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Bruhat Order for Two Flags and a Line

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Abstract. The classical Ehresmann-Bruhat order describes the possible degenerations of a pair of flags in a linear space V under linear transformations of V; or equivalently, it describes the closure of an orbit of $\mathrm{GL}(V)$ acting diagonally on the product of two flag varieties.

We consider the degenerations of a triple consisting of two flags and a line, or equivalently the closure of an orbit of $\mathrm{GL}(V)$ acting diagonally on the product of two flag varieties and a projective space. We give a simple rank criterion to decide whether one triple can degenerate to another. We also classify the minimal degenerations, which involve not only reflections (i.e., transpositions) in the Weyl group S_n, $n = \dim(V)$, but also cycles of arbitrary length. Our proofs use only elementary linear algebra and combinatorics.

Keywords: quiver representations, multiple flags, degeneration, geometric order

1. Introduction

1.1. A line and two flags

We shall deal with certain configurations of linear subspaces in $\mathbb{C}^n$ (or any vector space V). A configuration $F = (A, B_\bullet, C_\bullet)$ consists of a line $A \subset \mathbb{C}^n$ and two flags of subspaces of fixed dimensions, $B_\bullet = (B_1 \subset B_2 \subset \cdots \subset \mathbb{C}^n)$ and $C_\bullet = (C_1 \subset C_2 \subset \cdots \subset \mathbb{C}^n)$. In this Introduction, we restrict ourselves to the case in which $B_\bullet, C_\bullet$ are full flags: $\dim B_i = \dim C_i = i$ for $i = 0, 1, 2, \ldots, n$.

Our aim is to describe such configurations up to a linear change of coordinates in $\mathbb{C}^n$, and the ways in which more generic configurations can degenerate to more special ones. One could ask this question for configurations of arbitrarily many flags; however in general it is ‘wild’ problem. The distinguishing feature of our case is that there are only *finitely many* configuration types $F = (A, B_\bullet, C_\bullet)$, as we showed in a previous work [5] with Weyman and Zelevinsky.[1]

For example, there exists a *most generic* type $F_{\max}$, which degenerates to all other types. It consists of those configurations which can be written in terms of some basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ as:

$$A = \langle v_1 + v_2 + \cdots + v_n \rangle, \quad B_i = \langle v_1, v_2, \ldots, v_i \rangle, \quad C_i = \langle v_n, v_{n-1}, \ldots, v_{n-i+1} \rangle.$$

(Here $\langle v_1, v_2, \ldots \rangle$ means the linear span of $v_1, v_2, \ldots$) There is also a *most special* configuration type $F_{\min}$:

$$A = \langle v_1 \rangle, \quad B_i = C_i = \langle v_1, v_2, \ldots, v_i \rangle.$$

The configurations of a more generic type can be made to degenerate to more special ones by letting some of the basis vectors v_i approach each other, so that in the limit some of the spaces A, B_i, C_j increase their intersections.[2] Geometrically, a configuration type is an orbit of $\mathrm{GL}_n(\mathbb{C})$ acting diagonally on the product $\mathbb{P}^{n-1}\times\mathrm{Flag}(\mathbb{C}^n)\times\mathrm{Flag}(\mathbb{C}^n)$, with $F_{\max}$ the open orbit and $F_{\min}$ the unique closed orbit. Degeneration of configuration types means the topological closure of a large orbit contains a smaller orbit.

We seek a simple combinatorial description of all degenerations. The trivial case of $n = 2$ is illustrated by a diagram in Section 1.3.

Our problem is directly analogous to the classical case in which the configurations consist of two flags only: $F = (B_\bullet, C_\bullet)$. This theory originated with Schubert and Ehresmann; a good introduction is [4]. In this case, the configurations (up to change of basis in $\mathbb{C}^n$) correspond to permutations $w \in S_n$: the configuration type F_w consists of the double flags which can be written as:

$$B_i = \langle v_1, v_2, \ldots, v_i\rangle, \quad C_i = \left\langle v_{w(1)}, v_{w(2)}, \ldots, v_{w(i)}\right\rangle$$

for some basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$. A configuration type F_w is a degeneration of another F_y exactly if $w \le y$ in the *Bruhat order* on S_n. Namely, $w \le y$ iff

$$\#(\,[i] \cap w[j]\,) \ge \#(\,[i] \cap y[j]\,)$$

for all $1 \le i, j \le n$, where $[i] := \{1, 2, \ldots, i\}$ and $w[j] := \{w(1), w(2), \ldots, w(j)\}$. This *tableau criterion* has the geometric meaning:

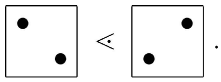
$$\#(\,[i]\cap w[j]\,) = \dim(B_i \cap C_j)$$

for $(B_\bullet, C_\bullet)$ of type F_w. The more special configuration F_w has larger intersections among its spaces than the more generic F_y.

We can also describe the classical Bruhat order in terms of its covers: $w \lessdot y$ iff $y = (i_0, i_1)\cdot w$ for some transposition $(i_0, i_1) \in S_n$ and $\ell(y) = 1 + \ell(w)$, where $\ell(w)$ is the number of inversions of w. We can picture this definition in terms of the permutation matrices $M_w = (m_{ij})$ and $M_y = (m'_{ij})$, where $m_{ij} := \delta_{w(i),j}$, $m'_{ij} := \delta_{y(i),j}$. Then $w \lessdot y$ means that we have a pair of entries $m_{i_0 j_0} = m_{i_1 j_1} = 1$ with (i_0, j_0) northwest of (i_1, j_1), and no other 1's in the rectangle $[i_0, i_1]\times[j_0, j_1]$; and we flip these two 'diagonal' entries in M_w to the corresponding anti-diagonal, obtaining M_y:

$$w = \begin{array}{c} \\ i_0 \\ \\ i_1 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} j_0 & & j_1 \end{array} \\ \boxed{\begin{array}{ccccccc} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & 0 & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{array}} \end{array} \lessdot\; y = \begin{array}{c} \\ i_0 \\ \\ i_1 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} j_0 & & j_1 \end{array} \\ \boxed{\begin{array}{ccccccc} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & 0 & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{array}} \end{array};$$

or in compact notation, with 1 replaced by $\bullet$ and all unaffected rows and columns omitted:



In terms of transpositions: $y = (i_0, i_1) \cdot w = w \cdot (j_0, j_1)$.

We give a full exposition and proof of these classical results in Sections 2.1 and 3.

1.2. Bruhat order

Let us return to our case of a line and two flags. As we showed in [5], we can index our configuration types by *decorated permutations* (w, Δ), where $\Delta = \{j_1 < j_2 < \cdots < j_t\}$ is any non-empty descending subsequence of w, meaning $w(j_1) > w(j_2) > \cdots > w(j_t)$. In the corresponding configuration $F_{w,\Delta}$, the permutation w describes the relative positions of $B_\bullet$ and $C_\bullet$ in terms of a basis $v_1, \ldots, v_n$, just as before; and Δ defines the extra line:

$$A = \langle v_{j_1} + v_{j_2} + \cdots + v_{j_t} \rangle.$$

Thus, the generic $F_{\max}$ is $F_{w,\Delta}$ for $w = w_0 = \underline{n, n-1, \ldots, 2, 1}$, the longest permutation, and $\Delta = \{1, 2, \ldots, n\}$. The most special $F_{\min}$ is $F_{w,\Delta}$ for $w = \mathrm{id} = \underline{1, 2, \ldots, n}$ and $\Delta = \{1\}$. We can picture a decorated permutation as a permutation matrix with circles around the positions $(w(j), j)$ for $j \in \Delta$. For example,  corresponds to $w = \underline{312}$, $\Delta = \{1, 3\}$.

We once again have a degeneration or Bruhat order, described combinatorially by a tableau criterion in terms of certain *rank numbers* which measure intersections of spaces in a configuration $(A, B_\bullet, C_\bullet)$ in $F_{w,\Delta}$. Namely, let

$$r_{ij}(w) := \dim(B_i \cap C_j) = \#(\,[i] \cap w[j]\,)$$

as before, and

$$\begin{aligned} r_{\langle ij \rangle}(w, \Delta) &:= \dim(B_i \cap C_j) + \dim(A \cap (B_i + C_j)) \\ &= \#(\,[i] \cap w[j]\,) + \delta_{ij}(w, \Delta), \end{aligned}$$

where

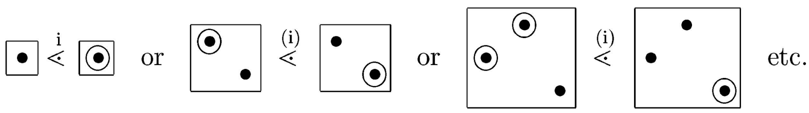
$$\delta_{ij}(w, \Delta) := \begin{cases} 1 & \text{if for all } k \in \Delta, \ \ k \le i \text{ or } w(k) \le j \\ 0 & \text{otherwise.} \end{cases}$$

We can realize this in terms of linear algebra by defining $\phi_{ij} : B_i \times C_j \to \mathbb{C}^n/A$, $(v_1, v_2) \mapsto v_1 + v_2 \bmod A$: then $r_{\langle ij \rangle}(w, \Delta) = \dim \operatorname{Ker} \phi_{ij}$. These definitions are suggested by quiver theory: see Section 1.4 below. We will show that our geometric degeneration order has the following combinatorial description:

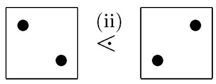
$$(w, \Delta) \leq (y, \Gamma) \Leftrightarrow \begin{cases} r_{ij}(w, \Delta) \geq r_{ij}(y, \Gamma) \\ r_{\langle ij \rangle}(w, \Delta) \geq r_{\langle ij \rangle}(y, \Gamma). \\ \quad \text{for all } 0 \leq i, j \leq n \end{cases}$$

Finally, we can classify the covers $(w, \Delta) \lessdot (y, \Gamma)$ of our new Bruhat order. Remarkably, in many of the cases below the pair $w < y$ is *not* a cover in the classical Bruhat order. We describe the covers in terms of certain flipping moves which we write in compact notation (again, with all unaffected rows and columns omitted). We describe how a more generic configuration $(A, B_\bullet, C_\bullet)$ (on the right) degenerates to a more special configuration (on the left).

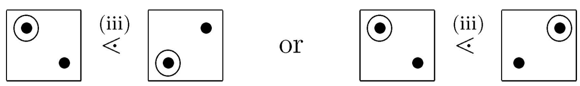
MOVE (i) The line A moves into one of the spaces $B_i + C_j$, leaving $B_\bullet, C_\bullet$ unchanged:



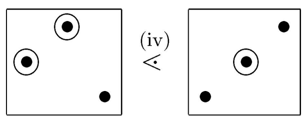
MOVE (ii) One of the B_i moves further into one of the C_j, leaving A unchanged:



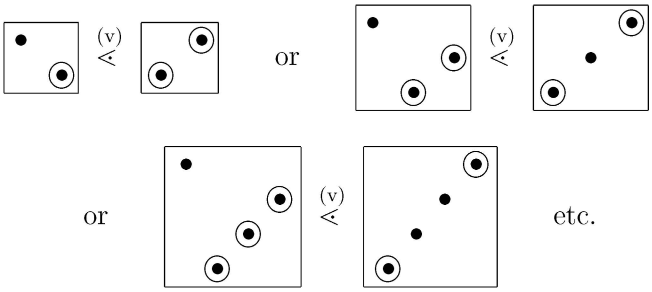
MOVE (iii) The line A lies in $B_i + C_j$. Then A moves into some $B_{i'} \subset B_i$, and so does the corresponding line in C_j. Alternatively, reverse the roles of B_i and C_j.



MOVE (iv) The line A lies in $B_i - C_j$, but not in $B_{i'} + C_{j'}$, where $B_{i'} \subset B_i$ and $C_{j'} \subset C_j$. Then A moves into $B_{i'} + C_{j'}$, and the corresponding line in $B_i + C_j$ moves with it.



