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Tight Graphs and Their Primitive Idempotents

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Abstract. In this paper, we prove the following two theorems.

Theorem 1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose E and F are primitive idempotents of Γ, with cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$, respectively. Then the following are equivalent.*
(i) *The entry-wise product $E \circ F$ is a scalar multiple of a primitive idempotent of Γ.*
(ii) *There exists a real number ϵ such that*

$$\sigma_i \rho_i - \sigma_{i-1}\rho_{i-1} = \epsilon(\sigma_{i-1}\rho_i - \sigma_i \rho_{i-1}) \quad (1 \leq i \leq d).$$

Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then Jurišić, Koolen and Terwilliger proved that the valency k and the intersection numbers a_1, b_1 satisfy

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq \frac{-ka_1b_1}{(a_1+1)^2}.$$

They defined Γ to be *tight* whenever Γ is not bipartite, and equality holds above.

Theorem 2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E and F denote nontrivial primitive idempotents of Γ.*
(i) *Suppose Γ is tight. Then E, F satisfy* (i), (ii) *in Theorem* 1 *if and only if E, F are a permutation of E_1, E_d.*
(ii) *Suppose Γ is bipartite. Then E, F satisfy* (i), (ii) *in Theorem* 1 *if and only if at least one of E, F is equal to E_d.*
(iii) *Suppose Γ is neither bipartite nor tight. Then E, F never satisfy* (i), (ii) *in Theorem* 1.

Keywords: tight graph, distance-regular, association scheme, Krein parameter

1. Introduction

Let Γ denote a distance-regular graph with vertex set X and diameter $d \geq 3$. Let E_0, $E_1, \ldots, E_d$ denote the primitive idempotents of Γ (see definitions in the next section). It is well-known

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q_{ij}^h E_h \quad (0 \leq i, j \leq d), \tag{1}$$

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where $\circ$ denotes the entry-wise matrix product and where q_{ij}^h are the Krein parameters. By (1), and since $E_0, E_1, \ldots, E_d$ are linearly independent we see that for all integers $i, j\,(0 \le i, j \le d)$, the following are equivalent.

(i) $E_i \circ E_j$ is a scalar multiple of a primitive idempotent of Γ.
(ii) $q_{ij}^h \neq 0$ for exactly one $h \in \{0, 1, \ldots, d\}$.

In this paper, we investigate pairs E_i and E_j for which (i), (ii) hold above. We state our main results in Theorems 1.1 and 1.3 below.

Theorem 1.1 *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Suppose E and F are primitive idempotents of Γ, with cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$, respectively. Then the following are equivalent.*
(i) *$E \circ F$ is a scalar multiple of a primitive idempotent of Γ.*
(ii) *There exists a real number ϵ such that*

$$\sigma_i\rho_i - \sigma_{i-1}\rho_{i-1} = \epsilon(\sigma_{i-1}\rho_i - \sigma_i\rho_{i-1}) \quad (1 \le i \le d).$$

Let Γ denote a distance-regular graph with diameter $d \ge 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. In [4], Jurišić, Koolen and Terwilliger proved that the valency k and the intersection numbers a_1, b_1 satisfy

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_d + \frac{k}{a_1 + 1}\right) \ge \frac{-ka_1b_1}{(a_1 + 1)^2}. \tag{2}$$

They defined Γ to be *tight* whenever Γ is not bipartite and equality holds in (2). They showed that Γ is tight precisely when Γ is "1-homogeneous" and $a_d = 0$. They also obtained the following characterization of tight graphs, which will be useful later.

Theorem 1.2 ([4]) *Let Γ denote a nonbipartite distance-regular graph with diameter $d \ge 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ and θ' denote eigenvalues of Γ, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. Let $\epsilon \in \mathbb{R}$. Then the following are equivalent.*
(i) *Γ is tight, θ, θ' are a permutation of θ_1, θ_d and $\epsilon = \dfrac{\sigma\rho - 1}{\rho - \sigma}$.*
(ii) *$\theta \neq \theta_0, \theta' \neq \theta_0$, and*

$$\sigma_i\rho_i - \sigma_{i-1}\rho_{i-1} = \epsilon(\sigma_{i-1}\rho_i - \sigma_i\rho_{i-1}) \quad (1 \le i \le d).$$

Combining Theorems 1.1 and 1.2, we obtain the following result.

