



Unitals in Finite Desarguesian Planes

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Abstract. We show that a suitable 2-dimensional linear system of Hermitian curves of $PG(2, q^2)$ defines a model for the Desarguesian plane $PG(2, q)$. Using this model we give the following group-theoretic characterization of the classical unitals. A unital in $PG(2, q^2)$ is classical if and only if it is fixed by a linear collineation group of order $6(q+1)^2$ that fixes no point or line in $PG(2, q^2)$.

Keywords: unitals, Hermitian curves, Desarguesian planes, unitary groups

1. Introduction

The combinatorial properties of the non-degenerate Hermitian curve lead to the concept of a unital (embedded) in a Desarguesian projective plane $PG(2, q^2)$, where $q = p^e$ for some prime p and some integer $e \geq 1$. A *unital* is a set \mathcal{U} of $q^3 + 1$ points such that every line meets \mathcal{U} in 1 or $q + 1$ points. The *classical* example of a unital comprises all points of a non-degenerate Hermitian curve of $PG(2, q^2)$, or equivalently, all absolute points of a (non-degenerate) unitary polarity of $PG(2, q^2)$. The linear group of $PG(2, q^2)$ fixing a classical unital \mathcal{U} is isomorphic to $PGU(3, q^2)$ and it acts on \mathcal{U} as $PGU(3, q^2)$ in its natural 2-transitive permutation representation. This property turns out to characterize the classical unital. In fact, Hoffer [17] (see also [5, 6]) proved that if a unital \mathcal{U} is fixed by a linear collineation group isomorphic to $PSU(3, q^2)$, then \mathcal{U} is classical. It has been conjectured that this holds true under some weaker condition, provided that \mathcal{U} is fixed by a large enough subgroup of $PGU(3, q^2)$. Previous results in this direction show that \mathcal{U} is classical if and only if the stabilizer of a point is isomorphic to the one-point stabilizer of $PGU(3, q^2)$ in its natural 2-transitive permutation representation (see [1–3, 12, 13, 18]). Another necessary and sufficient condition is that \mathcal{U} is fixed by a cyclic subgroup of $PGL(3, q^2)$ of order $q^2 - q + 1$ (see [10]). One of the non-sporadic maximal subgroups of $PGU(3, q^2)$ has order $6(q+1)^2$ and fixes a triangle but no point or line (see [15, 19]). Our main result is that the existence of such a subgroup in the linear collineation group fixing a unital \mathcal{U} of $PG(2, q^2)$ also implies that \mathcal{U} is classical.

Theorem 1.1 *Let \mathcal{U} be a unital in $PG(2, q^2)$ fixed by a linear collineation group K of order $6(q+1)^2$ that fixes no point or line in $PG(2, q^2)$. Then \mathcal{U} is classical, and one of the following two cases occurs.*

- (i) *K fixes a triangle and is isomorphic to the semidirect product of the subgroup M fixing each vertex of the triangle by the subgroup of order 6 acting on the vertices as the symmetric group. Moreover, M is the direct product of two cyclic homology groups of order $q+1$.*
- (ii) *K is the Hessian group of order 216, and $q=5$.*

Notice that the (full) *Hessian group* is the linear collineation group of order 216 which fixes a non-singular cubic curve by permuting among themselves the nine inflexional points. There is a single conjugacy class of such groups in $PG(2, q^2)$ with $p > 3$ (see [19], p. 217). In particular, the Hessian group is a subgroup of $PGU(3, q^2)$.

The proof of Theorem 1.1 requires not only finite geometry and finite group theory but also some combinatorial properties of the linear system of Hermitian curves. These properties are stated in Section 2.

2. A 2-dimensional linear system of Hermitian curves

A canonical form of the (non-degenerate) Hermitian curve in $PG(2, q^2)$ is

$$X^{q+1} + Y^{q+1} + Z^{q+1} = 0.$$

For any non-trivial triple (α, β, γ) of elements in $GF(q)$, called coefficients, the equation

$$\alpha X^{q+1} + \beta Y^{q+1} + \gamma Z^{q+1} = 0 \tag{1}$$

also defines a Hermitian curve $H(\alpha, \beta, \gamma)$ of $PG(2, q^2)$ which is still irreducible apart from the cases where some coefficient is zero. If (α, β, γ) ranges over the set of all non-trivial homogeneous triples of elements from $GF(q)$, the corresponding Hermitian curves form a 2-dimensional linear system Σ over $GF(q)$.