MOVE (v) The line A lies in $B_i + C_j$. Then B_i moves further into C_j, but A does *not* move with it, remaining outside $B_i \cap C_j$.



Note that the underlying permutations in this move may differ by an arbitrary-length cycle in S_n, not necessarily a transposition.

As in the classical case of two flags, certain regions enclosed by the affected dots must be empty for these moves to define covers $\lessdot$ (though they always define relations $<$). See Section 2.3. The above moves may seem complicated, but they are unavoidable in any computationally effective description: the minimal degenerations are what they are.

We showed in [5] that the number of parameters of a configuration type (i.e., its dimension when thought of as a GL_n-orbit in $\mathbb{P}^{n-1} \times \mathrm{Flag}(\mathbb{C}^n) \times \mathrm{Flag}(\mathbb{C}^n)$) is:

$$\dim(F_{w,\Delta}) = \binom{n}{2} + (n-1) + \ell(w) - \#\left\{ j \;\middle|\; \begin{array}{c} \text{for all } k \in \Delta, \\ k < j \text{ or } w(k) < w(j) \end{array} \right\}.$$

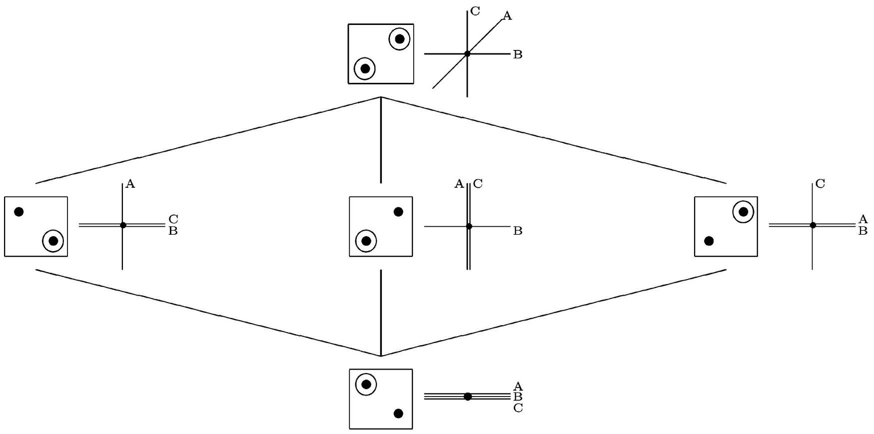
For example, $F_{\min}$ has dimension $\binom{n}{2} + (n-1) + 0 - (n-1) = \binom{n}{2}$. Indeed, the minimal orbit is isomorphic to $\mathrm{Flag}(\mathbb{C}^n)$.

It is easily seen from the description of the moves (i)–(v), together with the dot-vanishing conditions in Section 2.3, that each move increases the dimension by one. Thus, our Bruhat order is a poset ranked by $\dim(F) - \dim(F_{\min})$. (This is no longer true if $(B_\bullet, C_\bullet)$ are partial flags, and it is not clear whether our poset is ranked.)

We conjecture that a refinement of the move-labels (i)–(v) on the covers of our poset will give a lexicographic shelling similar to that of Edelman [3] for (undecorated) permutations.

1.3. *Examples $n = 2, 3$*

We illustrate our constructions in the simplest cases. Let $n = 2$. Then the Hasse diagram of our Bruhat order is:



Next to each decorated permutation, we have sketched the corresponding lines A, $B = B_1$, $C = C_1$ in $\mathbb{C}^2$, with $=\!=\!\!\,{}^{A}_{B}$ indicating that A and B coincide.

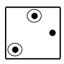
The elements of our poset correspond to the GL_2-orbits on $(\mathbb{P}^1)^3 = (\mathbb{P}^1) \times \mathrm{Flag}(\mathbb{C}^2) \times \mathrm{Flag}(\mathbb{C}^2)$: the minimal element is the full diagonal $\mathbb{P}^1 \subset (\mathbb{P}^1)^3$; the mid-level elements are the three partial diagonals, homeomorphic to $\mathbb{P}^1 \times \mathbb{C}$; and the maximal element is the generic orbit, homeomorphic to $\mathbb{P}^1 \times \mathbb{C} \times \mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C} \backslash \{\mathrm{pt}\}$.

Note that this maximal orbit is not a topological cell, even after fibering out $\mathbb{P}^1$. For general n, the maximal orbit is isomorphic to $GL_n/\mathbb{C}^\times = PGL_n$, having fundamental group $\mathbb{Z}/n\mathbb{Z}$. It will be a topic for another paper to understand the geometry of the orbit closures; however this example indicates that they have a non-trivial, but manageable topology.

Now let $n = 3$. We can enumerate the configuration types by counting the possible decorations (decreasing subsequences) of each permutation. The identity permutation has $n = 3$ decorations, the longest permutation has $2^n - 1 = 7$.

$$\begin{array}{ccccccccccccc} 3 & + & 4 & + & 4 & + & 5 & + & 5 & + & 7 & = 28 \\ \underline{123} & & \underline{213} & & \underline{132} & & \underline{231} & & \underline{312} & & \underline{321} & \end{array}$$

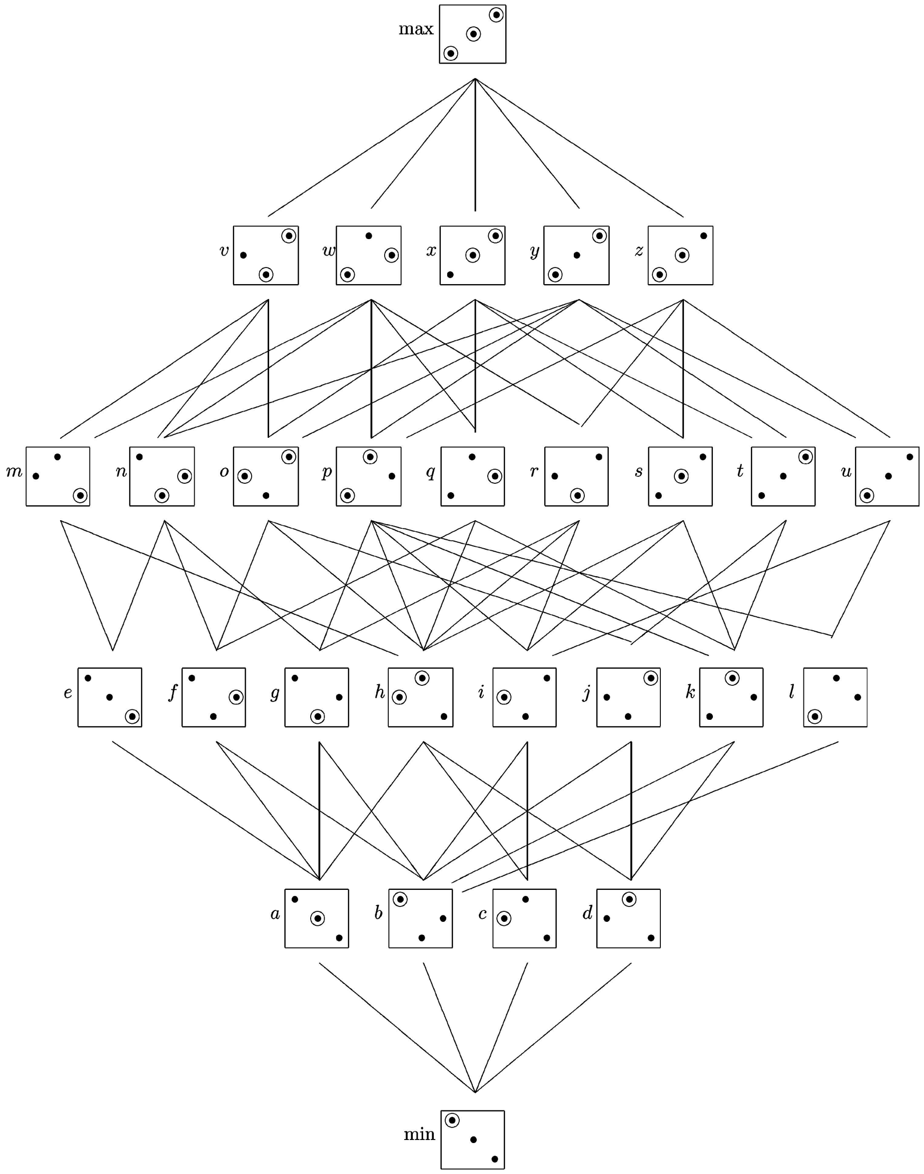
The Hasse diagram appears on the next page. We have labelled the elements min, $a, b, \ldots, x$, y, z, max, as indicated. For example,

$p = (\underline{312}, \{1, 2\}) =$  .

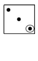
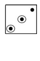
The 72 covering relations, each coming from a move of type (i)–(v), are:

min $\lessdot a$	min $\lessdot b$	min $\lessdot c$	min $\lessdot d$	$a \lessdot e$	$a \lessdot f$	$a \lessdot g$	$a \lessdot h$	
$b \lessdot f$	$b \lessdot g$	$b \lessdot i$	$b \lessdot j$	$b \lessdot k$	$b \lessdot l$	$c \lessdot h$	$c \lessdot i$	$c \lessdot l$
$d \lessdot h$	$d \lessdot j$	$d \lessdot k$	$e \lessdot m$	$e \lessdot n$	$f \lessdot n$	$f \lessdot o$	$f \lessdot q$	
$g \lessdot n$	$g \lessdot p$	$g \lessdot r$	$h \lessdot m$	$h \lessdot o$	$h \lessdot p$	$h \lessdot q$	$h \lessdot r$	$h \lessdot s$
$i \lessdot p$	$i \lessdot r$	$i \lessdot s$	$i \lessdot u$	$j \lessdot o$	$j \lessdot t$			
$k \lessdot p$	$k \lessdot q$	$k \lessdot s$	$k \lessdot t$	$l \lessdot p$	$l \lessdot u$	$m \lessdot v$	$m \lessdot w$	
$n \lessdot v$	$n \lessdot w$	$n \lessdot y$	$o \lessdot v$	$o \lessdot x$	$o \lessdot y$	$p \lessdot w$	$p \lessdot y$	$p \lessdot z$
$q \lessdot w$	$q \lessdot x$	$r \lessdot w$	$r \lessdot z$	$s \lessdot x$	$s \lessdot z$	$t \lessdot x$	$t \lessdot y$	
$u \lessdot y$	$u \lessdot z$	$v \lessdot$ max	$w \lessdot$ max	$x \lessdot$ max	$y \lessdot$ max			

The elements in the ith rank of the poset have orbit dimension $i + \dim(F_{\min}) = i + 3$.



As an illustration of the tableau criterion (i.e., rank numbers defining the Bruhat order), let us check that $e \not\leq z$: that is,

 $= (\underline{123}, \{3\}) = (A, B_\bullet, C_\bullet)$ and  $= (\underline{321}, \{1, 2\}) = (A', B'_\bullet, C'_\bullet)$

are unrelated elements in our poset, even though $\underline{123} < \underline{321}$ in the classical Bruhat order. Indeed, in the second configuration, $A' = \langle v_1 + v_2 \rangle \subset B'_2 = \langle v_1, v_2 \rangle$, and no degeneration of $(A', B'_\bullet, C'_\bullet)$ can destroy this containment. However, in the first configuration, $A = \langle v_3 \rangle \not\subset B_2 = \langle v_1, v_2 \rangle$. Thus $(\underline{123}, \{3\}) \not\leq (\underline{321}, \{1, 2\})$. In terms of our rank numbers: $r_{ij}(\underline{123}) \geq r_{ij}(\underline{321})$ for all i, j, in particular $r_{11}(\underline{123}) > r_{11}(\underline{321})$; but $r_{\langle 20 \rangle}(\underline{123}, \{3\}) < r_{\langle 20 \rangle}(\underline{321}, \{1, 2\})$.

1.4. Structure of the paper

Now we sketch our proof of the above results. After some easy geometric arguments, we reduce our claims to a rather difficult combinatorial lemma. The idea is to approximate the geometric degeneration order from above and below by combinatorially defined orders, and then show that these combinatorial bounds are equal.