Theorem 1.3 *Let Γ denote a distance-regular graph with diameter $d \ge 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E and F denote nontrivial primitive idempotents of Γ.*
(i) *Suppose Γ is tight. Then E, F satisfy* (i), (ii) *in Theorem* 1.1 *if and only if E, F are a permutation of E_1, E_d.*
(ii) *Suppose Γ is bipartite. Then E, F satisfy* (i), (ii) *in Theorem* 1.1 *if and only if at least one of E, F is equal to E_d.*
(iii) *Suppose Γ is neither bipartite nor tight. Then E, F never satisfy* (i), (ii) *in Theorem* 1.1.

2. Preliminaries

In this section, we review some definitions and basic concepts. For more background information, the reader may refer to the books of Bannai and Ito [1], Brouwer et al. [2] or Godsil [3].

Throughout, Γ will denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, path-length distance function ∂ and diameter $d := \max\{\partial(x, y) \mid x, y \in X\}$. We say Γ is *distance-regular* whenever for all integers $h, i, j\, (0 \leq h, i, j \leq d)$ and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* for Γ. We abbreviate $a_i := p^i_{1i} (0 \leq i \leq d)$, $b_i := p^i_{1i+1} (0 \leq i \leq d - 1)$, $c_i := p^i_{1i-1} (1 \leq i \leq d)$, and $k_i := p^0_{ii} (0 \leq i \leq d)$. Observe

$$c_i + a_i + b_i = k \quad (0 \leq i \leq d), \tag{3}$$

where $k := k_1 = b_0$, $c_0 := 0$ and $b_d := 0$.

It is known, by [1], (Chapter 3, Proposition 1.2)

$$k_i = \frac{b_0 b_1 b_2 \cdots b_{i-1}}{c_1 c_2 c_3 \cdots c_i} \quad (0 \leq i \leq d). \tag{4}$$

Hereon, we assume Γ is distance-regular.

Let $Mat_X(\mathbb{C})$ denote the algebra of matrices over $\mathbb{C}$ with rows and columns indexed by X. For each integer i $(0 \leq i \leq d)$, the *ith distance matrix* $A_i \in Mat_X(\mathbb{C})$ has x, y entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

Then

$$\begin{aligned} A_0 &= I, \\ A_0 + A_1 + \cdots + A_d &= J, \\ A_i^t &= A_i \quad (0 \leq i \leq d), \\ A_i A_j &= \sum_{h=0}^{d} p^h_{ij} A_h \quad (0 \leq i, j \leq d), \end{aligned}$$

where J denotes the all 1's matrix. The matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{C}$-algebra M, called the *Bose-Mesner algebra of* Γ. By [1], (Section 2.3)

M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$\begin{aligned} E_0 &= |X|^{-1} J, \\ E_0 + E_1 + \cdots + E_d &= I, \\ E_i^t &= E_i \quad (0 \le i \le d), \\ E_i E_j &= \delta_{ij} E_i \quad (0 \le i, j \le d). \end{aligned} \tag{5}$$

The $E_0, E_1, \ldots, E_d$ are called the *primitive idempotents* of Γ, and E_0 is called the *trivial idempotent*.

Set $A := A_1$ and define $\theta_0, \theta_1, \ldots, \theta_d \in \mathbb{C}$ such that

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

It is known $\theta_0 = k$, and that $\theta_0, \theta_1, \ldots, \theta_d$ are distinct real numbers. Moreover, $k \ge \theta_i \ge -k$ $(0 \le i \le d)$, see [1], (Theorem 1.3). We refer to θ_i as the *eigenvalue* of Γ associated with E_i $(0 \le i \le d)$. We call θ_0 the *trivial eigenvalue* of Γ. For each integer $i\,(0 \le i \le d)$, let m_i denote the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i). We observe from (5) that $m_0 = 1$.

Let θ denote an eigenvalue of Γ, let E denote the associated primitive idempotent, and let m denote the multiplicity of E. By [1], (Section 2.3) there exist $\sigma_0, \sigma_1, \ldots, \sigma_d \in \mathbb{R}$ such that

$$E = |X|^{-1} m \sum_{i=0}^{d} \sigma_i A_i.$$

It follows from [1], (Chapter 2, Proposition 3.3 (iii)) that $\sigma_0 = 1$. We call $\sigma_0, \sigma_1, \ldots, \sigma_d$ the *cosine sequence* of Γ associated with E (or θ). The cosine sequence associated with E_0 consists entirely of ones, see [2], (Section 4.1B). We shall often denote σ_1 by σ.