Throughout this paper we will consider the fundamental triangle T to be the one whose sides have equations $X=0$, $Y=0$ and $Z=0$. We label the vertices of this triangle $X_\infty = (1, 0, 0)$, $Y_\infty = (0, 1, 0)$ and $Z_\infty = (0, 0, 1)$. It is then easy to see that Σ contains exactly three Hermitian curves of rank 1, namely the sides of T . The points on each side of T , with the vertices deleted, are naturally partitioned into $q-1$ Baer sublines by thinking of the associated Miquelian inversive plane. Moreover, the rank 2 Hermitian curves in Σ are the $3(q-1)$ Hermitian curves obtained by forming the cone from any vertex of T to one of these Baer sublines on the opposite side of T . The remaining $(q-1)^2$ Hermitian curves in Σ have full rank, and each of these classical unitals meets each side of T in one of the above Baer sublines.

Now, we point out that Σ is invariant under a large linear collineation group of $PG(2, q^2)$. For any two non-zero elements u, v of $GF(q^2)$, the linear collineation $\varphi_{u,v} : X' = uX$,

$Y' = vY, Z' = Z$ sends the Hermitian curve $H(\alpha, \beta, \gamma)$ to the Hermitian curve with equation

$$\frac{\alpha}{u^{q+1}}X^{q+1} + \frac{\beta}{v^{q+1}}Y^{q+1} + \gamma Z^{q+1} = 0.$$

Since both u^{q+1} and v^{q+1} are in $GF(q)$, we see that $\varphi_{u,v}$ leaves the linear system Σ invariant. If u and v range over the set of all non-zero elements of $GF(q^2)$, we obtain a linear collineation group Γ of order $(q^2 - 1)^2$ which leaves Σ invariant. Moreover, if both u and v are $(q + 1) - st$ roots of unity, then $\varphi_{u,v}$ fixes each Hermitian curve in Σ , and such linear collineations form a subgroup Φ of order $(q + 1)^2$.

It is easy to see that the point orbits of Φ on $PG(2, q^2)$ consist of three fixed points, $3(q - 1)$ short orbits of size $q + 1$, and $(q - 1)^2$ full orbits of size $(q + 1)^2$. The fixed points are the vertices of T , and the orbits of size $q + 1$ are the above mentioned Baer sublines on the sides of T . Thus the rank 3 Hermitian curves in Σ are composed of three Φ -orbits of size $q + 1$ and $q - 2$ Φ -orbits of size $(q + 1)^2$. The rank 2 Hermitian curves in Σ consist of one fixed point under Φ , one Φ -orbit of size $q + 1$ and $q - 1$ Φ -orbits of size $(q + 1)^2$. The rank 1 Hermitian curves in Σ consist of two fixed points under Φ and $q - 1$ Φ -orbits of size $q + 1$. Hence, in all cases, each curve of Σ is partitioned into $q + 1$ Φ -orbits.

The above action of Φ on the curves in Σ allows us to define an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ in the following way. *Points* are represented by the point orbits under Φ , *lines* are represented by the Hermitian curves in Σ , and the incidence relation is the *natural inclusion*.

Theorem 2.1 *The incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a projective plane of order q .*

Proof: It is straightforward to check that $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ has $q^2 + q + 1$ points and the same number of lines. Moreover, each line is incident with $q + 1$ points. By Proposition 3.2.3 (m) of [11], it remains to show that any two distinct lines of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ have at most one common point. Let H_1 and H_2 be any two Hermitian curves in Σ sharing a point $P \in PG(2, q^2)$. Then the orbit $O(P)$ of P under Φ is in the intersection of H_1 and H_2 . If P is not on one of the sides of T , then $O(P)$ has length $(q + 1)^2$. On the other hand, two Hermitian curves have at most $(q + 1)^2$ common points by Bézout's Theorem. So, in this case, $H_1 \cap H_2 = O(P)$ and the result follows. Next suppose that P is a point on some side of T , but not equal to a vertex of T . Then $O(P)$ is the intersection of that side with H_1 (and H_2), and hence H_1 and H_2 share a common chord c . But then an easy computation shows that c is the full intersection of H_1 and H_2 , and again the result follows. Finally, suppose P coincides with a vertex of T , say Y_∞ . A simple computation shows that the intersection of H_1 and H_2 is the point Y_∞ in this case, proving the theorem. \square

Theorem 2.2 *The projective plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is isomorphic to $PG(2, q)$.*

Proof: Let σ defined as the map which sends the line of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ represented by $H(\alpha, \beta, \gamma)$ to the point $P = (\alpha, \beta, \gamma)$ of $PG(2, q^2)$. It is straightforward to check that three

lines of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ have a common point if and only if their images under σ are collinear. The result now follows using the self-duality of $PG(2, q)$. \square

3. Linear collineation groups of order $6(q + 1)^2$ in $PG(2, q^2)$ which fix a unital

In this section we obtain a technical result which will be useful in the proof of Theorem 1.1.