To begin, we distinguish in Section 2 three partial orders on decorated permutations (w, Δ). First, our geometric order $\overset{\text{deg}}{\leq}$ defined by degenerations of the corresponding configuration types $F_{w,\Delta}$. Second, the combinatorial order $\overset{\text{rk}}{\leq}$ defined in terms of the rank numbers $r_{ij}(w)$, $r_{\langle ij \rangle}(w, \Delta)$. Third, the order $\overset{\text{mv}}{\leq}$ generated by repeated application of our moves $\overset{\text{i}}{\lessdot}, \ldots, \overset{\text{v}}{\lessdot}$. We wish to show the equivalence of these three orders.

Some simple geometry and linear algebra in Section 4 suffices to show that:

$$(w, \Delta) \overset{\text{mv}}{\leq} (y, \Gamma) \Rightarrow (w, \Delta) \overset{\text{deg}}{\leq} (y, \Gamma) \Rightarrow (w, \Delta) \overset{\text{rk}}{\leq} (y, \Gamma).$$

That is, any move corresponds to a degeneration, and any degeneration increases the rank numbers. We are then left in Section 5 to show the purely combinatorial assertion:

$$(w, \Delta) \overset{\text{rk}}{\leq} (y, \Gamma) \Rightarrow (w, \Delta) \overset{\text{mv}}{\leq} (y, \Gamma).$$

Given a relation $(w, \Delta) \overset{\text{rk}}{<} (y, \Gamma)$, we find a move $(w, \Delta) \overset{\text{mv}}{\lessdot} (\tilde{w}, \tilde{\Delta})$ such that the smaller rank numbers of $(\tilde{w}, \tilde{\Delta})$ still dominate those of (y, Γ):

$$(w, \Delta) \overset{\text{mv}}{\lessdot} (\tilde{w}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (y, \Gamma).$$

Iterating this construction within our finite poset, we eventually get

$$(w, \Delta) \overset{\text{mv}}{\lessdot} (\tilde{w}_1, \tilde{\Delta}_1) \overset{\text{mv}}{\lessdot} \cdots \overset{\text{mv}}{\lessdot} (\tilde{w}_k, \tilde{\Delta}_k) = (y, \Gamma).$$

Throughout our proof, we work in the more general case where $B_\bullet, C_\bullet$ are arbitrary partial flags, with orbits indexed not by permutations but by double cosets of permutations, or "transport matrices", as defined in Section 2.1. Also, our proofs are characteristic-free: our vector spaces are over an arbitrary infinite field $\mathbf{k}$, not necessarily $\mathbb{C}$.

Our Rank Theorem, giving the equivalence of $\overset{\text{deg}}{\leq}$ and $\overset{\text{rk}}{\leq}$, is a strengthened converse to Proposition 4.5 in our work [5], which relied heavily on quiver theory. In the notation of [5], for a triple flag $X = (A, B_\bullet, C_\bullet)$, we have: $r_{ij}(X) = \dim \operatorname{Hom}(I_{\{(i,j)\}}, X)$ and $r_{\langle ij \rangle}(X) = \dim \operatorname{Hom}(I_{\{(i,r),(q,j)\}}, X)$.

Our approach is closely related to that of Zwara and Skowronski [8, 9]; Bongartz [1], Riedtmann [7] et al., who considered the degeneration order on quiver representations. However, in our special case, our results are sharper than those of the general theory. Our description of the covering moves (i)–(v) can be deduced (with non-trivial work) from Zwara's results on the extension order $\overset{\text{ext}}{\leq}$ in [8, 9]. But our results about the rank order $\overset{\text{rk}}{\leq}$ are considerably stronger than any of the corresponding general results: our order requires computing Hom with only a few indecomposables, rather than all.

2. Results

2.1. *Two flags*

In order to establish the notation for our main theorem in its full generality, we first state the classical theory for two flags.

Throughout this paper, all vector spaces are over a fixed field $\mathbf{k}$ of arbitrary characteristic, infinite but not necessarily algebraically closed; and we fix a vector space V of dimension n with standard basis $e_1, \dots, e_n$. Let $\mathbf{b} = (b_1, \dots, b_q)$ be a list of positive integers with sum equal to n: that is, a *composition* of n. We denote by Flag($\mathbf{b}$) the variety of partial flags $B_\bullet = (0 = B_0 \subset B_1 \subset \cdots \subset B_q = V)$ of vector subspaces in V such that

$$\dim(B_i/B_{i-1}) = b_i \quad (i = 1, \dots, q).$$

Flag($\mathbf{b}$) is a homogeneous space under the natural action of the general linear group $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbf{k})$.

Let us fix two compositions of n, $\mathbf{b} = (b_1, \dots, b_q)$ and $\mathbf{c} = (c_1, \dots, c_r)$. The Schubert (or Bruhat) decomposition classifies the orbits of $GL(V)$ acting diagonally on the double flag variety Flag($\mathbf{b}$) $\times$ Flag($\mathbf{c}$). We index these orbits by *transport matrices* $M = (m_{ij})$, which are $q \times r$ matrices of nonnegative integers m_{ij} with row sums $b_i = \sum_j m_{ij}$ and column sums $c_j = \sum_i m_{ij}$ (so that the sum of all entries is n). If $\mathbf{b} = \mathbf{c} = (1, \dots, 1) = (1^n)$, then Flag($\mathbf{b}$)$\times$Flag($\mathbf{c}$) consists of pairs of full flags, and each transport matrix is the permutation matrix $M = M_w$ corresponding to a $w \in S_n$, with $m_{w(i),i} = 1$ and $m_{ij} = 0$ otherwise.

Given a transport matrix M, we define the orbit $F_M \subset$ Flag($\mathbf{b}$) $\times$ Flag($\mathbf{c}$) as the following set of double flags $(B_\bullet, C_\bullet)$. Given any basis of V with the n vectors indexed as:

$$V = \left\langle v_{ijk} \;\middle|\; \begin{array}{c} (i, j) \in [q] \times [r] \\ 1 \leq k \leq m_{ij} \end{array} \right\rangle,$$

where $[q] = [1, q] := \{1, 2, \dots, q\}$, let

$$B_i := \langle v_{i'jk} \mid i' \leq i \rangle, \quad C_j := \langle v_{ij'k} \mid j' \leq j \rangle,$$

where $\langle\ \rangle$ denotes linear span. As the basis $\langle v_{ijk}\rangle$ varies, $(B_\bullet, C_\bullet)$ runs over all double flags in F_M. In the case $\mathbf{b} = \mathbf{c} = (1^n)$, with $M = M_w$, we may take $v_{ij1} = v_i$ for any basis $v_1, \dots, v_n$ of V, and obtain the configuration type F_w from the Introduction.

We can also describe this orbit by intersection conditions:

$$F_M = \{\,(B_\bullet, C_\bullet) \mid \dim(B_i \cap C_j) = r_{ij}(M)\,\},$$

where

$$r_{ij}(M) := \sum_{\substack{(k,l)\\ k\le i,\, l\le j}} m_{kl}$$

are the *rank numbers*. This characterization follows from Theorem 1 below.

These orbits cover the double flag variety:

$$\mathrm{Flag}(\mathbf{b})\times\mathrm{Flag}(\mathbf{c}) = \coprod_M F_M$$

where the union runs over all transport matrices M.

We shall need the following partial order on the matrix positions $(i, j) \in [1, q]\times[1, r]$: we write

$$(i, j) \le (i', j') \Leftrightarrow i \le i' \quad \text{and} \quad j \le j'.$$

That is, the northwest positions are small, the southeast positions large. Also, $(i, j) < (i', j')$ means $(i, j) \le (i', j')$ and $(i, j) \ne (i', j')$, a convention we will use when dealing with any partial order. Furthermore, for sets of positions $\Delta, \Delta' \subset [1, q]\times[1, r]$ we let:

$$\Delta \le \Delta' \Leftrightarrow \forall\,(i, j)\in\Delta\ \ \exists\,(i', j')\in\Delta' \quad \text{with} \quad (i, j) \le (i', j').$$

Now, the *degeneration order* or *Ehresmann-Bruhat order* on the set of all transport matrices describes how the orbits F_M touch each other:

$$M \overset{\deg}{\le} M' \Leftrightarrow \bar{F}_M \subset \bar{F}_{M'} \Leftrightarrow F_M \subset \bar{F}_{M'},$$

where $\bar{F}_M$ denotes the (Zariski) closure of F_M. Our goal is to give a combinatorial characterization of this geometric order.

First, we approximate the degeneration order on double flags by comparing rank numbers. We define:

$$M \overset{\mathrm{rk}}{\le} M' \Leftrightarrow r_{ij}(M) \ge r_{ij}(M') \quad \forall\,(i, j) \in [1, q]\times[1, r].$$

Second, we define certain moves on matrices which will turn out to be the covers of the degeneration order: that is, the relations $M \overset{\text{deg}}{\lessdot} M'$ such that $M \overset{\text{deg}}{<} M'' \overset{\text{deg}}{\leq} M' \;\Rightarrow\; M'' = M'$. Suppose we consider positions $(i_0, j_0) \leq (i_1, j_1)$ defining a rectangle

$$R = [i_0, i_1] \times [j_0, j_1] \subset [1, q] \times [1, r],$$

and we are given an M satisfying $m_{i_0 j_0}, m_{i_1 j_1} > 0$ and $m_{ij} = 0$ for all $(i, j) \in R$, $(i, j) \neq (i_0, j_0), (i_1, j_1), (i_0, j_1), (i_1, j_0)$. Then the *simple move* on the matrix M, at the rectangle R, is the operation which produces the new matrix:

$$M' = M - E_{i_0 j_j} - E_{i_1 j_1} + E_{i_0 j_1} + E_{i_1 j_0},$$

where E_{ij} denotes the coordinate matrix with 1 in position (i, j) and 0 elsewhere. We write $M \overset{\text{R}}{\lessdot} M'$ or $M \overset{\text{mv}}{\lessdot} M'$. Pictorially,

$$M = \begin{array}{c} \\ i_0 \\ \\ \\ \\ i_1 \\ \\ \end{array} \begin{array}{|ccccccc|} \hline & j_0 & & & j_1 & & \\ & \vdots & & & \vdots & & \\ \cdots & a & 0 \cdots 0 & & b & \cdots & \\ & 0 & & & 0 & & \\ & \vdots & 0 & & \vdots & & \\ & 0 & & & 0 & & \\ \cdots & c & 0 \cdots 0 & & d & \cdots & \\ & \vdots & & & \vdots & & \\ \hline \end{array} \overset{\text{R}}{\lessdot} M' = \begin{array}{c} \\ i_0 \\ \\ \\ \\ i_1 \\ \\ \end{array} \begin{array}{|ccccccc|} \hline & j_0 & & & j_1 & & \\ & \vdots & & & \vdots & & \\ \cdots & a-1 & 0 \cdots 0 & & b+1 & \cdots & \\ & 0 & & & 0 & & \\ & \vdots & 0 & & \vdots & & \\ & 0 & & & 0 & & \\ \cdots & c+1 & 0 \cdots 0 & & d-1 & \cdots & \\ & \vdots & & & \vdots & & \\ \hline \end{array}.$$

In the case where M is a permutation matrix, the simple move corresponds to multiplying by a transposition: if M is associated to w, then M' is associated to $w' = (i_0, i_1) \cdot w = w \cdot (j_0, j_1)$, and the vanishing conditions on m_{ij} assure that the Bruhat length $\ell(w') = \ell(w) + 1$.