Lemma 2.1 *Let Γ denote a distance-regular graph with diameter $d \ge 3$. For any $\theta, \sigma_0, \sigma_1, \ldots, \sigma_d \in \mathbb{C}$, the following are equivalent.*

(i) *$\sigma_0, \sigma_1, \ldots, \sigma_d$ is a cosine sequence of Γ and θ is the associated eigenvalue.*

(ii) *$\sigma_0 = 1$, and*

$$c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i \quad (0 \le i \le d), \tag{6}$$

where σ_{-1} and σ_{d+1} are indeterminates.

(iii) *$\sigma_0 = 1$, $k\sigma = \theta$ and*

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i \quad (1 \le i \le d), \tag{7}$$

where σ_{d+1} is an indeterminate.

Proof:

(i) $\Leftrightarrow$ (ii) See [2], (Proposition 4.1.1).
(ii) $\Leftrightarrow$ (iii) Follows from (3). □

Lemma 2.2 (Christoffel-Darboux Formula) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. For cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$ of Γ,*

$$(\sigma - \rho)\sum_{h=0}^{i} k_h \sigma_h \rho_h = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i}(\sigma_{i+1}\rho_i - \sigma_i \rho_{i+1}) \quad (0 \leq i \leq d), \tag{8}$$

where σ_{d+1} and ρ_{d+1} are indeterminates.

Proof: See [1], (Theorem 1.3) or [3], (Lemma 3.1). □

Corollary 2.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let σ_0, $\sigma_1, \ldots, \sigma_d$ denote a sequence of complex numbers. Then the following are equivalent.*
(i) $\sigma_0 = 1$, *and*

$$(\sigma - 1)\sum_{h=0}^{i} k_h \sigma_h = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i}(\sigma_{i+1} - \sigma_i) \quad (0 \leq i \leq d), \tag{9}$$

where σ_{d+1} is an indeterminate.
(ii) $\sigma_0, \sigma_1, \ldots, \sigma_d$ *is a cosine sequence of* Γ.

Proof:

(i) $\Rightarrow$ (ii) We show (7) holds. Pick an integer $i\,(1 \leq i \leq d)$. By (9) (with i replaced by $i-1$) we have

$$(\sigma - 1)\sum_{h=0}^{i-1} k_h \sigma_h = \frac{b_1 b_2 \cdots b_{i-1}}{c_1 c_2 \cdots c_{i-1}}(\sigma_i - \sigma_{i-1}). \tag{10}$$

Subtracting Eq. (10) from Eq. (9) and eliminating k_i from the result using (4), we get (7) as desired. Now $\sigma_0, \sigma_1, \ldots, \sigma_d$ is a cosine sequence by Lemma 2.1 (i) and (iii).
(ii) $\Rightarrow$ (i) Observe $\sigma_0 = 1$ by Lemma 2.1 (ii). To obtain (9), in Lemma 2.2 let $\rho_0, \rho_1, \ldots, \rho_d$ denote the cosine sequence for E_0, that is, $\rho_i = 1$ $(0 \leq i \leq d)$. □

The graph Γ is said to be *bipartite* whenever $a_i = 0$ for $0 \leq i \leq d$.

Lemma 2.4 *Let Γ denote a distance-regular graph with diameter $d \geq 3$, valency k, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ_d. Then the following are equivalent.*

(i) Γ *is bipartite.*
(ii) $\theta_d = -k$.
(iii) $\sigma_i = (-1)^i \quad (0 \le i \le d)$.

Proof:

(i) $\Leftrightarrow$ (ii) See [2], (Proposition 3.2.3).
(i), (ii) $\Rightarrow$ (iii) Follows easily from Lemma 2.1 (i), (ii).
(iii) $\Rightarrow$ (ii) Observe $\sigma_1 = -1$, and $\theta_d = \sigma_1 k$ by Lemma 2.1 (i), (iii), so $\theta_d = -k$. □

Lemma 2.5 *Let Γ denote a bipartite distance-regular graph with diameter $d \ge 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Pick any integer i $(0 \le i \le d)$. Then*
(i) $\theta_i = -\theta_{d-i}$,
(ii) $m_i = m_{d-i}$,
(iii) *Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ_i. Then the cosine sequence for θ_{d-i} is $\sigma_0, -\sigma_1, \sigma_2, -\sigma_3, \ldots, (-1)^d \sigma_d$.*

