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A Generalized Vandermonde Determinant

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Abstract. We prove two determinantal identities that generalize the Vandermonde determinant identity

$$\det\left(x_i^j\right)_{i,j=0,\ldots,m} = \prod_{0\leq i<j\leq m} (x_j - x_i).$$

In the first of our identities the set $\{0, \ldots, m\}$ indexing the rows and columns of the determinant is replaced by an arbitrary finite order ideal in the set of sequences of nonnegative integers which are 0 except for a finite number of components. In the second the index set is replaced by an arbitrary finite order ideal in the set of all partitions.

Keywords: Vandermonde, determinant, partition, ideal

In this paper we derive two determinantal identities that generalize the Vandermonde determinant identity

$$\det\left(x_i^j\right)_{i,j=0,\ldots,m} = \prod_{0\leq i<j\leq m} (x_j - x_i).$$

Let us denote by $\mathcal{S}$ the set of all sequences $(\lambda_0, \lambda_1, \ldots)$ of nonnegative integers with all but a finite number of the λ's equal to zero. For brevity we also use $(\lambda_0, \lambda_1, \ldots, \lambda_k)$ for the sequence $(\lambda_0, \lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$. $\mathcal{S}$ has a natural partial ordering: $\lambda \leq \mu$ if $\lambda_k \leq \mu_k$ for $k = 0, 1, 2, \ldots$. By an *order ideal* in $\mathcal{S}$ we mean a subset I of $\mathcal{S}$ such that if $\mu \in I$ and $\lambda \in \mathcal{S}$ and $\lambda \leq \mu$, then $\lambda \in I$. In the first of our identities the set $\{0, \ldots, m\}$ indexing the rows and columns of the Vandermonde determinant is replaced by an arbitrary finite order ideal in $\mathcal{S}$. By a *partition* we mean an element of $\mathcal{S}$ which is weakly decreasing. Let $\mathcal{P}$ denote the set of all partitions. An order ideal in $\mathcal{P}$ is a subset I of $\mathcal{P}$ with the property that if $\lambda, \mu \in \mathcal{P}$ and $\lambda \leq \mu$ and $\mu \in I$, then $\lambda \in I$. In the second identity the index set is an arbitrary finite order ideal in the set of all partitions. This identity involves the shift operator s which assigns to an element $\lambda = (\lambda_0, \lambda_1, \ldots)$ of $\mathcal{S}$ the sequence $s(\lambda) = (\lambda_1, \lambda_2, \ldots)$.

Our identities involve a two-dimensional array x_{ij}, $i, j \geq 0$, of indeterminates. For each pair $\lambda, \mu \in \mathcal{S}$ we define a monomial x_λ^μ in the x's by

$$x_\lambda^\mu = \prod_{k=0}^{\infty} x_{k,\lambda_k}^{\mu_k}.$$

Suppose that λ and μ are partitions with $\lambda < \mu$. We say that μ *covers* λ (and write $\lambda \prec \mu$) if λ and μ differ in exactly one coordinate. Suppose that μ covers λ and differs from λ in just the rth coordinate. Then we denote by $\Delta x(\mu, \lambda)$ the binomial difference $x_{r,\mu_r} - x_{r,\lambda_r}$.

Now we can state our results:

THEOREM 1. *Let I be a finite order ideal in $\mathcal{S}$. Then*

$$\det(x_\lambda^\mu)_{\lambda,\mu\in I} = \prod_{\substack{\lambda,\mu\in I\\ \lambda\prec\mu}} \Delta x(\mu, \lambda).$$

THEOREM 2. *Let I be a finite order ideal in $\mathcal{P}$. Then*

$$\det(x_\lambda^\mu)_{\lambda,\mu\in I} = \prod_{\lambda\in I} x_\lambda^{s(\lambda)} \prod_{\substack{\lambda,\mu\in I\\ \lambda\prec\mu}} \Delta x(\mu, \lambda).$$

One sees immediately that if I is the ideal consisting of the partitions (0), (1), ..., (m), then we recover the standard Vandermonde determinant as special cases of both Theorem 1 and Theorem 2. The proof of Theorem 1 is just like that of Theorem 2, only simpler. Thus in what follows we discuss only the proof of Theorem 2. Here is an example of Theorem 2.