We say M, M'' are related by the *move order*, if, starting with M, we can perform a sequence of simple moves on various rectangles $R_1, R_2, \ldots$ to obtain M'':

$$M \overset{\text{mv}}{\leq} M'' \Leftrightarrow M \overset{\text{R}_1}{\lessdot} M' \overset{\text{R}_2}{\lessdot} \cdots M''.$$

Theorem 1 (*Ehresmann-Chevalley*)
(a) *The three orders defined above are equivalent*:

$$M \overset{\text{deg}}{\leq} M' \Leftrightarrow M \overset{\text{rk}}{\leq} M' \Leftrightarrow M \overset{\text{mv}}{\leq} M'.$$

(b) *The relation* $M \overset{\text{deg}}{<} M'$ *is a cover exactly when* $M \overset{\text{mv}}{\lessdot} M'$.

We give a proof in Section 3. Once we have established the equivalences above, we call the common order the *Bruhat order*, written simply as $M \leq M'$.

2.2. *A line and two flags*

We now state our main theorems. We consider $\mathrm{GL}(V)$ acting diagonally on $\mathbb{P}(V)\times\mathrm{Flag}(\mathbf{b})\times\mathrm{Flag}(\mathbf{c})$, the variety of triples of a line and two flags. We showed in [5] that the orbits correspond to the *decorated matrices* (M, Δ), meaning that M is a transport matrix, and

$$\Delta = \{(i_1, j_1), \dots, (i_t, j_t)\} \subset [1, q] \times [1, r]$$

is a set of matrix positions satisfying:

$$i_1 < i_2 < \cdots < i_t, \quad j_1 > j_2 > \cdots > j_t, \quad \text{and} \quad m_{ij} > 0 \ \forall (i, j) \in \Delta.$$

That is, the positions $(i_1, j_1), (i_2, j_2), \dots$ proceed from northeast to southwest. We may concisely write down (M, Δ) by drawing a circle around the nonzero entries of M at the positions $(i, j) \in \Delta$.

The corresponding orbit $F_{M,\Delta}$ consists of the triple flags $(A, B_\bullet, C_\bullet)$ defined as follows. Given a basis $\langle v_{ijk}\rangle$ as above, the flags $B_\bullet, C_\bullet$ are defined exactly as before (and thus depend only on M); and the line is defined as $A := \langle \sum_{(i,j)\in\Delta} v_{ij1}\rangle$. Thus M indicates the relative positions of the two flags $B_\bullet, C_\bullet$, and Δ is a "decoration" on M indicating the position of the line A. Once again we have:

$$\mathbb{P}(V)\times\mathrm{Flag}(\mathbf{b})\times\mathrm{Flag}(\mathbf{c}) = \coprod_{(M,\Delta)} F_{M,\Delta},$$

where (M, Δ) runs over all decorated matrices. We also define the degeneration order $(M, \Delta) \overset{\text{deg}}{\leq} (M', \Delta')$ as before.

Next we define the rank order. For $(A, B_\bullet, C_\bullet) \in F_{M,\Delta}$, we define a new rank number:

$$\begin{aligned} r_{\langle ij\rangle}(M, \Delta) &:= \dim(B_i \cap C_j) + \dim(A \cap (B_i + C_j)) \\ &= r_{ij}(M) + \delta_{ij}(\Delta); \end{aligned}$$

where we define:

$$\begin{aligned} \delta_{ij}(\Delta) &:= \begin{cases} 1 & \text{if } \Delta \leq \{(i, r), (q, j)\} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } (i+1, j+1) \leq \Delta \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

We can extend this definition to $(i, j) \in [0, q]\times[0, r]$ by setting $r_{ij}(M) := 0$ if $i = 0$ or $j = 0$, so that

$$r_{\langle i0\rangle}(M, \Delta) = \dim(A \cap B_i) = \delta_{i0}(\Delta), \quad r_{\langle 0j\rangle}(M, \Delta) = \dim(A \cap C_j) = \delta_{0j}(\Delta).$$

Now we let:

$$(M,\Delta) \overset{\text{rk}}{\leq} (M',\Delta') \Leftrightarrow \begin{array}{c} r_{ij}(M) \geq r_{ij}(M') \\ r_{\langle ij\rangle}(M,\Delta) \geq r_{\langle ij\rangle}(M',\Delta') \end{array} \quad \forall (i,j) \in [0,q]\times[0,r].$$

Rank Theorem *The degeneration order and the rank order are equivalent*:

$$(M,\Delta) \overset{\text{deg}}{\leq} (M',\Delta') \Leftrightarrow (M,\Delta) \overset{\text{rk}}{\leq} (M',\Delta').$$

That is, the triple flag $(A, B_\bullet, C_\bullet)$ *is a degeneration of* $(A', B'_\bullet, C'_\bullet)$ *if and only if, for all* $(i,j) \in [0,q]\times[0,r]$,

$$\begin{array}{c} \dim(B_i \cap C_j) \geq \dim(B'_i \cap C'_j) \\ \dim(B_i \cap C_j) + \dim(A \cap (B_i + C_j)) \geq \dim(B'_i \cap C'_j) + \dim(A' \cap (B'_i + C'_j)). \end{array}$$

2.3. Simple moves

Below, we define simple moves of types (i)–(v) on a decorated matrix (M,Δ), each producing a new matrix (M',Δ'), so that we write $(M,\Delta) \overset{\text{mv}}{\lessdot} (M',\Delta')$. Given these moves, we define the move order $(M,\Delta) \overset{\text{mv}}{\leq} (M'',\Delta'')$ as before.

Move Theorem *The degeneracy order and the move order are equivalent*:

$$(M,\Delta) \overset{\text{deg}}{\leq} (M',\Delta') \Leftrightarrow (M,\Delta) \overset{\text{mv}}{\leq} (M',\Delta').$$

Again, we call the common order the *Bruhat order*.

Minimality Theorem *The relation* $(M,\Delta) \overset{\text{deg}}{<} (M',\Delta')$ *is a cover exactly when* $(M,\Delta) \overset{\text{mv}}{\lessdot} (M',\Delta')$ *for one of the simple moves* (i)–(v).

We introduce an operation which normalizes an arbitrary subset S of matrix positions into a decoration Δ of the prescribed form. For $S \subset [1,q]\times[1,r]$, let:

$$[S] := \{\, (i,j) \in S \mid (i,j) \not< (k,l)\ \forall (k,l) \in S \,\}$$

be the set of $\leq$-maximal positions in S. This operation is "explained" by the following lemma, proved in Section 5:

Lemma 1 (*Uncircling lemma*) *If* M *is a transport matrix,* S *a set of matrix positions with* $m_{ij} > 0$ *for all* $(i,j) \in S$, *and we define* $(A, B_\bullet, C_\bullet)$ *by the same formulas as for a decorated matrix* (*namely* $A := \langle \sum_{(i,j)\in S} v_{ij1} \rangle$), *then* $(A, B_\bullet, C_\bullet) \in F_{M,\Delta}$, *where* $\Delta = [S]$.

It remains to define the five types of simple moves $(M,\Delta) \overset{\text{(i)}}{\lessdot} (M',\Delta'), \ldots, (M,\Delta) \overset{\text{(v)}}{\lessdot} (M',\Delta')$. Although geometrically, it is natural to think of the more general configuration degenerating to the more special one, combinatorially it is more convenient to describe the modification of the more special (smaller) element (M, Δ) to obtain the more general (larger) element (M', Δ').

In each case, we indicate the matrix positions in (M, Δ) modified by the move, and we list the requirements on $M = (m_{ij})$ and Δ for the move to be valid. Then we specify the result of the move, (M', Δ').

(i) Suppose we have a position $(i_1, j_1) \not\leq \Delta$, with $m_{i_1 j_1} > 0$ and $m_{ij} = 0$ whenever $(i, j) < (i_1, j_1)$, $(i, j) \not\leq \Delta$. Then define:

$$M' := M, \quad \Delta' := [\,\Delta \cup \{(i_1, j_1)\}\,].$$

$$\begin{array}{c}(M, \Delta)\\ \boxed{\begin{array}{cccccc} & & & & & \circledast \\ & & 0 & \cdots & 0 & \\ & & \vdots & & \vdots & \\ & \circledast & 0 & & & \\ 0 \cdots & 0 & 0 & & \vdots & \\ \vdots & & & 0 & 0 & \\ 0 \cdots & & \cdots & 0 & a & \\ \circledast & & & & & \end{array}}\end{array} \overset{\text{(i)}}{\lessdot} \begin{array}{c}(M', \Delta')\\ \boxed{\begin{array}{cccccc} & & & & & \circledast \\ & & 0 & \cdots & 0 & \\ & & \vdots & & \vdots & \\ & * & 0 & & & \\ 0 \cdots & 0 & 0 & & \vdots & \\ \vdots & & & 0 & 0 & \\ 0 \cdots & & \cdots & 0 & \textcircled{a} & \\ \circledast & & & & & \end{array}}\end{array}$$

where $a = m_{i_1 j_1} > 0$, the symbol $*$ represents a matrix entry $m_{ij} > 0$, and a circle $\circledast$ around $* = m_{ij}$ means that $(i, j) \in \Delta$. The values of m_{ij} are not changed by the move.

(ii) Suppose we have positions $(i_0, j_0) < (i_1, j_1)$ with: $m_{i_0 j_0}, m_{i_1 j_1} > 0$, and $m_{ij} = 0$ whenever: $(i_0, j_0) < (i, j) < (i_1, j_1)$, $(i, j) \neq (i_0, j_1), (i_1, j_0)$. Suppose further that $(i_1, j_1) \notin \Delta$; $(i_0, j_1) \notin \Delta$ or $(i_1, j_0) \notin \Delta$; and $(i_0, j_0) \notin \Delta$ or $m_{i_0 j_0} > 1$. Then define:

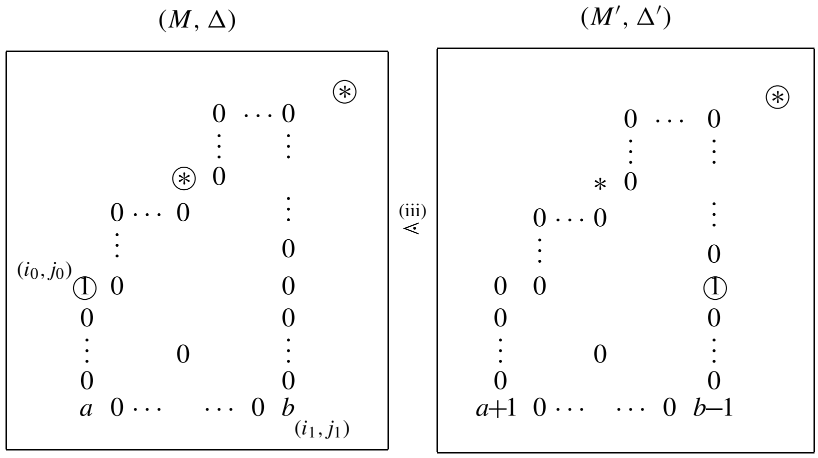
$$M' := M - E_{i_0 j_0} - E_{i_1 j_1} + E_{i_0 j_1} + E_{i_1 j_0}, \quad \Delta' := \Delta.$$

$$\begin{array}{c}(M, \Delta)\\ \boxed{\begin{array}{ccc} a & 0 \cdots 0 & b \\ 0 & & 0 \\ \vdots & 0 & \vdots \\ 0 & & 0 \\ c & 0 \cdots 0 & d \end{array}}\end{array} \overset{\text{(ii)}}{\lessdot} \begin{array}{c}(M', \Delta')\\ \boxed{\begin{array}{ccc} a{-}1 & 0 \cdots 0 & b{+}1 \\ 0 & & 0 \\ \vdots & 0 & \vdots \\ 0 & & 0 \\ c{+}1 & 0 \cdots 0 & d{-}1 \end{array}}\end{array}$$

Here $a = m_{i_0 j_0} > 0$, $d = m_{i_1 j_1} > 0$, a may be circled only if $a > 1$, at most one of b, c is circled, and d is not circled. The positions of circles are unchanged by the move.