Proof:

(i), (ii) See [2], (Proposition 3.2.3).
(iii) By Lemma 2.1 (ii) and recalling that $a_0, a_1, \ldots, a_d$ are all 0, it suffices to show

$$c_j(-1)^{j-1}\sigma_{j-1} + b_j(-1)^{j+1}\sigma_{j+1} = \theta_{d-i}(-1)^j\sigma_j \quad (0 \le j \le d). \tag{11}$$

By Lemma 2.1 (i), (ii), and since $\sigma_0, \sigma_1, \ldots, \sigma_d$ is a cosine sequence for θ_i,

$$c_j\sigma_{j-1} + b_j\sigma_{j+1} = \theta_i\sigma_j \quad (0 \le j \le d). \tag{12}$$

Evaluating (12) using Lemma 2.5 (i) we obtain (11), as desired. □

Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote a finite sequence of nonzero real numbers. By *the number of sign changes* in this sequence we mean the number of integers i $(0 \le i \le d-1)$ such that $\sigma_i\sigma_{i+1} < 0$. For an arbitrary finite sequence of real numbers, the number of sign changes in it is the number of sign changes in the sequence obtained by disregarding the zero terms.

Lemma 2.6 *Let Γ denote a distance-regular graph with diameter $d \ge 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote a cosine sequence of Γ, and let θ denote the corresponding eigenvalue. For any integer i $(0 \le i \le d)$, the following are equivalent.*
(i) $\theta = \theta_i$.
(ii) *$\sigma_0, \sigma_1, \ldots, \sigma_d$ has exactly i sign changes.*
Moreover, suppose $i \ge 1$, and that (i), (ii) *hold. Then the sequence $\sigma_0 - \sigma_1, \sigma_1 - \sigma_2, \ldots, \sigma_{d-1} - \sigma_d$ has exactly $i - 1$ sign changes.*

Proof: See [2], (Corollary 4.1.2) or [3], (Lemma 2.1). □

Lemma 2.7 *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with θ_1. Then*

$$\sigma_0 > \sigma_1 > \cdots > \sigma_d. \tag{13}$$

Proof: By Lemma 2.6, the sequence $\sigma_0 - \sigma_1, \sigma_1 - \sigma_2, \ldots, \sigma_{d-1} - \sigma_d$ has no sign changes. Recall, $\sigma_1 = \theta_1 k^{-1}$ and $\sigma_0 = 1$ by Lemma 2.1 (iii), so

$$1 = \sigma_0 > \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d. \tag{14}$$

Suppose (13) fails. Then there exists an integer i $(1 \leq i \leq d-1)$ such that

$$\sigma_{i-1} > \sigma_i = \sigma_{i+1}. \tag{15}$$

Setting $\sigma_i = \sigma_{i+1}$ in (7), we find in view of (15) that

$$\sigma_i = \sigma_{i+1} < 0. \tag{16}$$

Assume for now that $i = d-1$, so $\sigma_{d-1} = \sigma_d < 0$. Using (15), and by setting $i = d$ in (7) we obtain

$$\begin{aligned} 0 &= c_d(\sigma_{d-1} - \sigma_d) \\ &= k(\sigma - 1)\sigma_d, \end{aligned}$$

so $\sigma_d = 0$, a contradiction. Hence, $i < d-1$. We may now argue, by (14), (15), (7) and (16)

$$\begin{aligned} 0 &\geq -b_{i+1}(\sigma_{i+1} - \sigma_{i+2}) \\ &= -b_{i+1}(\sigma_{i+1} - \sigma_{i+2}) + c_{i+1}(\sigma_i - \sigma_{i+1}) \\ &= k(\sigma - 1)\sigma_{i+1} \\ &> 0, \end{aligned}$$

a contradiction. We now have (13), as desired. □

Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let M denote the Bose Mesner algebra of Γ. Since M is closed under the entrywise matrix product $\circ$, there exist scalars $q^h_{ij} \in \mathbb{C}$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q^h_{ij} E_h \quad (0 \leq i, j \leq d). \tag{17}$$

We call the q^h_{ij} the *Krein parameters* of Γ.

3. The main results

Let E and F denote primitive idempotents of a distance-regular graph Γ. In this section, we focus on finding necessary and sufficient conditions such that $E \circ F$ is a scalar multiple of a primitive idempotent of Γ.

