \mu_{k+1}(j) = 1$), and $P_j x_k$ denoting the shortest distance projection of the current iterate x_k onto C_j. The weights and the relaxation coefficient may vary at each step. Conditions on the choice of $\lambda_{k+1}, \mu_{k+1}(j)$ and J_{k+1} for each $k = 0, 1, \ldots$ should then be found to assure the (usually weak) convergence of the sequence $\{x_k\}_{k=0}^{+\infty}$ to a point in C^*.

In this paper we present an adaptation of the iteration scheme given by (1), in order to improve possibly the speed of convergence. The reason to do this is the following. In the construction of the sequence $\{x_k\}_{k=0}^{+\infty}$ that we present we know that, when x_k has been found, the set C^* is part of a half-space Q_k with border line S_k through the point x_k; however, when the next iteration point x_{k+1} is determined from x_k as in (1), there is no guarantee that x_{k+1} also belongs to that half-space; indeed, the position of x_{k+1} in (1) is also determined by the sets C_j for all j in J_{k+1}, and a control in advance of all r-tuples of sets $\{C_t\}_{t=1}^m$ seems not to be possible.

The adaptation that we present goes as follows. Instead of determining the next iteration point directly by (1), we use the right hand side of (1) to determine, for $k \ge 1$, an intermediate point v_{k+1}, given by

$$v_{k+1} = x_k + \lambda_{k+1} \sum_{j \in J_{k+1}} \mu_{k+1}(j)(P_j x_k - x_k) \tag{2}$$

(we put $v_1 \equiv x_1$); v_{k+1} belongs to the border line S_{k+1} of a new half-space Q_{k+1} containing C^*. Using an easy criterion, we check whether or not v_{k+1} also is an element of Q_k, i.e., whether or not v_{k+1} is "on the good side" of x_k. If it is, we put $x_{k+1} \equiv v_{k+1}$; if not, we use the points v_{k+1}, v_k, x_k and x_{k-1} to determine as next iteration point x_{k+1} a point in the intersection of S_k and S_{k+1}. As such, the more involved calculations connected with the adaptation procedure to determine $x_{k+1} \neq v_{k+1}$ generally have to be carried out only at a limited number of steps during the iteration, and so the time needed for it may be kept small.

Besides the above-mentioned general description of determining the elements of the iteration sequence $\{x_k\}_{k=0}^{+\infty}$, we also have to clarify how the corresponding subsets $J_{k+1} \subset J$ and the corresponding weights $\{\mu_{k+1}(j)\}_{J \in J_{k+1}}$ will be choosen. We may state this control strategy under the following form:

(Control 1). *There exists some positive integer M such that, for each $n \in Z^+$,*

$$J \subset \cup_{k=0}^{M-1} J_{n+k},$$

i.e., all the sets $\{C_t\}_{t=1}^m$ are selected at least once within any M consecutive iterations.

(Control 2). *For some positive number δ, $0 < \delta < 1/r$, when at step $k+1$ ($k = 0, 1, \dots$) the subset $J_{k+1} \subset J$ is involved, and hence also the sets $\{C_j\}_{j \in J_{k+1}}$ and the weights $\{\mu_{k+1}(j)\}_{j \in J_{k+1}}$, take $\mu_{k+1}(j) \geq \delta$ for each $j \in J_{k+1}$; only when $x_k \in C_j$ we may take either $\mu_{k+1}(j) = 0$ or $\mu_{k+1}(j) \neq 0$.*

At this point we have introduced the necessary elements to state the main result of our paper. In order to separate it from the more technical matters involved with the convergence procedure, we state it here as Theorem 1.

Theorem 1. *Suppose that in a real Hilbert space $(H, \langle\ ,\ \rangle, ||\quad ||)$ a finite number $\{C_t\}_{t=1}^m$ of closed convex sets are given, with $C^* \equiv \cap_{t=1}^m C_t \neq \emptyset$. Starting from some point x_o in H, construct a sequence $\{x_k\}_{k=0}^{+\infty}$ in the following manner : when x_k has been found, put*

$$v_{k+1} = x_k + \lambda_{k+1} \sum_{j \in J_{k+1}} \mu_{k+1}(j)(P_j x_k - x_k),$$

where J_{k+1} and $\mu_{k+1}(j)$ fulfill the conditions mentioned in (Control 1) and (Control 2), and with λ_{k+1} given by

$$\lambda_{k+1} = \frac{\sum_{j \in J_{k+1}} \mu_{k+1}(j)||x_k - P_j x_k||^2}{||x_k - \sum_{j \in J_{k+1}} \mu_{k+1}(j) P_j x_k||^2}.$$

Put $x_1 \equiv v_1$. For $k = 1, 2, \dots$ compute $\alpha_k \equiv \langle x_{k-1} - v_k, v_{k+1} - v_k \rangle$. When $\alpha_k \leq 0$, take $x_{k+1} \equiv v_{k+1}$. When $\alpha_k > 0$, compute

$$\beta_k \equiv ||v_k - x_{k-1}||^2,$$
$$\gamma_k \equiv ||v_{k+1} - x_k||^2,$$
$$\zeta_k \equiv \beta_k \gamma_k - \alpha_k^2,$$

and put

$$x_{k+1} = v_{k+1} + \frac{\alpha_k \gamma_k}{\zeta_k}(v_k - x_{k-1}) + \frac{\alpha_k^2}{\zeta_k}(v_{k+1} - x_k).$$

Then the sequence $\{x_k\}_{k=0}^{+\infty}$ is weakly convergent to a point in C^*.