Example. Let I be the order ideal of $\mathcal{P}$ consisting of the partitions (0), (1), (2), (11), and (21). To simplify the notation for this example, let us replace the variables

$$x_{1,0}, x_{1,1}, x_{1,2}, \ldots \quad \text{by} \quad y_0, y_1, y_2, \ldots.$$

If we order the rows and columns as we did the five partitions above, then the matrix in our theorem is

$$M = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0y_0 & x_0^2y_0 \\ 1 & x_1 & x_1^2 & x_1y_0 & x_1^2y_0 \\ 1 & x_2 & x_2^2 & x_2y_0 & x_2^2y_0 \\ 1 & x_1 & x_1^2 & x_1y_1 & x_1^2y_1 \\ 1 & x_2 & x_2^2 & x_2y_1 & x_2^2y_1 \end{bmatrix}.$$

Theorem 2 states that

$$\det M = x_1x_2(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)^2(y_1 - y_0)^2.$$

Here the monomial factor x_1 arises from the partition (11), and x_2 arises from (21). The six binomial factors arise from the six covering pairs {(0), (1)}, {(0), (2)}, {(1), (2)}, {(1), (11)}, {(2), (21)}, and {(11), (21)}.

Implicit in the proof we will give is a method for computing the determinant. We now illustrate the method for this example. Let I be the ideal in the example. The largest index of any nonzero part of a partition in I is 1, and the largest value of such a part is 1. The two partitions in I that achieve this maximum are (11) and (21). To reduce M to block-triangular form, we subtract from the (11) and (21) rows appropriate combinations of rows corresponding to partitions which they cover. In this case we subtract the (1) row from the (11) row and the (2) row from the (21) row. Then M becomes

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0y_0 & x_0^2y_0 \\ 1 & x_1 & x_1^2 & x_1y_0 & x_1^2y_0 \\ 1 & x_2 & x_2^2 & x_2y_0 & x_2^2y_0 \\ 0 & 0 & 0 & x_1(y_1-y_0) & x_1^2(y_1-y_0) \\ 0 & 0 & 0 & x_2(y_1-y_0) & x_2^2(y_1-y_0) \end{bmatrix}.$$

We factor out $x_1(y_1-y_0)$ from the fourth row and $x_2(y_1-y_0)$ from the last row. The determinant is now the product of two Vandermonde determinants, and we easily verify that Theorem 2 is correct in this case.

The general proof is just a more elaborate version of the preceding computation. It requires a simple property of Vandermonde matrices given in the lemma below.

LEMMA. *Let V_m denote the Vandermonde matrix $\left(x_i^j\right)_{i,j=0,\dots,m}$. Then there is a linear combination of the first m rows of V_m which, when subtracted from the last row, reduces the last row to*

$$(0, 0, \dots, 0, (x_m - x_0)(x_m - x_1)\cdots(x_m - x_{m-1})).$$

Proof of Lemma. Since the upper-left m-by-m minor of V_m is also Vandermonde, it is nonsingular. Thus a linear combination of the first m rows can reduce the last row to the form $(0, \dots, 0, c)$ for some c. But then we have

$$\det V_m = c \det V_{m-1}.$$

The lemma then follows immediately from the standard Vandermonde determinant formula. □

Proof of Theorem 2. Let I be an order ideal in the set of all partitions. Let M be the matrix $(x_\lambda^\mu)_{\lambda,\mu\in I}$. If I is empty or consists of just the all-zero partition, the result is trivial. We therefore assume that I contains a nonzero partition.

Let r be the largest integer such that I contains a partition λ with $\lambda_r > 0$. We may assume by induction that Theorem 2 is true for order ideals containing fewer partitions than I. Let m be the largest value of λ_r for all partitions λ in I. Then $m > 0$.

Let J be the subset of I consisting of those partitions λ of I such that $\lambda_r < m$. Let K be the set of those partitions in I for which $\lambda_r = m$. Then I is the disjoint union of the nonempty sets J and K, so that J and K determine a block decomposition of M into the four blocks M_{JJ}, M_{JK}, M_{KJ}, and M_{KK}.