The normalizer of Φ contain linear collineations $g : X' = Y, Y' = Z, Z' = X$ of order 3 and $h : X' = Y, Y' = X, Z' = Z$ of order 2, and these collineations generate a group Sym_3 acting on the vertices of the fundamental triangle as the symmetric group on three objects. Let $Sym_3\Phi$ denote the group of order $6(q + 1)^2$ which is the semidirect product of Φ by Sym_3 . We similarly let $Sym_3\Gamma$ denote the semidirect product of Γ by Sym_3 , which is the full linear collineation stabilizer of the triangle T . Our purpose is to prove the following result.

Theorem 3.1 *Let K be a linear collineation group of order $6(q + 1)^2$ of $PG(2, q^2)$ which fixes a unital. If K has no fixed point or line in $PG(2, q^2)$, then either K is conjugate to $Sym_3\Phi$ under $PGL(3, q^2)$, or $q = 5$ and K is the Hessian group of order 216.*

We need the following lemma.

Lemma 3.2 *A minimal normal subgroup of K is solvable.*

Proof: For q even, K has a subgroup of odd order and index 2. As any group of odd order is solvable by the famous Feit-Thomson theorem [14], it follows that K is solvable. Hence, simple subgroups of K have prime order, and every minimal normal subgroup of K is elementary abelian. For q odd, assume to the contrary that a minimal normal subgroup N of K is the direct product of nonabelian simple groups. Consider the normal subgroup $M = K \cap PSL(3, q^2)$ of K . Then N is a normal subgroup of M . The subgroups of $PSL(3, q^2)$ containing no normal elementary abelian subgroup are listed, for instance, in [8]. Since, in our case, the characteristic p of the underlying field $GF(q^2)$ does not divide the order of the subgroup, apart from the case where $p = 3$ and $|N| = 3n$ with $(3, n) = 1$, the possibilities for N are the alternating groups A_5, A_6, A_7 and $PSL(2, 7)$ (see Theorem 1.1 in [8]). Note that each of these groups contains at least two elementary abelian subgroups of order 4 such that no involution of one commutes with an involution of the other. Moreover, since no two distinct involutory homologies fixing a unital have the same center (or the same axis) in this case (see [5]), the centers and the axes of three pairwise commuting linear collineations of order 2 are the vertices and the sides of a triangle. Let Δ denote the set of the centers of the involutory collineations in M . By the previous remarks, Δ contains the six vertices of two triangles such that no vertex of one lies on a side of the other. Hence, Δ contains a non-degenerate quadrangle. This implies that the identity is the only linear collineation fixing Δ pointwise. As a consequence, the centralizer of N in M is trivial, and thus M is isomorphic to a subgroup of $Aut(N)$ containing N . On the other hand, the automorphism groups of the non-abelian simple groups listed above are well known and have order 120, 1440, 5040 and 336, respectively. Thus the possibilities for the order of M are 60, 120, 360,

720, 1440, 2520, 5040, 168, and 336, but none of these values is consistent with the fact that M has order $2(q + 1)^2$ or $6(q + 1)^2$. \square

Lemma 3.3 *If K has no fixed point or line, then either K fixes a triangle, or $q = 5$ and K is the Hessian group of order 216.*

Proof: Assume that K fixes no triangle, else we are done. By the previous lemma, a minimal normal subgroup N of K must be solvable. Since K has even order, it contains a perspectivity; more precisely, an involutory homology for q odd, and an involutory elation for q even. Since K fixes no point or line, the centers and axes of the perspectivities in K generate a projective subplane π_0 of $PG(2, q^2)$ (see [5] and [6]). Of course, it is possible that $\pi_0 = PG(2, q^2)$. Moreover, it is easy to see that no proper subplane of π_0 is fixed by K . Our assumptions on K show that it faithfully acts on π_0 as a strongly irreducible collineation group, and thus Hering's main result in [16] implies that the solvable normal subgroup N of K is an elementary abelian group of order 9 which coincides with its centralizer in K . In particular, $3 \mid (q + 1)$. Moreover, K contains a subgroup of order 18 consisting of N and nine involutory perspectivities such that the centers of them have the same configuration as the nine inflexions of a non-singular cubic curve. That is, each line joining two centers passes through a third center. Since N is sharply transitive on the set of these 9 centers, any element of K of order 3 not contained in N fixes three collinear centers, and hence is a homology. This shows that K contains a homology of order 3, as 27 divides the order of K . Now, by Theorem 9 in [19] for q odd, and by Theorem 3 in [15], K is the Hessian group of order 216 and $q = 5$. \square