The technical development of the procedure itself is going on in a Hilbert space $(\mathcal{H}, \langle\langle\ ,\ \rangle\rangle, |||\ \ |||)$ associated to $(H, \langle\ ,\ \rangle, ||\ \ ||)$. The association of a new Hilbert space $\mathcal{H}$ to the given Hilbert space H, leading to a transformation of a parallel iteration method for finding a point in $C^* \subset H$ to a semi-sequential iteration method in $\mathcal{H}$ goes back to Pierra [16] for fixed weights and for $r = m$, and has been extended for variable weights in [11] and [8] (again, only for the case $r \equiv m$ or, what is the same, for $J_{k+1} \equiv J$ for all k).

The paper is organized as follows. In Section 2 we describe the construction of the associated Hilbert space $\mathcal{H}$ and the corresponding subsets in it; this description is an adaptation of the one appearing in [8], but is given here in its adapted form for the sake of readers' convenience. In Section 3 we construct the transformed sequence $\{\widehat{x_k}\}_{k=0}^{+\infty}$ in $\mathcal{H}$ of the wanted sequence $\{x_k\}_{k=0}^{+\infty}$ in H, using geometrical conditions for the determination of the relaxation parameter and for the position of the current iteration point $\widehat{x_k}$. Finally, in Section 4 we show weak convergence of the sequence $\{x_k\}_{k=0}^{+\infty}$ in H to a point of C^*.

2. The associated Hilbert space $\mathcal{H}$

Let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$ and norm $||\ \ ||$ derived from $\langle\ ,\ \rangle$. Suppose that in H m closed convex subsets $\{C_t\}_{t=1}^{m}$ are given with nonempty intersection $C^* \equiv \cap_{t=1}^{m} C_t$; let $J = \{1, 2, \dots, m\}$. In order to find a point in C^* in an iterative manner, use will be made of a parallel computer with r processors, with $r \leq m$. Assume that, during the iteration process, we obtain at step k ($k = 0, 1, \dots$) a point x_k (starting from some given point x_0), and that to obtain the next iterate x_{k+1} we need an "intermediate" point v_{k+1} by using r sets among the m sets $\{C_t\}_{t=1}^{m}$. Denoting for short by J_{k+1} ($k = 0, 1, \dots$) the subset of J consisting of the r indices determining the r sets $(C_j)_{j \in J_{k+1}}$ used to obtain v_{k+1} from x_k, and denoting by $P_j x_k$ the shortest distance projection of x_k onto the corresponding set C_j for $j \in J_{k+1}$, assume that v_{k+1} is given by

$$v_{k+1} = x_k + \lambda_{k+1} \sum_{j \in J_{k+1}} \mu_{k+1}(j)(P_j x_k - x_k), \tag{3}$$

with λ_{k+1} a (variable) positive relaxation coefficient and with $\{\mu_{k+1}(j)\}_{j \in J_{k+1}}$ a set of r (variable) nonnegative weights (i.e., $\mu_{k+1}(j) \geq 0$ for each j, and $\sum_{j \in J_{k+1}} \mu_{k+1}(j) = 1$). We investigate this intermediate step in a newly defined Hilbert space $\mathcal{H}$, the construction of which is described now.

Consider the closed interval [0,1] with its Borel measurable sets and with the corresponding Lebesgue measure m. When H is made into a measurable space by considering the Borel measurable sets corresponding to the norm topology on H, let $\mathcal{L}([0,1],H)$ be the set of all measurable functions from [0,1] to H, and denote by $L^2([0,1],H)$ the set of equivalence classes in $\mathcal{L}([0,1],H)$ of functions φ in $\mathcal{L}([0,1],H)$ such that

$$|||\varphi|||^2 \equiv \int_0^1 ||\varphi(t)||^2 dm(t) < \infty. \tag{4}$$

Putting, for functions φ and ψ in $\mathcal{L}([0,1],H)$

$$\langle\langle \varphi, \psi \rangle\rangle = \int_0^1 \langle \varphi(t), \psi(t) \rangle dm(t),$$

we obtain that $L^2([0,1],H)$ is a Hilbert space with norm $||| \quad |||$ derived from the inner product $\langle\langle \ , \ \rangle\rangle$. This space $L^2([0,1],H)$ is our new Hilbert space associated to H; it will be denoted for short by $\mathcal{H}$.

For $i = 1,2\ldots$ we denote by (J_i, μ_i) the set J_i endowed with a (probability) measure μ_i such that $\mu_i(j) \geq 0$ for $j \in J_i$ and $\sum_{j\in J_i} \mu_i(j) = 1$. These nonnegative numbers denote the weights used in (3) to obtain v_i from x_{i-1}. Suppose that for some fixed $i \in \{1,2,\ldots\}$ we have a set $\{\mu_i(j)\}_{j\in J_i}$ of nonnegative weights. Denoting the indices in J_i in ascending order by $\{i_1,\ldots,i_r\}$, let then T_i be the function from (J_i,μ_i) into the set of subsets of [0,1] defined as follows

$$T_i(i_1) =]0, \mu_i(i_1)],$$

and, for $j = 2,3,\ldots,r$

$$T_i(i_j) =]\mu_i(i_1) + \ldots + \mu_i(i_{j-1}), \mu_i(i_1) + \ldots + \mu_i(i_j)].$$

Together with the singleton number $\{0\}$, the sets $T_i(i_j)$ $(j = 1,\ldots,r)$ form a disjoint covering of [0,1]; as some of the numbers $\mu_i(i_j)$ may be zero, some of the sets $T_i(i_j)$ may be empty.

A first important subset of $\mathcal{H}$, denoted by D, consists of the set of equivalence classes which correspond to constant functions in $\mathcal{L}([0,1],H)$. D may be obtained by the natural imbedding $q: H \to \mathcal{H}$, which associates to each v in H the element $q(v)$ in $\mathcal{H}$ defined by $q(v)(t) = v$ for all $t \in [0,1]$; as such, $D \equiv q(H)$. It has been shown in [8, Lemma 1] that D is a closed linear subspace of $\mathcal{H}$.