Lemma 3.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. For all integers $h, i, j\,(0 \leq h, i, j \leq d)$, the following are equivalent.*
(i) *$E_i \circ E_j$ is a scalar multiple of E_h.*
(ii) *$q^r_{ij} = 0$ for all $r \in \{0, 1, \ldots, d\} \backslash h$.*

Proof: Follows immediately from (17) and the linear independence of $E_0, E_1, \ldots, E_d$. □

Lemma 3.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E, F and H denote primitive idempotents of Γ, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$; $\rho_0, \rho_1, \ldots, \rho_d$* and *$\gamma_0, \gamma_1, \ldots, \gamma_d$ denote the associated cosine sequences. Then the following are equivalent.*
(i) *$E \circ F$ is a scalar multiple of H.*
(ii) *$\gamma_i = \sigma_i \rho_i \quad (0 \leq i \leq d)$.*
Moreover, suppose (i), (ii) *hold. Then the scalar referred to in* (i) *is equal to*

$$m_\sigma m_\rho m_\gamma^{-1} |X|^{-1}, \tag{18}$$

where m_σ, m_ρ and m_γ denote the multiplicity of E, F and H respectively.

Proof: Observe

$$E = |X|^{-1} m_\sigma \sum_{i=0}^{d} \sigma_i A_i, \tag{19}$$

$$F = |X|^{-1} m_\rho \sum_{i=0}^{d} \rho_i A_i, \tag{20}$$

$$H = |X|^{-1} m_\gamma \sum_{i=0}^{d} \gamma_i A_i. \tag{21}$$

(i) ⇒ (ii) By assumption, there exists $\alpha \in \mathbb{C}$ such that

$$E \circ F = \alpha H. \tag{22}$$

Eliminating E, F and H in (22) using (19)–(21), and evaluating the result we find

$$m_\sigma m_\rho \sigma_i \rho_i = \alpha |X| m_\gamma \gamma_i \quad (0 \leq i \leq d). \tag{23}$$

Setting $i = 0$ in (23), and recalling $\sigma_0 = 1$, $\rho_0 = 1$ and $\gamma_0 = 1$, we find

$$\alpha = m_\sigma m_\rho m_\gamma^{-1} |X|^{-1}. \tag{24}$$

Eliminating α in (23) using (24), we obtain

$$\sigma_i \rho_i = \gamma_i \quad (0 \leq i \leq d),$$

as desired.

(ii) $\Rightarrow$ (i) By (19)–(21),

$$\begin{aligned} E \circ F &= |X|^{-2} m_\sigma m_\rho \sum_{i=0}^{d} \sigma_i \rho_i A_i \\ &= |X|^{-2} m_\sigma m_\rho \sum_{i=0}^{d} \gamma_i A_i \\ &= \alpha H, \end{aligned}$$

where α is as in (24). □

In the next lemma, we consider some examples.

Lemma 3.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$.*

(i) $E_0 \circ E_i = |X|^{-1} E_i \quad (0 \leq i \leq d)$.

(ii) *Suppose Γ is bipartite. Then*

$$E_d \circ E_i = |X|^{-1} E_{d-i} \quad (0 \leq i \leq d).$$

Proof:

(i) Follows easily from (5).

(ii) By Lemma 2.4, the cosine sequence for E_d is $1, -1, 1, \ldots, (-1)^d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for E_i. By Lemma 2.5, the cosine sequence for E_{d-i} is $\sigma_0, -\sigma_1, \sigma_2, \ldots, (-1)^d \sigma_d$. Combining the above information with Lemma 3.2, we find $E_d \circ E_i$ is a scalar multiple of E_{d-i}. The scalar is $|X|^{-1}$ by (18), Lemma 2.5 (ii), and the fact that $m_0 = 1$. □

Theorem 3.4 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E and F denote primitive idempotents of Γ, with cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$, respectively. The following are equivalent.*

(i) *$E \circ F$ is a scalar multiple of a primitive idempotent of Γ.*

(ii) *There exists a real number ϵ such that*

$$\sigma_i \rho_i - \sigma_{i-1} \rho_{i-1} = \epsilon (\sigma_{i-1} \rho_i - \sigma_i \rho_{i-1}) \quad (1 \leq i \leq d). \tag{25}$$

Proof:

(i) ⇒ (ii) Suppose $E \circ F$ is a scalar multiple of a primitive idempotent H of Γ. Let $\gamma_0, \gamma_1, \ldots, \gamma_d$ be the cosine sequence for H. First, assume $E \neq F$ and set

$$\epsilon = \frac{\sigma\rho - 1}{\rho - \sigma}. \tag{26}$$

Pick an integer i $(1 \leq i \leq d)$. Observe by Lemma 3.2, Corollary 2.3, Lemma 2.2 and Eq. (26),

$$\begin{aligned}
\sigma_i \rho_i - \sigma_{i-1}\rho_{i-1} &= \gamma_i - \gamma_{i-1} \\
&= (\gamma - 1)\frac{c_1 c_2 \cdots c_{i-1}}{b_1 b_2 \cdots b_{i-1}} \sum_{h=0}^{i-1} k_h \gamma_h \\
&= (\sigma\rho - 1)\frac{c_1 c_2 \cdots c_{i-1}}{b_1 b_2 \cdots b_{i-1}} \sum_{h=0}^{i-1} k_h \sigma_h \rho_h \\
&= (\sigma\rho - 1)\left(\frac{\sigma_i \rho_{i-1} - \sigma_{i-1}\rho_i}{\sigma - \rho}\right) \\
&= \epsilon(\sigma_{i-1}\rho_i - \sigma_i \rho_{i-1}),
\end{aligned}$$

as desired.

Next suppose $E = F$. Then $\sigma_i = \rho_i$ for $0 \leq i \leq d$, so in view of Lemma 3.2,

$$\gamma_i = \sigma_i^2 \quad (0 \leq i \leq d).$$

Observe $\gamma_0, \gamma_1, \ldots, \gamma_d$ are all non-negative, so $H = E_0$ by Lemma 2.6. In particular, $\gamma_i = 1$ for $0 \leq i \leq d$, so

$$\sigma_i^2 = 1 \quad (0 \leq i \leq d).$$

Now, (25) holds for all $\epsilon \in \mathbb{R}$.

(ii) ⇒ (i) Set

$$\gamma_i := \sigma_i \rho_i \quad (0 \leq i \leq d). \tag{27}$$

By Lemma 3.2, it suffices to show $\gamma_0, \gamma_1, \ldots, \gamma_d$ is a cosine sequence. By Corollary 2.3, this will occur if we can show

$$(\gamma - 1)\sum_{h=0}^{i} k_h \gamma_h = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i}(\gamma_{i+1} - \gamma_i) \tag{28}$$

for $0 \leq i \leq d$, where γ_{d+1} is indeterminate. Setting $i = 1$ in (25), we obtain

$$\sigma\rho - 1 = \epsilon(\rho - \sigma). \tag{29}$$

By Lemma 2.2,

$$(\sigma - \rho)\sum_{h=0}^{i} k_h \sigma_h \rho_h = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i}(\sigma_{i+1}\rho_i - \sigma_i \rho_{i+1}) \tag{30}$$

for $0 \leq i \leq d$. We first show (28) holds at $i = d$. Observe that the right side of (30) vanishes at $i = d$. Combining this observation with (27) and (29), we find

$$(\gamma - 1)\sum_{h=0}^{d} k_h \gamma_h = (\sigma\rho - 1)\sum_{h=0}^{d} k_h \sigma_h \rho_h \tag{31}$$

$$= \epsilon(\rho - \sigma)\sum_{h=0}^{d} k_h \sigma_h \rho_h \tag{32}$$

$$= 0, \tag{33}$$

so (28) holds at $i = d$.

Next, we show (28) holds for $i < d$. Multiplying Eq. (30) by ϵ, and simplifying the resulting equation using (25) and (29), we obtain

$$(\sigma\rho - 1)\sum_{h=0}^{i} k_h \sigma_h \rho_h = \frac{b_1 b_2 \cdots b_i}{c_1 c_2 \cdots c_i}(\sigma_{i+1}\rho_{i+1} - \sigma_i \rho_i), \tag{34}$$

which implies (28) in view of (27). We have shown (28) for $0 \leq i \leq d$, so $\gamma_0, \gamma_1, \ldots, \gamma_d$ is a cosine sequence by Corollary 2.3. □

We conclude by determining which nontrivial primitive idempotents E and F satisfy (i), (ii) of Theorem 3.4. We consider three cases.