We are now in a position to prove Theorem 3.1. Assume that K is not the Hessian group of order 216, and take the fixed triangle of K from the previous lemma as the fundamental triangle $X_\infty, Y_\infty, Z_\infty$ of the coordinate system in $PG(2, q^2)$. Then K becomes a subgroup of the group $Sym_3\Gamma$ introduced earlier. Let M be the subgroup of K of index ≤ 6 which fixes each vertex of the fundamental triangle T . To show that M contains a nontrivial homology group M_1 with center $X_\infty = (1, 0, 0)$ and axis $X = 0$, consider the permutation representation L of M on the line $X = 0$. We have to prove that $|L| < |M|$. From the fixed points of M , we know that L is a subgroup of the affine linear group $AGL(1, q^2)$ acting on the line $X = 0$. Since L fixes both vertices of T on this line, L is a subgroup of $GL(1, q^2)$. Since $|GL(1, q^2)| = q^2 - 1$ and $(q + 1)^2$ divides $|M|$, we cannot have $|L| = |M|$, and the claim follows.

Since M_1 leaves the unital \mathcal{U} invariant, M_1 acts semiregularly on the nonfixed points of $l \cap \mathcal{U}$ for any nontangent line l through the center X_∞ . Allowing l to vary and assuming for the moment that $X_\infty \in \mathcal{U}$, we obtain $|M_1|$ divides both q and $q - 1$, contradicting the fact that $|M_1| \neq 1$. Thus we conclude $X_\infty \notin \mathcal{U}$, and the $q + 1$ feet of X_∞ (points of \mathcal{U} lying on the tangents to \mathcal{U} passing through X_∞) must be fixed by M_1 . This implies $X = 0$ meets \mathcal{U} in some chord r . Since L fixes r and acts on it semi-regularly, we have that $|L| \leq q + 1$. Hence the homology group M_1 with center X_∞ and axis $X = 0$ has order at least $q + 1$. On the other hand, $q + 1$ is the maximum order of a homology group fixing a unital. Thus $|M_1| = q + 1$. This implies $|L| = q + 1$, $|M| = (q + 1)^2$, and $K = Sym_3M$. Note that M_1 consists of all collineations $X' = uX, Y' = Y, Z' = Z$ with u ranging over the $(q + 1)$ -st

roots of unity. Replacing $X = 0$ with $Y = 0$, we also find a homology group of order $q + 1$ with center Y_∞ and axis $Y = 0$. Thus M also contains the subgroup M_2 consisting of all collineations $X' = X$, $Y' = vY$, $Z' = Z$ where v ranges over the $(q + 1)$ -st roots of unity. It is easy to see that M is the direct product of M_1 and M_2 , and hence M coincides with the group Φ introduced in Section 2.

4. A group-theoretic characterization of the classical unital

In this section we give a proof for Theorem 1.1. Recall that \mathcal{U} denotes a unital in $PG(2, q^2)$ fixed by the linear collineation group K of order $6(q + 1)^2$, which we assume fixes no point or line. First suppose that $q = 5$ and hence K is a Hessian group of order 216 by Theorem 3.1. As we have already noticed, there is only one Hessian group of order 216, up to projective equivalence in $PG(2, 25)$. By a computer-aided exhaustive search with the software package MAGMA [9], it is then possible to check that the classical one is the only unital fixed by the Hessian group of order 216. Hence we may assume $q \neq 5$, and thus by the previous section we know K fixes a triangle. Moreover, we may take a projective frame such that K is given by $Sym_3\Phi$, where Φ was introduced in Section 2. Looking at the description of the Φ -orbits previously given, it is clear that any unital fixed by Φ must consist of three Φ -orbits of size $q + 1$ and $q - 2$ Φ -orbits of size $(q + 1)^2$. Consider the resulting $q + 1$ Φ -orbits comprising \mathcal{U} , and let Δ denote the corresponding set of $q + 1$ points in the projective plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ defined in Section 2. To show \mathcal{U} is a (non-degenerate) Hermitian curve, it suffices to show Δ is a line of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$. That is, we must show every line of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ meets Δ nontrivially, or equivalently, show that every Hermitian curve $H(\alpha, \beta, \gamma)$ intersects \mathcal{U} nontrivially. Hence the argument is completed by the following result.

Theorem 4.1 *Each Hermitian curve $H(\alpha, \beta, \gamma)$ meets \mathcal{U} nontrivially.*

Proof: We note first that each Hermitian curve $H(\alpha, \beta, \gamma)$ is a codeword in the $GF(p)$ -code generated the lines of $PG(2, q^2)$. This is certainly true for a reducible $H(\alpha, \beta, \gamma)$ as its point-set is a union of lines, while, for irreducible Hermitian curves, the claim is a result due to Blokhuis, Brouwer and Wilbrink [7] (see also [4]). Now, Theorem 4.1 follows by the argument given in the proof of Proposition 3.2 of [10]. \square

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