The other important class of subsets of $\mathcal{H}$ is defined as follows. Suppose that at a fixed step i $(i = 1,2,\ldots)$ in the iteration (3) use is made of the sets $\{C_j\}_{j\in J_i}$ and the weights $\{\mu_i(j)\}_{j\in J_i}$, and as above let $J_i = \{i_1,\ldots,i_r\}$. Whenever we choose an element v_{i_j} in each set $C_{i_j}(j = 1,\ldots,r)$, we may define a function $\varphi^i : [0,1] \to H$ by putting $\varphi^i(t) = v_{i_j}$, for all $t \in T_i(i_j)$, and by letting $\varphi^i(0)$ be that chosen element v_{i_s} in C_{i_s} with the smallest index $s \in \{i_1, i_2,\ldots,i_r\}$ such that $T_i(i_s) \neq \emptyset$. Hence, φ^i is a piecewise constant function defined on [0,1], with values lying in some or all of the chosen sets $\{C_{i_j}\}_{j=1}^r$; clearly, $\varphi^i \in \mathcal{H}$. Considering the set of all such functions

φ^i as explained above, for fixed i (and hence for fixed J_i and for fixed $\{C_{i_j}\}_{j=1}^r$), for fixed $\{\mu_i(i_j)\}_{j=1}^r$, and for all possible choices of elements v_{i_j} in the corresponding sets $\{Ci_j\}_{j=1}^r$, we denote the set of equivalence classes of these functions in $\mathcal{H}$ by $F_{\mu_i}^{Ji}$. Analogous as in [8, Lemma 2] it may be shown that each set $F_{\mu_i}^{Ji}$ is a closed convex subset of $\mathcal{H}$.

For given v in $H, q(v)$ is a point in $D \subset \mathcal{H}$. For fixed i, fixed J_i, fixed $\{C_j\}_{j\in J_i}$, and fixed $\{\mu_i(j)\}_{j\in J_i}$, the point $q(v)$ may be projected onto the set $F_{\mu_i}^{Ji}$. Denoting the corresponding projection by $P_{F_{\mu_i}^{J_i}}[q(v)]$, this projection point is a piecewise constant function on [0,1], taking on the subintervals of [0,1] determined by weights $\{\mu_i(j)\}_{j\in J_i}$ a constant value in the sets $\{C_j\}_{j\in J_i}$. As in Proposition 1 in [8], it may be proved that these constant values (otherwise said, the "components") of $P_{F_{\mu_i}^{J_i}}[q(v)]$ are given by $\{P_j(v)\}_{j\in J_i}$ (where $P_j(v)$ denotes the projection (in H) of the point v onto the set C_j for $j \in J_i$). In the same manner (cfr. [8, Proposition 2]), when a function $\varphi \in F_{\mu_i}^{J_i}$ is given, and this function has components $\{w_j\}_{j\in J_i}$ in the corresponding sets $\{C_j\}_{j\in J_i}$ in H, then $P_D(\varphi)$ is the image under the natural imbedding q of H in $\mathcal{H}$ of the point $\sum_{j\in J_i}\mu_i(j)w_j$ in H; i.e.,

$$P_D(\varphi) = q(\sum_{j\in J_i}\mu_i(j)w_j).$$

We now have the necessary equipment to interprete the intermediate iteration step (3) in H as a corresponding step in $\mathcal{H}$. When v_{k+1} is obtained from x_k in H as in (3), which implies that sets $\{C_j\}_{j\in J_{k+1}}$, and values λ_{k+1} and $\{\mu_{k+1}(j)\}_{j\in J_{k+1}}$ are given, we can form the transformed point $\widehat{x_k} \equiv q(x_k)$ in $\mathcal{H}$ of x_k in H; from $\widehat{x_k}$ a point $\widehat{v_{k+1}}$ in $\mathcal{H}$ may be obtained in two steps:

$$\widehat{y_{k+1}} = \widehat{x_k} + \lambda_{k+1}(P_{F_{\mu_{k+1}}^{J_{k+1}}}(\widehat{x_k}) - \widehat{x_k}), \tag{5}$$

$$\widehat{v_{k+1}} = P_D(\widehat{y_{k+1}}). \tag{6}$$

Then $\widehat{v_{k+1}}$ is precisely the image under the imbedding q of the point v_{k+1} given by (3); i.e., $\widehat{v_{k+1}} \equiv q(v_{k+1})$. The interesting part of course is that, conversely, the steps ((5) + (6)) in $\mathcal{H}$ may be used to determine values of λ_{k+1} and $\{\mu_{k+1}(j)\}_{j\in J_{k+1}}$, for instance by imposing geometrical conditions in $\mathcal{H}$, and that v_{k+1} in (3) is then determined by $v_{k+1} = q^{-1}(\widehat{v_{k+1}})$. We finally remark that the result of obtaining $\widehat{v_{k+1}}$ from $\widehat{x_k}$ as in ((5) + (6)) also may be written in one step, namely

$$\widehat{v_{k+1}} = \widehat{x_k} + \lambda_{k+1}(P_D(P_{F_{\mu_{k+1}}^{J_{k+1}}}(\widehat{x_k})) - \widehat{x_k}). \tag{7}$$