(iii)(a) Suppose $(i_0, j_0) < (i_1, j_1)$ with $(i_0, j_0) \in \Delta$, $m_{i_0 j_0} = 1$, $m_{i_1 j_1} > 0$, and $m_{ij} = 0$ whenever: $(i_0, j_0) < (i, j) < (i_1, j_1)$, $(i, j) \neq (i_1, j_0)$; and whenever $(i, j) \leq (i_0, j_1)$, $(i, j) \not\leq \Delta$. Then define:

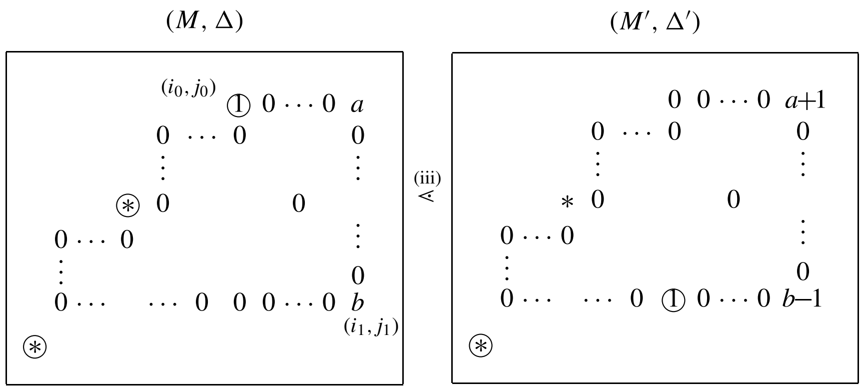
$$M' := M - E_{i_0 j_0} - E_{i_1 j_1} + E_{i_0 j_1} + E_{i_1 j_0}, \quad \Delta' := [\,\Delta \cup \{(i_0, j_1)\}\,].$$



Here the ① on the left is at position (i_0, j_0), $a = m_{i_1 j_0}$, and $b = m_{i_1 j_1} > 0$.

(iii)(b) The transpose of move (iii)(a). Suppose $(i_0, j_0) < (i_1, j_1)$ with $(i_0, j_0) \in \Delta$, $m_{i_0 j_0} = 1$, $m_{i_1 j_1} > 0$, and $m_{ij} = 0$ whenever: $(i_0, j_0) < (i, j) < (i_1, j_1)$, $(i, j) \neq (i_0, j_1)$; and whenever $(i, j) \leq (i_1, j_0)$, $(i, j) \not\leq \Delta$. Then define:

$$M' := M - E_{i_0 j_0} - E_{i_1 j_1} + E_{i_0 j_1} + E_{i_1 j_0}, \quad \Delta' := [\,\Delta \cup \{(i_1, j_0)\}\,].$$

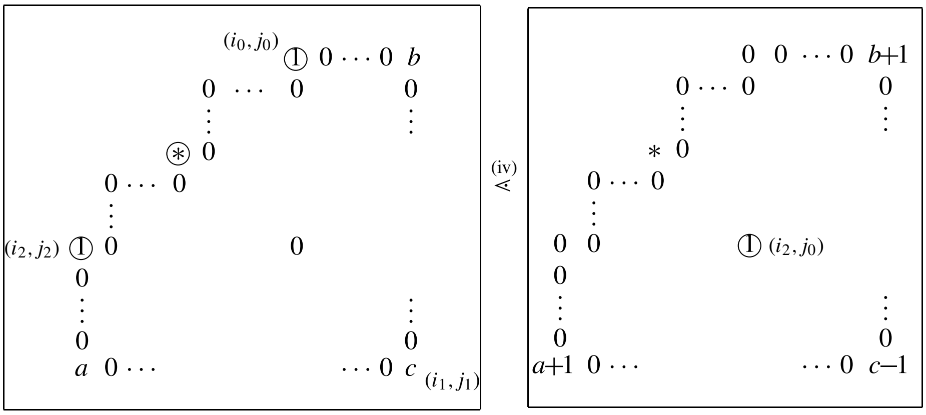


Here the ① on the left is at position (i_0, j_0), $a = m_{i_0 j_1}$, and $b = m_{i_1 j_1} > 0$.

(iv)(a) Suppose $i_0 < i_2 < i_1$ and $j_2 < j_0 < j_1$ with $(i_0, j_0), (i_2, j_2) \in \Delta$, $m_{i_0 j_0} = m_{i_2 j_2} = 1$, $m_{i_1 j_1} > 0$, and $m_{ij} = 0$ whenever: $(i_0, j_2) < (i, j) < (i_1, j_1)$ and $(i, j) \not\leq \Delta$ and $(i, j) \neq (i_0, j_1), (i_1, j_2)$. Then define:

$$M' := M - E_{i_0 j_0} - E_{i_1 j_1} - E_{i_2 j_2} + E_{i_1 j_2} + E_{i_2 j_0} + E_{i_0 j_1}, \quad \Delta' := [\Delta \cup \{(i_2, j_0)\}].$$

(M, Δ) (M', Δ')



Here the coordinate markings indicate that on the left, the ①'s are at positions (i_0, j_0), (i_2, j_2), and $c = m_{i_1 j_1} > 0$. The ① on the right is at position (i_2, j_0).

(iv)(bc) Suppose we have $(i_0, j_0) < (i_1, j_1)$ with $m_{i_0 j_0}, m_{i_1, j_1} > 0$, and $m_{ij} = 0$ whenever: $(i_0, j_0) < (i, j) < (i_1, j_1)$, except for $(i, j) = (i_0, j_1), (i_1, j_0)$ and one other position as specified below. Further suppose one of the following cases:

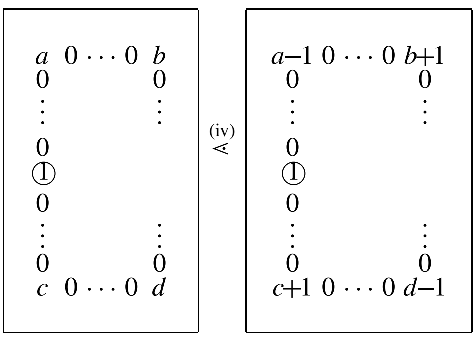
(b) we have $i_0 < i_2 < i_1$ with $(i_2, j_0) \in \Delta$, $m_{i_2 j_0} = 1$, and $(i_0, j_1) \not\leq \Delta$; or
(c) we have $j_0 < j_2 < j_1$ with $(i_0, j_2) \in \Delta$, $m_{i_0 j_2} = 1$, and $(i_1, j_0) \not\leq \Delta$.

Then define:

$$M' := M - E_{i_0 j_0} - E_{i_1 j_1} + E_{i_0 j_1} + E_{i_1 j_0}, \quad \Delta' := \Delta.$$

This is the same as move (ii), except that it occurs in the presence of a ① at (i_0, j_2) or (i_2, j_0).

(b) (M, Δ) (M', Δ')



Here b must not be circled, and there must be no circled element weakly southeast of b. The move does not change the circled positions.

(c)

$$(M,\Delta)\qquad\qquad\qquad (M',\Delta')$$

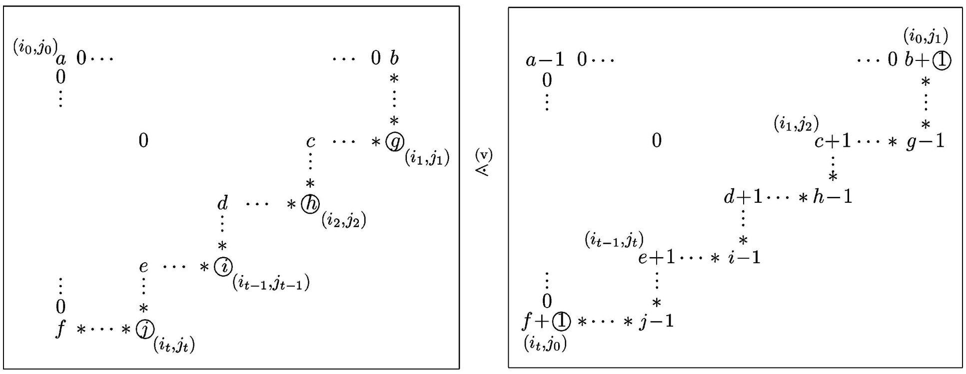
$$\begin{array}{|ccccccc|}\hline a & 0\cdots 0 & \textcircled{1} & 0\cdots 0 & b \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ c & 0\cdots & & \cdots 0 & d \\ \hline \end{array} \overset{\text{(iv)}}{<} \begin{array}{|ccccc|}\hline a-1 & 0\cdots 0 & \textcircled{1} & 0\cdots 0 & b+1 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ c+1 & 0\cdots & & \cdots 0 & d-1 \\ \hline \end{array}$$

Here c must not be circled, and there must be no circled element weakly southeast of c. The move does not change the circled positions.

(v) Suppose, for $t \geq 1$, we have $i_0 < i_1 < \cdots < i_t$, and $j_1 > j_2 > \cdots > j_t > j_0$, with $(i_1, j_1), \ldots, (i_t, j_t) \in \Delta$, $m_{i_0 j_0} > 0$, and $m_{ij} = 0$ whenever: $(i_0, j_0) < (i, j) \leq (i_s - 1, j_s - 1)$ for some $s = 1, \ldots, t$. Then define:

$$M' := M - E_{i_0 j_0} - E_{i_1 j_1} - \cdots - E_{i_t j_t} + E_{i_0 j_1} + E_{i_t j_0},$$
$$\Delta' := D \setminus \{(i_0, j_0), (i_1, j_1), \ldots, (i_t, j_t)\} \cup \{(i_0, j_1), (i_t, j_0)\}.$$

$$(M,\Delta)\qquad\qquad\qquad\qquad (M',\Delta')$$



Here $m+\textcircled{1}$ means $m+1$ circled, and $*$ means a value $m_{ij} \geq 0$, which is unchanged by the move. Note that, in contrast to moves (i)–(iv), we have $\Delta' < \Delta$.

2.4. *Strategy of proof*

We will prove the Rank and Move Theorems for triple flags by means of three "chain lemmas."

Lemma 2 $(M, \Delta) \overset{\text{mv}}{\leq} (M', \Delta') \Rightarrow (M, \Delta) \overset{\text{deg}}{\leq} (M', \Delta')$

For $(M, \Delta) \overset{\text{mv}}{\lessdot} (M', \Delta')'$, we will give an explicit degeneration of (M', Δ') to (M, Δ).

Lemma 3 $(M, \Delta) \overset{\text{deg}}{\leq} (M', \Delta')' \Rightarrow (M, \Delta) \overset{\text{rk}}{\leq} (M', \Delta')$

This will follow from general principles of algebraic geometry.

Lemma 4 $(M, \Delta) \overset{\text{rk}}{\leq} (M', \Delta') \Rightarrow (M, \Delta) \overset{\text{mv}}{\leq} (M', \Delta')$

This is a purely combinatorial result. Given $(M, \Delta) \overset{\text{rk}}{\leq} (M', \Delta')$, we construct $(\tilde{M}, \tilde{\Delta})$ with $(M, \Delta) \overset{\text{mv}}{\lessdot} (\tilde{M}\ \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

3. Proof of Theorem 1

In order to prepare and illuminate the proofs of Lemmas 2–4 for triple flags, we give the precisely analogous arguments for the classical case of two flags, thereby proving Theorem 1.