Theorem 3.5 *Let Γ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E and F denote nontrivial primitive idempotents of Γ. The following are equivalent.*
(i) *$E \circ F$ is a scalar multiple of a primitive idempotent of Γ.*
(ii) *E, F are a permutation of E_1, E_d.*
Suppose (i), (ii) *hold, and let H denote the primitive idempotent of Γ such that $E \circ F$ is a scalar multiple of H. Then the eigenvalue associated with H is θ_{d-1}. Moreover,*

$$k\theta_{d-1} = \theta_1 \theta_d. \tag{35}$$

Proof:

(i) ⇔ (ii) Follows from Theorem 1.2 and Theorem 3.4.

If (i), (ii) hold then the eigenvalues of Γ associated with E and F are a permutation of θ_1 and θ_d. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$ denote the cosine sequences for θ_1

and θ_d, respectively. By Lemma 2.6, $\sigma_0, \sigma_1, \dots, \sigma_d$ has 1 sign change and $\rho_0, \rho_1, \dots, \rho_d$ has d sign changes. Combining this with Lemma 3.2 and Lemma 2.7, we observe that the cosine sequence for H, namely $\sigma_0\rho_0, \sigma_1\rho_1, \dots, \sigma_d\rho_d$ has $d-1$ sign changes. By Lemma 2.6, the eigenvalue associated with H is θ_{d-1}. Finally, applying Lemma 2.1 (iii), we get $\sigma_1 = \theta_1 k^{-1}$, $\rho_1 = \theta_d k^{-1}$, and $\sigma_1\rho_1 = \theta_{d-1}k^{-1}$, giving (35), as desired. □

Theorem 3.6 *Let Γ denote a bipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E and F denote nontrivial primitive idempotents of Γ. The following are equivalent.*
(i) *$E \circ F$ is a scalar multiple of a primitive idempotent of Γ.*
(ii) *At least one of E, F is equal to E_d.*

Proof:

(i) $\Rightarrow$ (ii) Let $\sigma_0, \sigma_1, \dots, \sigma_d$ and $\rho_0, \rho_1, \dots, \rho_d$ denote the cosine sequences for E and F, respectively. By Theorem 3.4, there exists a real number ϵ such that

$$\sigma_i\rho_i - \sigma_{i-1}\rho_{i-1} = \epsilon(\sigma_{i-1}\rho_i - \sigma_i\rho_{i-1}) \quad (1 \leq i \leq d). \tag{36}$$

First assume $E = F$. Setting $i = 1$ in (36), we observe $\sigma_1^2 = \sigma_0^2 = 1$. Observe $\sigma_1 \neq 1$ since E is nontrivial, so $\sigma_1 = -1$. Now $E = E_d$ by Lemma 2.4 (i), (ii).

Next assume $E \neq F$. Setting $i = 1, 2$ in (36), we get

$$\sigma\rho - 1 = \epsilon(\rho - \sigma), \tag{37}$$

$$\sigma_2\rho_2 - \sigma\rho = \epsilon(\sigma\rho_2 - \sigma_2\rho). \tag{38}$$

Let θ and θ' denote the eigenvalues for E and F, respectively. Then by Lemma 2.1 (iii),

$$\sigma = \frac{\theta}{k}, \quad \rho = \frac{\theta'}{k}. \tag{39}$$

Recalling that $a_0, a_1, \dots, a_d$ are 0 since Γ is bipartite, and setting $i = 1$ in (6) and (3), we get

$$\sigma_2 = \frac{\theta^2 - k}{k(k-1)}, \quad \rho_2 = \frac{\theta'^2 - k}{k(k-1)}. \tag{40}$$

Eliminating ϵ in (38) using (37), and simplifying the result using (39) and (40), we obtain

$$(\theta^2 - k^2)(\theta'^2 - k^2) = 0.$$

Observe $\theta \neq k$ and $\theta' \neq k$, since E and F are nontrivial, so one of θ, θ' is equal to $-k$. Thus, one of E, F is equal to E_d.

(ii) $\Rightarrow$ (i) Follows immediately from Lemma 3.3 (ii). □

Theorem 3.7 *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$, *and suppose* Γ *is neither tight nor bipartite. Let* E *and* F *denote nontrivial primitive idempotents of* Γ. *Then* $E \circ F$ *is never a scalar multiple of a primitive idempotent of* Γ.

Proof: Immediate from Theorem 1.2 and Theorem 3.4. □

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