3. Construction of the adapted sequence in $\mathcal{H}$

Let $\mathcal{H}$ be the Hilbert space with inner product $\langle\langle\ ,\ \rangle\rangle$ and norm $|||\quad|||$ derived from $\langle\langle\ ,\ \rangle\rangle$, introduced in Section 2, with the closed linear subspace D and an uncountably infinite family of closed convex sets $(F_{\mu_\omega}^{J_\omega})_{\omega\in\Omega}$. For the moment, the explicit choice of the subsets $J_\omega \subset J$ and of the weights $\{\mu_\omega(j)\}_{j\in J_\omega}$ is not involved ; hence, for notational reasons we denote these sets momentarily for short by $\{F_\omega\}_{\omega\in\Omega}$. We know that in the original Hilbert space H the set $C^* \equiv \cap_{t=1}^m C_t$ is nonempty, and that $q(C^*) \subset D \cap (\cap_{\omega\in\Omega} F_\omega)$. Hence, we act as if we want to obtain, in an iterative manner, a point in $D \cap (\cap_{\omega\in\Omega} F\omega)$. Denote by P_D and P_{F_ω} the shortest distance projection operators in $\mathcal{H}$ onto D and F_ω, respectively.

Let $\{\widehat{x_o}\}$ be a given starting point for a sequence $\{\widehat{x_n}\}_{n=0}^{+\infty}$ in D, and choose a set F_1 among the sets $(F_\omega)_{\omega\in\Omega}$. At the projection point $P_{F_1}\widehat{x_o}$ of $\widehat{x_o}$ onto F_1 there is a supporting (also called tangent) hyperplane in $\mathcal{H}$, denoted by $\mathcal{P}_1$, that intersects D along a hyperplane S_1 of D. On the halfline in $\mathcal{H}$ emanating from $\widehat{x_0}$ and going through $P_{F_1}\widehat{x_0}$ we choose a point $\widehat{y_1}$ such that the projection of $\widehat{y_1}$ onto D, denoted by $\widehat{v_1}$, is a point of S_1. As $\widehat{y_1}$ is given by

$$\widehat{y_1} = \widehat{x_o} + \lambda_1(P_{F_1}\widehat{x_0} - \widehat{x_0}) \tag{8}$$

for some suitable positive relaxation coefficient λ_1, the point $\widehat{v_1}$ is given by

$$\widehat{v_1} = \widehat{x_0} + \lambda_1(P_D(P_{F_1}\widehat{x_0}) - \widehat{x_0}). \tag{9}$$

We put $\widehat{x_1} \equiv \widehat{v_1}$. It is easy to check that, for any point $\hat{x}$ in S_1, the vector $\widehat{v_1} - \hat{x}$ is orthogonal onto the vector $\widehat{v_1} - \widehat{x_0}$.

Denoting for an ordered pair $(\hat{a}, \hat{b})$ of different points in D

$$S(\hat{a}, \hat{b}) = \{\hat{w} \in D : \langle\langle \hat{w} - \hat{b}, \hat{b} - \hat{a}\rangle\rangle \geq 0\}, \tag{10}$$

$S(\hat{a}, \hat{b})$ is a closed half-space of D, and $\hat{b}$ is the projection of $\hat{a}$ onto $S(\hat{a}, \hat{b})$. Hence, in our construction we can say that S_1 is the "border line" of $S(\widehat{x_0}, \widehat{v_1})$, and $\widehat{v_1}$ is the projection of $\widehat{x_0}$ onto $S(\widehat{x_0}, \widehat{v_1})$. Besides, due to the manner in which $\widehat{v_1}$ has been constructed we know that

$$D \cap F_1 \subset S(\widehat{x_0}, \widehat{v_1}). \tag{11}$$

Choosing another set F_2 in $\{F_\omega\}_{\omega\in\Omega}$, we construct from $\widehat{x_1} \equiv \widehat{v_1}$ a point $\widehat{v_2}$ in D in an analogous manner as given by (8) and (9). Hence

$$\widehat{v_2} = \widehat{x_1} + \lambda_2(P_D(P_{F_2}\widehat{x_1}) - \widehat{x_1}), \tag{12}$$

and λ_2 is determined such that $\widehat{v_2}$ belongs to the intersection S_2 of D and the hyperplane $\mathcal{P}_2$ of $\mathcal{H}$ supporting the set F_2 at the point $P_{F_2}\widehat{x_1}$. Analogously as above we can say that S_2 is the "border line" of the half-space $S(\widehat{x_1}, \widehat{v_2})$ of D.

In view of our aim to reach a point in $D\cap(\cap_{\omega\in\Omega}F_\omega)$, and of (11), we want that the next iteration point $\widehat{x_2}$ of the sequence $\{\widehat{x_n}\}_{n=0}^{+\infty}$ that we want to construct should belong to $S(\widehat{x_0},\widehat{v_1})$. Hence, when $\langle\langle\widehat{x_0}-\widehat{v_1},\widehat{v_2}-\widehat{v_1}\rangle\rangle\leq 0,\widehat{v_2}$ is a point that belongs to $S(\widehat{x_0},\widehat{v_1})$ (intuitively, it is "on the good side" of S_1), and in that case we take $\widehat{x_2}\equiv\widehat{v_2}$. However, when the condition about $\widehat{v_2}$ is not true, i.e., when $\langle\langle\widehat{x_0}-\widehat{v_1},\widehat{v_2}-\widehat{v_1}\rangle\rangle>0$, we take as next iteration point $\widehat{x_2}$ a point in the intersection of S_1 and S_2. An explicit description of how $\widehat{x_2}$ is constructed will be given in the general procedure further on.