Lemma 5 $M \overset{\text{mv}}{\leq} M' \Rightarrow M \overset{\text{deg}}{\leq} M'$

Proof: Given $M \overset{\text{mv}}{\lessdot} M'$, it suffices to find a one-parameter algebraic family of double flags $(B_\bullet(\tau), C_\bullet(\tau))$, indexed by $\tau \in \mathbf{k}$, such that:

$$(B_\bullet(\tau), C_\bullet(\tau)) \in F_{M'} \text{for} \tau \neq 0,$$
$$(B_\bullet(0), C_\bullet(0)) \in F_M.$$

Consider a basis of V indexed according to $M = (m_{ij})$ as:

$$V = \langle e_{ijk} \mid (i, j) \in [1, q] \times [1, r], 1 \leq k \leq m_{ij} \rangle,$$

and define a set of vectors indexed according to $M' = (m'_{ij})$,

$$\{v_{ijk}(\tau) \mid (i, j) \in [1, q] \times [1, r], 1 \leq k \leq m'_{ij}\}$$

as follows. Let us use the symbol $e_{ij\max}$ to mean that the third subscript in e_{ijk} has as large a value as possible, namely max $= m_{ij}$; and similarly $v_{ij\max}$ means max $= m'_{ij}$. Now let:

$$v_{i_0 j_1 \max}(\tau) := e_{i_0 j_0 \max} + \tau e_{i_1 j_1 \max}$$
$$v_{i_1 j_0 \max}(\tau) := e_{i_0 j_0 \max}$$
$$v_{ijk}(\tau) := e_{ijk} \qquad \text{otherwise.}$$

For $\tau \neq 0$, let $B_i(\tau) := \langle v_{i'jk} \mid i' \leq i \rangle$, $C_j(\tau) := \langle v_{ij'k} \mid j' \leq j \rangle$. For $\tau \neq 0$, the set $\{v_{ijk}\}$ forms a basis of V, and with respect to this basis $(B_\bullet(\tau), C_\bullet(\tau)) \in F_{M'}$.

Now define $B_i(0) := \lim_{\tau\to 0} B_i(\tau)$ and $C_j(0) := \lim_{\tau\to 0} C_j(\tau)$, the limits in the Zariski topology.[3] We proceed to evaluate these limits, showing that, with respect to the basis $\langle e_{ijk}\rangle$, we have $(B_\bullet(0), C_\bullet(0)) \in F_M$.

As $\tau \to 0$, the vectors $\{ v_{i'jk}(\tau) \mid i' \le i_0 \}$ remain linearly independent, and $v_{i_0 j_1 \max}(\tau) \to e_{i_0 j_0 \max}$, so we have:

$$B_{i_0}(0) = \langle v_{i'jk}(0) \mid i' \le i_0\rangle = \langle e_{i'jk} \mid i' \le i_0\rangle.$$

Furthermore, we can take linear combinations of basis vectors to find, for arbitrary $\tau \neq 0$:

$$\begin{aligned} B_{i_1}(\tau) &= \big\langle e_{i_0 j_0 \max} + \tau e_{i_1 j_1 \max},\ e_{i_0 j_0 \max}\big\rangle \\ &\quad \oplus \big\langle v_{i'jk} \mid i' \le i_1,\ (i,j,k) \neq \big(i_0, j_1, m'_{i_0 j_1}\big), \big(i_1, j_0, m'_{i_1, j_0}\big)\big\rangle \\ &= \big\langle e_{i_0 j_0 \max},\ e_{i_1 j_1 \max}\big\rangle \\ &\quad \oplus \big\langle e_{i'jk} \mid i' \le i_0, (i,j,k) \neq \big(i_0, j_0, m_{i_0 j_0}\big), \big(i_1, j_1, m_{i_1, j_1}\big)\big\rangle \\ &= \langle e_{i'jk} \mid i' \le i_1\rangle. \end{aligned}$$

Since the final basis is constant with respect to τ, the limit space $B_{i_1}(0)$ exists and has the same basis. Similarly for any i, j we have $B_i(0) = \langle e_{i'jk} \mid i' \le i\rangle$, $C_j(0) = \langle e_{ij'k} \mid j \le j'\rangle$. □

Lemma 6 $M \overset{\deg}{\le} M' \ \Rightarrow\ M \overset{\mathrm{rk}}{\le} M'$

Proof: Since $M \overset{\deg}{\le} M'$, there exists an algebraic family with $(B_\bullet(\tau), C_\bullet(\tau)) \in F_{M'}$ for $\tau \neq 0$, and $(B_\bullet(0), C_\bullet(0)) \in F_M$.

For any fixed d, the condition $\dim(B_i \cap C_j) \ge d$ defines a closed subvariety of $\mathrm{Flag}(\mathbf{b}) \times \mathrm{Flag}(\mathbf{c})$ (cut out by the vanishing of certain determinants in the homogeneous coordinates of the flag varieties). Hence if the generic elements of the family satisfy $r_{ij}(M') \ge d$, then so does the limit element at $\tau = 0$. That is, the rank numbers $r_{ij}(M')$ can only get larger upon degeneration. □

Lemma 7 $M \overset{\mathrm{rk}}{\le} M' \ \Rightarrow\ M \overset{\mathrm{mv}}{\le} M'$.
In fact, if $M \overset{\mathrm{rk}}{<} M'$, then we can perform a simple move on M to obtain a matrix $\tilde{M}$ satisfying

$$M \overset{\mathrm{mv}}{\lessdot} \tilde{M} \overset{\mathrm{rk}}{\le} M'.$$

Proof: Denote $M = (m_{ij})$, $r_{ij} = r_{ij}(M)$, $M' = (m'_{ij})$, $r'_{ij} = r_{ij}(M')$, so that $r_{ij} \ge r'_{ij}$ by assumption. Consider the lexicographically first position (k_0, l_0) where the matrices differ: that is, $m_{ij} = m'_{ij}$ for $i < k_0$ and for $i = k_0$, $j < l_0$; and $m_{k_0 l_0} > m'_{k_0 l_0}$, making $r_{k_0 l_0} > r'_{k_0 l_0}$. Consider a rectangle $[k_0, k_1] \times [l_0, l_1]$ as large as possible such that $r_{ij} > r'_{ij}$ for $(i, j) \in [k_0, k_1 - 1] \times [l_0, l_1 - 1]$. (Such a rectangle is not necessarily unique.)

Claim: $m_{i_1 j_1} > 0$ for some $(i_1, j_1) \in [k_0 + 1, k_1] \times [l_0 + 1, l_1]$. Otherwise we would have $m_{ij} = 0$ for all $(i, j) \in [k_0 + 1, k_1] \times [l_0 + 1, l_1]$. By the maximality of the rectangle, we have $r_{il_1} = r'_{il_1}$ and $r_{k_1 j} = r'_{k_1 j}$ for some $(i, j) \in [k_0, k_1 - 1] \times [j_0, j_2 - 1]$. By the definition of the rank numbers and the vanishing of the m_{kl}, we have:

$$r_{k_1 l_1} = r_{i l_1} + r_{k_1 j} - r_{ij} < r'_{i l_1} + r'_{k_1 j} - r'_{ij} \leq r'_{k_1 l_1}.$$

This contradiction proves our claim.

We may assume the (i_1, j_1) found above is $\leq$-minimal, so that $m_{ij} = 0$ for $(i, j) \in [k_0 + 1, i_1] \times [l_0 + 1, j_1]$, $(i, j) \neq (i_1, j_1)$. By moving right or down from (k_0, l_0) to a position (i_0, j_0), we can get:

$$m_{i_0 j_0}, m_{i_1 j_1} > 0, \quad m_{ij} = 0 \quad \text{for } (i_0, j_0) < (i, j) < (i_1, j_1), (i, j) \neq (i_0, j_1), (i_1, j_0)$$
$$r_{ij} > r'_{ij} \quad \text{for } (i, j) \in [i_0, i_1 - 1] \times [j_0, j_1 - 1].$$

These are all the necessary conditions to perform our simple move: $\tilde{M} := M - E_{i_0 j_0} - E_{i_1 j_1} + E_{i_1 j_0} + E_{i_0 j_1}$. Denoting $\tilde{r}_{ij} = r_{ij}(\tilde{M})$, we have:

$$\tilde{r}_{ij} = \begin{cases} r_{ij} - 1 \geq r'_{ij} & \text{if } (i, j) \in [i_0, j_1 - 1] \times [j_0, j_1 - 1] \\ r_{ij} \geq r'_{ij} & \text{otherwise,} \end{cases}$$

showing that $\tilde{M} \overset{\text{rk}}{\leq} M'$. □

Thus, all three orders are identical, denoted $M \leq M'$. It only remains to prove that the simple moves give the covers of this order.

Suppose $M \overset{R}{\lessdot} M'$ for a rectangle $R = [i_0, i_1] \times [j_0, j_1]$, and $M < \tilde{M} \leq M'$. Without loss of generality, we may assume $M \underset{S}{\lessdot} \tilde{M}$ for some other rectangle S. Then it is clear that

$$r_{ij}(M) \geq r_{ij}(\tilde{M}) \geq r_{ij}(M').$$

But

$$r_{ij}(M) = \begin{cases} r_{ij}(M') + 1 & \text{for } (i, j) \in [i_0, i_1 - 1] \times [j_0, j_1 - 1] \\ r_{ij}(M') & \text{otherwise,} \end{cases}$$

and similarly for S. Hence $S \subset R$, but because $m_{ij} = 0$ for all $(i, j) \in R$, we must have $S = R$, and $\tilde{M} = M'$.

This completes the proof of Theorem 1.

4. Geometry of decorated matrices

In this section, we prove the first two chain lemmas for triple flags, the ones involving geometric arguments. We follow the same lines of argument as for two flags.

Let (M, Δ), (M', Δ') be decorated matrices indexing the orbits in $\mathbb{P}^{n-1} \times \mathrm{Flag}(\mathbf{b}) \times \mathrm{Flag}(\mathbf{c})$, and denote as above $M = (m_{ij})$, $r_{ij} = r_{ij}(M)$, etc., as well as $r_{\langle ij \rangle} = r_{\langle ij \rangle}(M, \Delta)$, etc. (Later we will also use $\tilde{r}_{ij} = r_{ij}(\tilde{M})$, etc.) Let us first prove the Uncircling Lemma:

Proof of Lemma 1: Suppose we have $(i_0, j_0), (i_1, j_1) \in S$ with $(i_0, j_0) < (i_1, j_1)$. Let $S' = S \setminus \{(i_0, j_0)\}$, and denote by $F_{M,S}$, $F_{M,S'}$ the GL(V)-orbits defined analogously to $F_{M,\Delta}$. It suffices to show that $F_{M,S} = F_{M,S'}$.

Let $\langle e_{ijk} \rangle$ be a basis of V indexed according to M (that is, $k \leq m_{ij}$), and let $B_i := \langle e_{i'jk} \mid i' \leq i \rangle$, $C_j := \langle e_{ij'k} \mid j' \leq j \rangle$, and $A := \langle \sum_{(i,j) \in S} e_{ij1} \rangle$. By definition, $(A, B_\bullet, C_\bullet) \in F_{M,S}$.

Now let us write the same $(A, B_\bullet, C_\bullet)$ in terms of a new basis $\langle v_{ijk} \rangle$, defined by $v_{i_1 j_1 1} := e_{i_1 j_1 1} + e_{i_0 j_0 1}$ and $v_{ijk} = e_{ijk}$ otherwise. We have

$$\begin{aligned} B_{i_1} &= \left\langle e_{i_0 j_0 1},\, e_{i_1 j_1 1} \right\rangle \oplus \langle e_{ijk} \mid i \leq i_1,\ (i,j,k) \neq (i_0, j_0, 1), (i_1, j_1, 1) \rangle \\ &= \langle e_{i_0 j_0 1},\, e_{i_1 j_1 1} + e_{i_0 j_0 1} \rangle \oplus \langle e_{ijk} \mid i \leq i_1,\ (i,j,k) \neq (i_0, j_0, 1), (i_1, j_1, 1) \rangle \\ &= \langle v_{ijk} \mid i \leq i_1 \rangle \end{aligned}$$

Similarly $B_i = \langle v_{i'jk} \mid i' \leq i \rangle$ and $C_j = \langle v_{ij'k} \mid j' \leq j \rangle$ for all i, j. Finally,

$$\begin{aligned} A &= \left\langle e_{i_0 j_0 1} + e_{i_1 j_1} + \sum_{(i,j) \in S''} e_{ij1} \right\rangle \\ &= \left\langle v_{i_1 j_1} + \sum_{(i,j) \in S''} v_{ij1} \right\rangle = \left\langle \sum_{(i,j) \in S'} v_{ij1} \right\rangle, \end{aligned}$$

where $S'' = S \setminus \{(i_0, j_0), (i_1, j_1)\}$. Hence $(A, B_\bullet, C_\bullet) \in F_{M,S'}$, proving the Lemma.