Next we show how one may subtract a linear combination of J rows from K rows in order to make the block M_{KJ} zero. Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r-1}, m)$ be any partition in K. Consider the submatrix of M whose rows are indexed by the partitions $(\lambda_0, \dots, \lambda_{r-1}, k)$, $k = 0, \dots, m$. These partitions are part of I since I is an order ideal. Moreover, those with $k < m$ are precisely the partitions of J that are covered by λ. Each column of this submatrix is one of the columns

$$\left[x^j_{r,0}, \dots, x^j_{r,m}\right]^t$$

multiplied by a monomial in the other x's. Hence, by applying the same row operations given by the lemma, we can reduce the row with index λ so that all its entries are zero except those in the columns from K. If $\mu \in K$, then the (λ, μ) entry of M, which was originally x^μ_λ, is replaced by

$$x^{\mu_0}_{0,\lambda_0} x^{\mu_1}_{1,\lambda_1} \cdots x^{\mu_{r-1}}_{r-1,\lambda_{r-1}} (x_{r,m} - x_{r,0})(x_{r,m} - x_{r,1}) \cdots (x_{r,m} - x_{r,m-1}).$$

Let us denote by M' the KK block after the preceding reduction. Then

$$\det M = \det M_{JJ} \cdot \det M'.$$

By induction we can use Theorem 2 to compute det M_{JJ}.

Now we consider M'. Let ν denote the partition $(m, m, \dots, m)$ with $r + 1$ initial parts equal to m and the rest of the parts equal to 0. For $\lambda \in K$ let λ' denote the partition obtained by subtracting m from the parts of λ. Then $\lambda = \lambda' + \nu$. Define the order ideal $K' = \{\lambda' | \lambda \in K\}$. Suppose that, for $\mu \in K$, we divide the μth row of M' by

$$x^m_{0,\mu_0} \cdots x^m_{r-1,\mu_{r-1}} (x_{r,m} - x_{r,0}) \cdots (x_{r,m} - x_{r,m-1}) = x^{s(\nu)}_\mu \prod_{\substack{\lambda \in J \\ \lambda \prec \mu}} \Delta x(\mu, \lambda).$$

Then we obtain a matrix M'' with

$$\det M = \det M_{JJ} \cdot \left[\prod_{\mu \in K} x^{s(\nu)}_\mu\right] \cdot \left[\prod_{\substack{\lambda \in J, \mu \in K \\ \lambda \prec \mu}} \Delta x(\mu, \lambda)\right] \cdot \det M''.$$

Suppose that we introduce new variables $y_{i,j} = x_{i,m+j}$. Then M'' becomes the matrix

$$\left(y^{\mu'}_{\lambda'}\right)_{\lambda', \mu' \in K'},$$

so we can apply Theorem 2 (inductively) to evaluate det M''. After applying Theorem 2 and replacing the y's by the x's, we find that

$$\det M'' = \prod_{\lambda \in K} x_\lambda^{s(\lambda-\nu)} \prod_{\substack{\lambda,\mu\in K\\ \lambda\prec\mu}} \Delta x(\mu, \lambda).$$

Also,

$$\det M_{JJ} = \prod_{\lambda \in J} x_\lambda^{s(\lambda)} \prod_{\substack{\lambda,\mu\in J\\ \lambda\prec\mu}} \Delta x(\mu, \lambda).$$

Theorem 2 follows by combining the preceding three equations. □

Remark. The study of these determinants arose as follows. Let m be a positive integer, and let P_m denote the set of partitions of m, i.e., partitions λ with $\sum_{k\geq 0} \lambda_k = m$. The set of partitions P_m is not an order ideal. However, if we apply to each $\lambda \in P_m$ the operator which subtracts 1 from each of the nonzero parts of λ, then we obtain an order ideal Q_m. For any two partitions λ, μ let $\lambda \cdot \mu = \sum_{k\geq 0} \lambda_k \mu_k$. In [1] it was necessary to show that the matrix $(q^{\lambda\cdot\mu})_{\lambda,\mu\in Q_m}$ was nonsingular for certain values of q. Out of curiosity we computed the determinants of some of these matrices and were surprised by the factorization of these determinants into products of powers of q and powers of binomials of the form $q^i - 1$. This led to the discovery of Theorem 2, which immediately implies the observed factorization when we replace the variables $x_{i,j}$ with q^i.

Reference

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