In a general manner, when $\widehat{x_0},\widehat{v_1},\widehat{x_1},\widehat{v_2},\widehat{x_2},\ldots\widehat{v_{k-1}},\widehat{x_{k-1}},\widehat{v_k},\widehat{x_k}$ have already been obtained, we first determine $\widehat{v_{k+1}}$ by

$$\widehat{v_{k+1}}=\widehat{x_k}+\lambda_{k+1}(P_D(P_{F_{k+1}}\widehat{x_k})-\widehat{x_k}), \tag{13}$$

for a suitable value of λ_{k+1} such that $\widehat{v_{k+1}}$ belongs to the intersection S_{k+1} of D and the hyperplane $\mathcal{P}_{k+1}$ of $\mathcal{H}$ supporting the set F_{k+1} at the point $P_{F_{k+1}}\widehat{x_k}$. When $\langle\langle\widehat{x_{k-1}}-\widehat{v_k},\widehat{v_{k+1}}-\widehat{v_k}\rangle\rangle\leq 0$, put $\widehat{x_{k+1}}\equiv\widehat{v_{k+1}}$. Otherwise, take as $\widehat{x_{k+1}}$ a suitable point in the intersection of the hyperplanes S_k and S_{k+1} of D. In this way the sequence $\{\widehat{x_k}\}_{k=0}^{+\infty}$ in $\mathcal{H}$ is constructed.

The suitable value λ_{k+1} such that the condition stated in the lines following (13) should be true, may be derived from [10, Lemma 1]. Adapted to our notation, this value is given by

$$\lambda_{k+1}=\frac{|||P_{F_{k+1}}\widehat{x_k}-\widehat{x_k}|||^2}{|||P_D(P_{F_{k+1}}\widehat{x_k})-\widehat{x_k}|||^2}. \tag{14}$$

The determination of $\widehat{x_{k+1}}$ when $\langle\langle\widehat{x_{k-1}}-\widehat{v_k},\widehat{v_{k+1}}-\widehat{v_k}\rangle\rangle>0$ is described in the following procedure.

Procedure to determine $\widehat{x_{k+1}}$:
Suppose that

$$\alpha_k\equiv\langle\langle\widehat{x_{k-1}}-\widehat{v_k},\widehat{v_{k+1}}-\widehat{v_k}\rangle\rangle>0.$$

Put

$$\begin{aligned}\beta_k&\equiv|||\widehat{v_k}-\widehat{x_{k-1}}|||^2\\ \gamma_k&\equiv|||\widehat{v_{k+1}}-\widehat{x_k}|||^2\\ \zeta_k&\equiv\beta_k\gamma_k-\alpha_k^2.\end{aligned}$$

Then $\widehat{x_{k+1}}$ is determined by

$$\widehat{x_{k+1}}=\widehat{v_{k+1}}+\frac{\alpha_k\gamma_k}{\zeta_k}(\widehat{v_k}-\widehat{x_{k-1}})+\frac{\alpha_k^2}{\zeta_k}(\widehat{v_{k+1}}-\widehat{x_k}). \tag{15}$$

Indeed, a straightforward computation shows that

$$\langle\langle\widehat{x_{k+1}}-\widehat{v_{k+1}},\widehat{v_{k+1}}-\widehat{x_k}\rangle\rangle=0 \tag{16}$$

and

$$\langle\langle \widehat{x_{k+1}} - \widehat{v_k}, \widehat{v_k} - \widehat{x_{k-1}} \rangle\rangle = 0. \tag{17}$$

Hence, $\widehat{x_{k+1}}$ belongs to the intersection of the hyperplanes S_{k+1} and S_k of D. As also $\widehat{x_k}$ belongs to S_k, we also have

$$\langle\langle \widehat{x_{k-1}} - \widehat{v_k}, \widehat{v_k} - \widehat{x_k} \rangle\rangle = 0. \tag{18}$$

From the determination of $\widehat{x_{k+1}}$ in the case that $\widehat{x_{k+1}} \neq \widehat{v_{k+1}}$ several inequalities may be derived, which will be needed in the sequel. We discuss them now.

We first remark that, when convergence of the sequence $\{\widehat{x_k}\}_{k=0}^{+\infty}$ has not yet been obtained at step $k+1$, we not only have that α_k is strictly positive (by assumption), but that also β_k and γ_k are strictly positive. Also ζ_k is strictly positive (from which it follows that $\widehat{x_{k+1}}$ in (15) is always well-defined); indeed, as we also have $\alpha_k = \langle \widehat{x_{k-1}} - \widehat{v_k}, \widehat{v_{k+1}} - \widehat{x_k} \rangle$, and due to the Cauchy–Schwarz inequality, $\zeta_k = 0$ would imply that the half-spaces $S(\widehat{x_{k-1}}, \widehat{v_k})$ and $S(\widehat{x_k}, \widehat{v_{k+1}})$ have an empty intersection; but this is impossible as the nonempty subset $D \cap (\cap_{\omega\in\Omega} F_\omega)$ should be part of both $S(\widehat{x_{k-1}}, \widehat{v_k})$ and $S(\widehat{x_k}, \widehat{v_{k+1}})$ by construction. We know by construction that $D \cap (\cap_{i=1}^{k+1} F_i)$ is a subset of the intersection of the half-spaces $S(\widehat{x_{k-1}}, \widehat{v_k})$ and $S(\widehat{x_k}, \widehat{v_{k+1}})$. In particular, for any $\hat{z}$ in $D \cap (\cap_{i=1}^{k+1} F_i)$ we have, due to a well-known property of projections

$$\langle\langle \hat{z} - \widehat{v_{k+1}}, \widehat{v_{k+1}} - \widehat{x_k} \rangle\rangle \geq 0 \tag{19}$$

and

$$\langle\langle \hat{z} - \widehat{v_k}, \widehat{v_k} - \widehat{x_{k-1}} \rangle\rangle \geq 0. \tag{20}$$

Using (16) we obtain that for such $\hat{z}$

$$\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_{k+1}} - \widehat{x_k} \rangle\rangle = \langle\langle \hat{z} - \widehat{v_{k+1}}, \widehat{v_{k+1}} - \widehat{x_k} \rangle\rangle$$

and so, using (19) we obtain

$$\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_{k+1}} - \widehat{x_k} \rangle\rangle \geq 0. \tag{21}$$

Analogously, using (17) we have

$$\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_k} - \widehat{x_{k-1}} \rangle\rangle = \langle\langle \hat{z} - \widehat{v_k}, \widehat{v_k} - \widehat{x_{k-1}} \rangle\rangle$$

which, in view of (20), leads to

$$\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_k} - \widehat{x_{k-1}} \rangle\rangle \geq 0. \tag{22}$$

Using these inequalities we can prove the following result, wich will be crucial for the convergence of the sequence.