Proof of Lemma 2: We follow the method in the proof of Lemma 5. For each type of move $(M, \Delta) \overset{\mathrm{mv}}{<} (M', \Delta')$ listed in Section 2.3, we start with a basis $V = \langle e_{ijk} \mid k \leq m_{ij} \rangle$ indexed according to M; then define another set of vectors $\langle v_{ijk}(\tau) \mid k \leq m'_{ij} \rangle$ indexed according to M' which forms a basis when $\tau \neq 0$.

We then define a family of triple flags for $\tau \neq 0$ by:

$$\begin{aligned} &B_i(\tau) := \langle v_{i'jk}(\tau) \mid i' \leq i \rangle,\ C_j(\tau) := \langle v_{ij'k}(\tau) \mid j' \leq j \rangle, \\ &A(\tau) := \left\langle \sum_{(i,j) \in \Delta'} v_{ij1}(\tau) \right\rangle, \end{aligned}$$

so that by definition $(A(\tau), B_\bullet(\tau), C_\bullet(\tau)) \in F_{M',\Delta'}$ with respect to the basis $\langle v_{ijk}(\tau) \rangle$. Furthermore, it will be clear that as $\tau \to 0$, we have

$$(A(\tau), B_\bullet(\tau), C_\bullet(\tau)) \to (A(0), B_\bullet(0), C_\bullet(0)),$$

where

$$B_i(0) := \langle e_{i'jk} \mid i' \le i \rangle, \quad C_j(0) := \langle e_{ij'k} \mid j' \le j \rangle.$$

$$A(0) := \left\langle \sum_{(i,j)\in\Delta} e_{ij1} \right\rangle.$$

This shows that $\lim_{\tau\to 0}(A(\tau), B_\bullet(\tau), C_\bullet(\tau)) \in F_{M,\Delta}$, and proves the Lemma. We will give details only in the last, most complicated case.

(i) Let $S := \{(i, j) \in \Delta \mid (i, j) < (i_1, j_1)\}$. Now define: $v_{i_1 j_1 1} := \tau e_{i_1 j_1 1} + \sum_{(i,j)\in S} e_{ij1}$ and $v_{ijk} := e_{ijk}$ otherwise. (In each of the following cases, we implicitly define $v_{ijk} := e_{ijk}$ for those (i, j, k) which are not otherwise specified.)

(ii) Since $\Delta = \Delta'$, the line A is unchanged in the move. Thus we may take v_{ijk} exactly as in the proof of Lemma 5.

(iii) Let $S := \{(i, j) \in \Delta \mid (i, j) < (i_0, j_1)\}$. For (iii)(a), define: $v_{i_1 j_0 \max} := e_{i_0 j_0 1}$ and $v_{i_0 j_1 1} := \tau e_{i_1 j_1 \max} + \sum_{(i,j)\in S} e_{ij1}$. Transpose for (iii)(b).

(iv)(a) Let $S := \{(i, j) \in \Delta \mid (i, j) < (i_2, j_0)\}$. Now define: $v_{i_1 j_2 \max} := e_{i_2 j_2 1} + \tau e_{i_1 j_1 \max}$, $v_{i_0 j_1 \max} := e_{i_0 j_0 1} + \tau e_{i_1 j_1 \max}$, and $v_{i_2 j_0 1} := \sum_{(i,j)\in S} e_{ij1}$.

(iv)(bc) Same as (ii).

(v) Let $S := \Delta \setminus \{(i_1, j_1), \dots, (i_t, j_t)\}$. Define:

$$\begin{aligned}
v_{ij1} &:= \tau e_{ij1} \quad \text{for } (i, j) \in S; \quad v_{i_s j_s k} := e_{i_s, j_s, k+1} \quad \text{for } s = 1, 2, \dots, t;\\
v_{i_0 j_1 1} &:= e_{i_0 j_0 \max} + \tau \textstyle\sum_{s=1}^{t} e_{i_s j_s 1}, \quad v_{i_0 j_1 k} := e_{i_0, j_1, k-1} \quad \text{for } k > 1;\\
v_{i_t j_0 1} &:= -e_{i_0 j_0 \max}, \quad v_{i_t j_0 k} := e_{i_t, j_0, k-1} \quad \text{for } k > 1;\\
v_{i_s j_{s+1} \max} &:= e_{i_0 j_0 \max} + \tau^2 e_{i_1 j_1 1} + \tau^2 e_{i_2 j_2 1} + \cdots + \tau^2 e_{i_s j_s 1}\\
&\quad + \tau e_{i_{s+1} j_{s+1} 1} + \cdots + \tau e_{i_t j_t 1} \quad \text{for } s = 1, 2, \dots, t-1.
\end{aligned}$$

The crucial part of the transition matrix between $\langle v_{ijk} \rangle$ and $\langle e_{ijk} \rangle$, containing all the nonzero "non-diagonal" coefficients, is:

	$v_{i_0 j_1 1}$	$v_{i_1 j_2 \max}$	$v_{i_2 j_3 \max}$	$\cdots$	$v_{i_{t-1} j_t \max}$	$v_{i_t j_0 1}$
$e_{i_0 j_0 \max}$	1	1	1	$\cdots$	1	-1
$e_{i_1 j_1 1}$	τ	τ^2	τ^2	$\cdots$	τ^2	0
$e_{i_2 j_2 1}$	τ	τ	τ^2	$\cdots$	τ^2	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$e_{i_{t-1} j_{t-1} 1}$	τ	τ	τ	$\cdots$	τ^2	0
$e_{i_t j_t 1}$	τ	τ	τ	$\cdots$	τ	0

Let us focus on these basis vectors, suppressing all other vectors with ellipsis marks .... We compute, as $\tau \to 0$:

$$\begin{aligned} B_{i_0}(\tau) &= \big\langle v_{i_0 j_1 1}, \ldots \big\rangle \to \big\langle e_{i_0 j_0 \max}, \ldots \big\rangle, \\ B_{i_1}(\tau) &= \big\langle v_{i_0 j_1 1}, v_{i_1 j_2 \max}, \ldots \big\rangle = \big\langle v_{i_0 j_1 1},\ \tau^{-1}\big(v_{i_0 j_1 1} - v_{i_1 j_2 \max}\big), \ldots \big\rangle \\ &= \big\langle v_{i_0 j_1 1},\ (1-\tau) e_{i_1 j_1 1}, \ldots \big\rangle \to \big\langle e_{i_0 j_0 \max}, e_{i_1 j_1 1}, \ldots \big\rangle, \end{aligned}$$

and similarly for the other B_i. Also:

$$\begin{aligned} C_{j_0}(\tau) &= \big\langle v_{i_t j_0 1}, \ldots \big\rangle \to \big\langle e_{i_0 j_0 \max}, \ldots \big\rangle, \\ C_{j_t}(\tau) &= \big\langle v_{i_t j_0 1}, v_{i_{t-1} j_t \max}, \ldots \big\rangle = \big\langle -v_{i_t j_0 1},\ \tau^{-1}(v_{i_{t-1} j_t \max} + v_{i_t j_0 1}), \ldots \big\rangle \\ &= \big\langle v_{i_t j_0 1},\ \tau e_{i_1 j_1 1} + \cdots + \tau e_{i_{t-1} j_{t-1} 1} + e_{i_t j_t 1}, \ldots \big\rangle \to \big\langle e_{i_0 j_0 \max}, e_{i_t j_t 1}, \ldots \big\rangle, \end{aligned}$$

and similarly for the other C_j. Finally,

$$\begin{aligned} A(\tau) &= \left\langle v_{i_0 j_1 1} + v_{i_t j_0 1} + \sum_{(i,j)\in S} v_{ij1} \right\rangle \\ &= \left\langle \tau e_{i_1 j_1} + \cdots + \tau e_{i_t j_t 1} + \sum_{(i,j)\in S} \tau e_{ij1} \right\rangle \\ &= \left\langle e_{i_1 j_1} + \cdots + e_{i_t j_t 1} + \sum_{(i,j)\in S} e_{ij1} \right\rangle. \end{aligned}$$

In each case the $\tau \to 0$ limit is the desired subspace, an element of $F_{M,\Delta}$ with respect to the basis $\langle e_{ijk} \rangle$.

Proof of Lemma 3: We must show that $r_{\langle ij \rangle}(A, B_\bullet, C_\bullet) \geq d$ is a closed condition on $\mathbb{P}(V) \times \mathrm{Flag}(\mathbf{b}) \times \mathrm{Flag}(\mathbf{c})$. This is clear if $i = 0$ or $j = 0$. Given $(i, j) \in [1, q] \times [1, r]$ and a triple flag $X = (A, B_\bullet, C_\bullet)$, define the linear map $\phi_{ij}^X : B_i \times C_j \to V/A$ by $(v_1, v_2) \mapsto v_1 + v_2 \bmod A$. Then ϕ_{ij}^X depends algebraically on X: indeed, we may write ϕ_{ij}^X in coordinates as a matrix of size $(n-1) \times (b_i + c_j)$ with entries depending polynomially on the homogeneous coordinates of X. Thus $\dim \mathrm{Ker}(\phi_{ij}^X) \geq d$ is a closed condition on X. However,

$$\begin{aligned} \dim \mathrm{Ker}(\phi_{ij}^X) &= \dim B_i + \dim C_j - \mathrm{rank}\left(\phi_{ij}^X\right) \\ &= \dim B_i + \dim C_j - \dim((B_i + C_j)/A) \\ &= \dim B_i + \dim C_j - \dim(B_i + C_j) + \dim(A \cap (B_i + C_j)) \\ &= \dim(B_i \cap C_j) + \dim(A \cap (B_i + C_j)) \\ &= r_{\langle ij \rangle}(A, B_\bullet, C_\bullet) \end{aligned}$$

5. Combinatorics of decorated matrices

To prove the Rank and Move Theorems, it only remains to show Lemma 4.

Thus, assume we are given $(M, \Delta) \overset{\text{rk}}{<} (M', \Delta')$. We wish to show that we can perform one of the moves (i)–(v) on (M, Δ) to obtain a decorated matrix $(\tilde{M}, \tilde{\Delta})$ satisfying:

$$(M, \Delta) \overset{\text{mv}}{\lessdot} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta').$$

We give an algorithm producing $(\tilde{M}, \tilde{\Delta})$.

For cases (i)–(v)(a), we assume $\Delta \leq \Delta'$. This is equivalent to $\delta_{ij}(\Delta) \geq \delta_{ij}(\Delta')$ for all (i, j). Note that if $(\tilde{M}, \tilde{\Delta})$ is any decorated matrix satisfying $\tilde{M} \overset{\text{rk}}{\leq} M'$ and $\tilde{\Delta} \leq \Delta'$, then $(\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

Case (i). Assume $M = M'$. Then $\Delta < \Delta'$, and we may choose a minimal (i_1, j_1) with $(i_1, j_1) \not\leq \Delta$, $(i_1, j_1) \leq \Delta'$. Now we may apply move (i) to the appropriate block $[i_0, i_1] \times [j_0, j_1]$ of (M, Δ), obtaining $(\tilde{M}, \tilde{\Delta})$. Since $M = \tilde{M} = M'$ and $\Delta < \tilde{\Delta} \leq \Delta'$, we have:

$$(M, \Delta) \overset{\text{(i)}}{\lessdot} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta').$$

For the rest of the cases, we assume $M \neq M'$, so that $M \overset{\text{rk}}{<} M'$ in the rank order for two flags. We may then apply the proof of Lemma 7 to find $(i_0, j_0), (i_1, j_1)$ such that:

$$\begin{aligned} &m_{i_0 j_0}, m_{i_1 j_1} > 0, \quad m_{ij} = 0 \text{ for } (i_0, i_1) < (i, j) < (i_1, j_1), (i, j) \neq (i_0, j_1), (i_1, j_0) \\ &\text{and} \quad r_{ij} > r'_{ij} \quad \text{for } (i, j) \in R, \quad \text{where } R := [i_0, i_1 - 1] \times [j_0, j_1 - 1]. \end{aligned}$$

Henceforth, in cases (ii)–(v)(a), we will assume as given such positions $(i_0, j_0), (i_1, j_1)$.