Proposition 1. *For any $\hat{z}$ in $D \cap (\cap_{i=1}^{k+1} F_i)$ we have*

$$\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{x_{k+1}} - \widehat{x_k} \rangle\rangle \geq 0. \tag{23}$$

Proof. We have to consider two cases, depending on the fact that $\widehat{x_{k+1}}$ and $\widehat{v_{k+1}}$ either are different or coincide.

In the case that $\widehat{x_{k+1}} \neq \widehat{v_{k+1}}$, we replace in the second half of the inner product (23) the term $\widehat{x_{k+1}}$ by its expression (15). This leads to

$$\begin{aligned}&\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{x_{k+1}} - \widehat{x_k}\rangle\rangle \\ &= \langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_{k+1}} - \widehat{x_k}\rangle\rangle + \frac{\alpha_k\gamma_k}{\zeta_k}\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_k} - \widehat{x_{k-1}}\rangle\rangle \\ &+ \frac{\alpha_k^2}{\zeta_k}\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{v_{k+1}} - \widehat{x_k}\rangle\rangle,\end{aligned}$$

and each term on the right-hand side is nonnegative, due to (21), (22) and the strict positivity of the quantities $\alpha_k, \gamma_k, \zeta_k$.

In het case that $\widehat{x_{k+1}} = \widehat{v_{k+1}}, \widehat{x_{k+1}}$ is the projection of $\widehat{x_k}$ onto the half-space $S(\widehat{x_k}, \widehat{x_{k+1}})$, and $D \cap (\cap_{i=1}^{k+1} F_i)$ is by construction a subset of that half-space. Hence, again using a property of projections, we see that also in this case the inequality (23) is true. □

Considering the complete sequence $\{\widehat{x_k}\}_{k=0}^{+\infty}$ we can restate Proposition 1 in the following form.

Proposition 1'. *For any* $\hat{z}$ *in* $D \cap (\cap_{i=1}^{\infty} F_i)$ *we have*

$$\langle\langle \hat{z} - \widehat{x_{k+1}}, \widehat{x_{k+1}} - \widehat{x_k}\rangle\rangle \geq 0 \tag{23'}$$

for all $k = 0, 1, 2, \ldots$.

As a consequence of Proposition 1' we can derive the following results. For each point $\hat{z}$ in $D \cap (\cap_{i=1}^{\infty} F_i)$ we have

$$\begin{aligned}&|||\widehat{x_n} - \hat{z}|||^2 \\ &= |||\widehat{x_n} - \widehat{x_{n+1}}|||^2 + |||\widehat{x_{n+1}} - \hat{z}|||^2 + 2\langle\langle \widehat{x_n} - \widehat{x_{n+1}}, \widehat{x_{n+1}} - \hat{z}\rangle\rangle,\end{aligned}$$

which, due to (23'), leads to

$$|||\widehat{x_n} - \hat{z}|||^2 \geq |||\widehat{x_n} - \widehat{x_{n+1}}|||^2 + |||\widehat{x_{n+1}} - \hat{z}|||^2. \tag{24}$$

This implies that the sequence $\{|||\widehat{x_n} - \hat{z}|||\}_{n=0}^{+\infty}$ is a descending sequence of positive numbers, and so it converges to some limit, say $d(\hat{z})$. As also $|||\widehat{x_{n+1}} - \hat{z}||| \to d(\hat{z})$ we derive, again from (24), that

$$|||\widehat{x_n} - \widehat{x_{n+1}}||| \to 0 \text{ for } n \to +\infty. \tag{25}$$

In view of (16) we know that the vectors $\widehat{x_{k+1}} - \widehat{v_{k+1}}$ and $\widehat{v_{k+1}} - \widehat{x_k}$ are mutually orthogonal (this is true in particular when $\widehat{x_{k+1}} \equiv \widehat{v_{k+1}}$). The Pythagorean theorem leads to

$$|||\widehat{x_k} - \widehat{x_{k+1}}|||^2 = |||\widehat{x_k} - \widehat{v_{k+1}}|||^2 + |||\widehat{v_{k+1}} - \widehat{x_{k+1}}|||^2,$$

from which we deduce, in view of (25), that

$$|||\widehat{x_k} - \widehat{v_{k+1}}||| \to 0 \text{ for } k \to +\infty. \tag{26}$$

As $\widehat{v_{k+1}}$ belongs to the hyperplane $\mathcal{P}_{k+1}$ of $\mathcal{H}$ that is orthogonal to $\widehat{x_k} - P_{F_{k+1}}\widehat{x_k}$, again applying the Pythagorean theorem gives us

$$|||\widehat{x_k} - \widehat{v_{k+1}}|||^2 = |||\widehat{x_k} - P_{F_{k+1}}\widehat{x_k}|||^2 + |||P_{F_{k+1}}\widehat{x_k} - \widehat{v_{k+1}}|||^2,$$

and using (26) we may deduce

$$|||\widehat{x_k} - P_{F_{k+1}}\widehat{x_k}|||^2 \to 0 \text{ for } k \to +\infty. \tag{27}$$

From the fact that the sequence $\{|||\widehat{x_n} - \hat{z}|||\}_{n=0}^{+\infty}$ is descending we also conclude that the sequence $\{\widehat{x_n}\}_{n=0}^{+\infty}$ is bounded; indeed, for any $\hat{z}$ in $D \cap (\cap_{i=1}^{\infty} F_i)$ we have

$$|||\widehat{x_n}||| \leq |||\widehat{x_n} - \hat{z}||| + |||\hat{z}||| \leq |||\widehat{x_0} - \hat{z}||| + |||\hat{z}|||.$$

In order to determine completely the sequence $\{\widehat{x_k}\}_{k=0}^{+\infty}$ in $\mathcal{H}$, we now have to clarify how the corresponding subsets $J_{k+1} \subset J$ and the corresponding weights $\{\mu_{k+1}(j)\}_{j \in J_{k+1}}$ may be chosen.

As our original aim is to find a point in $C^* \equiv \cap_{t=1}^{m} C_t$, it is clear that each set C_i should be involved an infinite number of times while constructing the sequence $\{x_k\}_{k=0}^{+\infty}$ (or $\{\widehat{x_k}\}_{k=0}^{+\infty}$).

An easy way to express this, while still leaving some flexibility, is given by the following control strategy.

(Control 1). *There exists some positive integer M such that, for each $n \in Z^+$*

$$J \subset \cup_{k=0}^{M-1} J_{n+k},$$

i.e., all the sets $\{C_t\}_{t=1}^{m}$ are selected at least once within any M consecutive iterations.

In order that at step $k+1$ the sets $\{C_j\}_{j \in J_{k+1}}$ really should be activated, the corresponding weights may not be zero. Again using an easy way leaving enough flexibility leads to the following second part of the control strategy.

(Control 2). *For some positive number $\delta, 0 < \delta < 1/r$, when at step $k+1$ $(k = 0, 1, \dots)$ the subset $J_{k+1} \subset J$ is involved, and hence also the sets $\{C_j\}_{j \in J_{k+1}}$ and the weights $\{\mu_{k+1}(j)\}_{j \in J_{k+1}}$, take $\mu_{k+1}(j) \geq \delta$ for each $j \in J_{k+1}$; only when $x_k \in C_j$ we may take either $\mu_{k+1}(j) = 0$ or $\mu_{k+1}(j) \neq 0$.*

The results obtained above in this Section and involving the notation F_{k+1} or F_ω, may now be rewritten using the respective notations $F_{\mu_{k+1}}^{J_{k+1}}$ and $F_{\mu_\omega}^{J_\omega}$; this is straightforward. At this moment only two of them will be mentioned explicitly, as they connect results in $\mathcal{H}$ and H. As we know from [8] that

$$|||\widehat{x_k} - P_{F_{\mu_{k+1}}^{J_{k+1}}}|||^2 = \sum_{j\in J_{k+1}} \mu_{k+1}(j)||x_k - P_j x_k||^2,$$

and

$$|||\widehat{x_k} - P_D(P_{F_{\mu_{k+1}}^{J_{k+1}}}\widehat{x_k})|||^2 = ||x_k - \sum_{j\in J_{k+1}} \mu_{k+1}(j) P_j x_k||^2,$$

the expression of λ_{k+1} in (14) may be given using distances in H, while from (27) we derive that

$$\sum_{j\in J_{k+1}} \mu_{k+1}(j)||x_k - P_j x_k||^2 \to 0 \text{ when } k \to +\infty. \quad (28)$$

4. Convergence of the sequence in the original Hilbert space

From the sequence $\{\widehat{x_k}\}_{k=0}^{+\infty}$ in the associated Hilbert space $\mathcal{H}$, constructed by choosing the sets and the weights as described in Section 3, we get the sequence $\{x_k\}_{k=0}^{+\infty}$ in the original Hilbert space H by putting $x_k \equiv q^{-1}(\widehat{x_k})$. We have to show that this sequence $\{x_k\}_{k=0}^{+\infty}$ is weakly convergent to a point in $C^* \equiv \cap_{t=1}^m C_t$.

For given vectors x and y in H with images $q(x)$ and $q(y)$ in $D \subset \mathcal{H}$ we know from [8] that $\langle\langle q(x), q(y)\rangle\rangle = \langle x, y\rangle$, and $|||q(x) - q(y)||| = ||x - y||$. In particular, for the sequence $\{x_k\}_{k=0}^{+\infty}$ in H we know that

$$||x_n - x_{n+1}|| \to 0 \text{ for } n \to +\infty \quad (29)$$

(from (25)), that the sequence $\{x_k\}_{k=0}^{+\infty}$ is bounded, and that for each point z in C^* the sequence of positive numbers $\{||x_n - z||\}_{n=0}^{+\infty}$ is convergent to some limit denoted by $d(z)$. Using the above facts, we first show that the sequence $\{x_k\}_{k=0}^{+\infty}$ has a subsequence that weakly converges to a point in C^*.

Let i be a fixed index in $J \equiv \{1, \ldots, m\}$. As i is involved an infinite number of times, the sequence $\{x_k\}_{k=0}^{+\infty}$ has a subsequence, that we denote by $\{x_n^i\}_{n=1}^{+\infty}$, with corresponding subsets $J_n^i \subset J$ and weights $\{\mu_n^i(j)\}_{j\in J_n^i}$ such that $i \in J_n^i$ and $\mu_n^i(i) \geq \delta$ (for this last fact, except possibly when $x_n^i \in C_i$, i.e., when $P_i x_n^i = x_n^i$, in which case we may put $\mu_n^i(i) = 0$). As the sequence $\{x_k\}_{k=0}^{+\infty}$ is bounded, the subsequence $\{x_n^i\}_{n=1}^{+\infty}$ is also bounded, and so it contains a subsequence, that we denote by $\{x_{n_p}^i\}_{p=1}^{+\infty}$, such that this subsequence $\{x_{n_p}^i\}_{p=1}^{+\infty}$ is weakly convergent to a point $a^i \in H$. In view of (28) and of the choice of $\mu_{n_p}^i(i)$ we have that

$$||x_{n_p}^i - P_i x_{n_p}^i|| \to 0 \text{ when } p \to +\infty, \quad (30)$$

and this in turn immediately implies that also the sequence $\{P_i x^i_{n_p}\}_{p=1}^{+\infty}$ is weakly convergent to a^i. But the closed convex set C_i is also weakly closed, and so $a^i \in C_i$.

Let now j be any index in $\{1, \dots, m\}, j \neq i$. Within any M iterations after (and including) the iteration that gave us the point $x^i_{n_p}$, the index j is involved at least once. Picking within each such M-iteration one element of the sequence $\{x_k\}_{k=0}^{+\infty}$ for which the index j was involved, we obtain a subsequence $\{x^j_{m_p}\}_{p=1}^{+\infty}$; for each index m_p there exists an index n_p such that $n_p \leq m_p \leq n_p + M - 1$. For this subsequence $\{x^j_{m_p}\}_{p=1}^{+\infty}$ we have analogously that

$$||x^j_{m_p} - P_j x^j_{m_p}|| \to 0 \text{ when } p \to +\infty. \tag{31}$$

As we may derive from (29) that

$$\lim_{k \to +\infty} ||x_{k+M} - x_k|| = 0,$$

we conclude in particular that

$$\lim_{p \to +\infty} ||x^i_{n_p} - x^j_{m_p}|| = 0. \tag{32}$$

The above analysis allows us to prove that also the sequence $\{P_j x^j_{m_p}\}_{p=1}^{+\infty}$ is weakly convergent to the above-mentioned point a^i. Indeed, for any $b \in H$ we have

$$|\langle P_j x^j_{m_p} - a^i, b\rangle| \leq |\langle P_j x^j_{m_p} - x^j_{m_p}, b\rangle| + |\langle x^j_{m_p} - x^i_{n_p}, b\rangle| + |\langle x^i_{n_p} - a^i, b\rangle|,$$

and each term on the right-hand side is convergent to zero when $p \to +\infty$, respectively by (31), by (32), and by the fact that $\{x^i_{n_p}\}_{p=1}^{+\infty}$ is weakly convergent to a^i. We conclude in particular that $a^i \in C_j$, as each element $P_j x^j_{m_p}$ belongs to C_j. As this is true for any index j in $\{1, \dots, m\}$, we may conclude that $a^i \in \cap_{j=1}^m C_j \equiv C^*$.

We may resume the foregoing in the following lemma.

Lemma 1. *With the choices of $J_{k+1}, \{\mu_{k+1}(j)\}_{j \in J_{k+1}}$ and λ_{k+1} for all $k = 0, 1 \dots$ as mentioned, respectively in (Control 1), (Control 2) and (28), the sequence $\{x_k\}_{k=0}^{+\infty}$ has a subsequence that weakly converges to a point in C^*.*

We remark that the foregoing arguments may also be used to show that, whenever a subsequence of $\{x_k\}_{k=0}^{+\infty}$ is weakly convergent to a point $a \in H$, the weak limit a belongs to C^*. So we may state

Lemma 2. *The weak limit of any subsequence of $\{x_k\}_{k=0}^{+\infty}$ that weakly converges, belongs to C^*.*

To obtain the result that the sequence $\{x_k\}_{k=0}^{+\infty}$ itself is weakly convergent to a point in C^*, it is now sufficient to show that each weakly convergent subsequence of $\{x_k\}_{k=0}^{+\infty}$ is weakly convergent to the same point a. So, let us suppose that there exists subsequences $\{x_{n_k}\}_{k=0}^{+\infty}$ and $\{x_{n_{k'}}\}_{k'=0}^{+\infty}$, converging weakly to points a and a' respectively. Due to Lemma 2, both a and a' belong to C^*. For $k \in Z^+$ we obtain

$$||x_{n_k} - a'||^2 - ||x_{n_k} - a||^2$$

$$= \langle x_{n_k} - a + a - a', x_{n_k} - a + a - a' \rangle - ||x_{n_k} - a||^2,$$

which, after expanding the inner product, leads to

$$||x_{n_k} - a'||^2 - ||x_{n_k} - a||^2 = 2\langle x_{n_k} - a, a - a' \rangle + ||a - a'||^2. \tag{33}$$

Similarly, for $k' \in Z^+$ we get

$$||x_{n_{k'}} - a||^2 - ||x_{n_{k'}} - a'||^2 = 2\langle x_{n_{k'}} - a', a' - a \rangle + ||a' - a||^2. \tag{34}$$

As remarked at the beginning of this section, the sequences $\{||x_n - a||\}_{n=1}^{+\infty}$ and $\{||x_n - a'||\}_{n=1}^{+\infty}$ are both convergent with respective limits $d(a)$ and $d(a')$. In particular, we get

$$\lim_{n \to +\infty} (||x_n - a|| - ||x_n - a'|| = d(a) - d(a'). \tag{35}$$

Taking in (33) and (34) the limit, respectively for $k \to +\infty$ and for $k' \to +\infty$ we obtain, in view of (35)

$$d(a')^2 - d(a)^2 = 0 + ||a - a'||^2,$$

and

$$d(a)^2 - d(a')^2 = 0 + ||a' - a||^2,$$

from which we conclude that $a = a'$. Hence the result, leading to Theorem 1 as stated in the Introduction.

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