Case (ii). Assume $(i_0, j_0), (i_1, j_1) \notin \Delta$; or $(i_0, j_0) \in \Delta$, $m_{i_0 j_0} > 1$.

If $(i_0, j_1), (i_1, j_0) \in \Delta$, then we may apply move (i) to the block $[i_0, i_1] \times [j_0, j_1]$ of (M, Δ), obtaining $\tilde{M} = M$, $\tilde{\Delta} = [\Delta \cup \{(i_1, j_1)\}]$. Clearly for all (i, j), $\tilde{r}_{ij} = r_{ij} \geq r'_{ij}$; and for $(i, j) \notin$, $\tilde{r}_{\langle ij \rangle} = r_{\langle ij \rangle} \geq r'_{\langle ij \rangle}$; while for $(i, j) \in R$, $\tilde{r}_{\langle ij \rangle} = r_{ij} \geq r'_{ij} + 1 \geq r'_{\langle ij \rangle}$. Thus $(M, \Delta) \overset{\text{(i)}}{\lessdot} (\tilde{M}, \tilde{\Delta}) \leq (M', \Delta')$.

If, on the other hand, $(i_0, j_1) \notin \Delta$ or $(i_1, j_0) \notin \Delta$, then we may apply move (ii) to the block $[i_0, i_1] \times [j_0, j_1]$ of (M, Δ), obtaining $(M, \Delta) \overset{\text{(ii)}}{\lessdot} (\tilde{M}, \tilde{\Delta})$. For any (i, j) we have $\tilde{M} \overset{\text{rk}}{\leq} M'$ by Lemma 7, and by assumption $\tilde{\Delta} = \Delta \leq \Delta'$, so $(M, \Delta) \overset{\text{(ii)}}{\lessdot} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

Case (iii). (a) Assume $(i_0, j_0) \in \Delta$, $m_{i_0 j_0} = 1$, and $(i_0, j_1) \leq \Delta'$.

If $m_{ij} > 0$, for some (i, j) in the rectangle $[1, i_0] \times [j_0, j_1]$ and with $(i, j) \not\leq \Delta$, we may apply move (i) to some block in this rectangle, obtaining $(M, \Delta) \overset{\text{(i)}}{\lessdot} (\tilde{M}, \tilde{\Delta})$ with $M = \tilde{M} \overset{\text{rk}}{\leq} M'$ and $\Delta < \tilde{\Delta} \leq \Delta'$. Thus $(M, \Delta) \overset{\text{(i)}}{\lessdot} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

Otherwise $m_{ij} = 0$ for all (i, j) in $[1, i_0] \times [j_0, j_1]$ with $(i, j) \not\leq \Delta$, so we have all the conditions necessary to apply move (iii)(a) to the block $[i_0, i_1] \times [j_0, j_1]$ of (M, Δ), obtaining $(M,\Delta)\overset{\text{(iii)}}{<}(\tilde{M},\tilde{\Delta})$. As before, $\tilde{M}\overset{\text{rk}}{\le} M'$ by Lemma 7, and $\tilde{\Delta}\le\Delta\cup\{(i_0,j_1)\}$ $\le\Delta'$ by assumption, so $(M,\Delta)\overset{\text{(iii)}}{<}(\tilde{M},\tilde{\Delta})\overset{\text{rk}}{\le}(M',\Delta')$.

Case (iii)(b). Assume $(i_0,j_0)\in\Delta$, $m_{i_0j_0}=1$, and $(i_1,j_0)\le\Delta'$. We can repeat the arguments of the previous case with rows and columns transposed.

Case (iv)(a). Assume $(i_0,j_0)\in\Delta$, $m_{i_0j_0}=1$, and $(i_1,j_0),(i_0,j_1)\not\le\Delta'$; and furthermore that $m_{ij}=m'_{ij}$ for $i\le i_0$ and for $i=i_0$, $j<j_0$. (In terms of the proof of Lemma 7, the last condition means $(i_0,j_0)=(k_0,l_0)$.)

Our assumptions imply:

$$\Delta'\le\{(i_0-1,r),(q,j_0-1),(i_1-1,j_1-1)\}.$$

Since $(i_0,j_0)\in\Delta\le\Delta'$, there must exist $(s_1,t_1)\in\Delta'\cap R$.

Claim: $r_{i,j_0-1}>r'_{i,j_0-1}$ for $i\in[s_1,i_1-1]$. Otherwise, if $r_{i,j_0-1}=r'_{i,j_0-1}$ for some $i\in[s_1,i_1-1]$, we would have:

$$\begin{aligned} r_{it_1}&=r_{i,j_0-1}+r_{i_0-1,t_1}-r_{i_0-1,j_0-1}+1\\ &\le r'_{i,j_0-1}+r'_{i_0-1,t_1}-r'_{i_0-1,j_0-1}+\sum_{(i_0,j_0)\le(k,l)\le(i,t_1)}m'_{kl}\\ &=r'_{it_1}. \end{aligned}$$

This contradicts $r_{ij}>r'_{ij}$ within R, establishing the claim.

Now take a rectangle $S:=[s_0,i_1-1]\times[t_0,j_0-1]$ as large as possible such that $r_{ij}>r'_{ij}$ for $(i,j)\in S$. Further, take (s_0,t_0) to be lexicographically minimal with the above property, so that $s_0\le s_1$ by the above Claim.

Claim: There exists $(i_2,j_2)\in S$ such that $m_{i_2j_2}>0$. Otherwise, we would have $m_{ij}=0$ within S. By the maximality of S, there exists $(i,j)\in S$ so that $r_{i,t_0-1}=r'_{i,t_0-1}$ and $r_{s_0-1,j}=r'_{s_0-1,,j}$. Then

$$r_{ij}=r_{s_0-1,j}+r_{i,t_0-1}-r_{s_0-1,t_0-1}\le r'_{s_0-1,j}+r'_{i,t_0-1}-r'_{s_0-1,t_0-1}\le r'_{ij}.$$

This contradicts $r_{ij}>r'_{ij}$ within S, establishing the claim.

Thus, we may choose (i_2,j_2) in S with $m_{i_2j_2}>0$ and $m_{ij}=0$ for $(i_2,j_2)<(i,j)\le(i_1-1,j_0-1)$. In fact, choose (i_2,j_2) to be as northeast as possible with these properties, so that $m_{ij}=0$ for $(i,j)\in[s_0,i_2-1]\times[j_2+1,j_0-1]$. If $(i_2,j_2)\notin\Delta$ or $m_{i_2j_2}>1$, we have all the necessary conditions to apply move (ii) to the block $[i_2,i_1]\times[j_2,j_1]$ or $[i_2,i_1]\times[j_2,j_0]$ of (M,Δ), and finish as in case (ii) above, obtaining

$$(M,\Delta)\overset{\text{(ii)}}{<}(\tilde{M},\tilde{\Delta})\overset{\text{rk}}{\le}(M',\Delta').$$

If $m_{ij}>0$ for some (i,j) in the rectangle $[1,s_0-1]\times[j_2,j_0-1]$ with $(i,j)\not\le\Delta$, then we may apply move (i) to some block within this rectangle. Since the southeast corner

$(s_0-1, j_0-1) \leq (s_1, t_1) \in \Delta'$, we obtain:

$$(M, \Delta) \overset{\text{(i)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta').$$

Thus, we have reduced to the case where $(i_2, j_2) \in \Delta$, $m_{i_2 j_2} = 1$, and $m_{ij} = 0$ or $(i, j) \in \Delta$ for all $(i, j) \in [1, i_2-1] \times [j_2, j_0-1]$.

If $m_{i_1 j_0} > 0$, we may apply move (iii) to the block $[i_2, i_1] \times [j_2, j_0]$ of (M, Δ). Then clearly $\tilde{r}_{ij} \geq r'_{ij}$, since the block lies inside S. For (i, j) outside $[i_0, i_2-1] \times [j_2, j_0-1]$, we have

$$(i+1, j+1) \leq \tilde{\Delta} \Leftrightarrow (i+1, j+1) \leq \Delta \Rightarrow (i+1, j+1) \leq \Delta',$$

so

$$\tilde{r}_{\langle ij \rangle} = \tilde{r}_{ij} + \delta_{ij}(\tilde{\Delta}) = \tilde{r}_{ij} + \delta_{ij}(\Delta) \geq r'_{ij} + \delta_{ij}(\Delta') = r'_{\langle ij \rangle}.$$

For $(i, j) \in [i_0, s_0-1] \times [j_2, j_0-1]$, we have

$$(i+1, j+1) \leq (s_0, j_0) \leq (s_1, t_1) \in \Delta',$$

and $\tilde{r}_{ij} = r_{ij}$, so $\tilde{r}_{\langle ij \rangle} \geq r_{ij} \geq r'_{ij} = r'_{\langle ij \rangle}$. For $(i, j) \in [s_0, i_2-1] \times [j_2, j_0-1] \subset S$, we have $\tilde{r}_{\langle ij \rangle} \geq \tilde{r}_{ij} = r_{ij} \geq r'_{ij} + 1 \geq r'_{\langle ij \rangle}$. Therefore

$$(M, \Delta) \overset{\text{(iii)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta').$$

Finally, suppose $m_{i_1 j_0} = 0$ or $(i, j) \in \Delta$, and as before $(i_2, j_2) \in \Delta$, $m_{i_2 j_2} = 1$, and $m_{ij} = 0$ for all $(i, j) \in [i_0+1, i_2-1] \times [j_2+1, j_0-1]$. Then we may apply move (iv)(a) to the rectangle $[i_2, i_1] \times [j_2, j_1]$ in (M, Δ). Now, for (i, j) outside the region

$$[i_2, i_1-1] \times [j_2, j_1-1] \ \cup \ [i_0, i_1-1] \times [j_0, j_1-1] \subset S \cup R$$

we have $\tilde{r}_{ij} = r_{ij} \geq r'_{ij}$; whereas for (i, j) inside this region we have $\tilde{r}_{ij} = r_{ij} - 1 \geq r'_{ij}$.

To check $\tilde{r}_{\langle ij \rangle} \geq r'_{\langle ij \rangle}$, we repeat the argument we used to show $(M, \Delta) \overset{\text{(iii)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$ immediately above. For (i, j) outside the rectangle $[i_0, i_2-1] \times [j_2, j_0-1]$, we have $(i+1, j+1) \leq \Delta \Leftrightarrow (i+1, j+1) \leq \tilde{\Delta} \Rightarrow (i+1, j+1) \leq \Delta'$, so clearly $\tilde{r}_{\langle ij \rangle} \geq r'_{\langle ij \rangle}$ as above. Similarly repeat the arguments for (i, j) inside $[i_0, i_2-1] \times [j_2, j_0-1]$, to obtain

$$(M, \Delta) \overset{\text{(iv)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta').$$

Case (iv)(bc). Assume $(i_0, j_0) \in \Delta$, $m_{i_0 j_0} = 1$, and $(i_1, j_0), (i_0, j_1) \not\leq \Delta'$; but that $m_{ij} \neq m'_{ij}$ for some $i \leq i_0$ or for $i = i_0$, $j < j_0$. From the proof of Lemma 7, the last condition means that there exists $k_0 < i_0$ with $m_{k_0 j_0} > 0$, or there exists $l_0 < j_0$ with $m_{i_0 l_0} > 1$. Assume the first alternative (the other one being merely a transpose).

By increasing k_0 if necessary, we may assume that $m_{ij} = 0$ for $(i, j) \in [k_0, i_1]\times[j_0, j_1]$ except that definitely $m_{ij} > 0$ for $(i, j) = (k_0, j_0), (i_0, j_0), (i_1, j_1)$, and possibly $m_{ij} > 0$ for $(i, j) = (i_1, j_0), (i_0, j_1)$ and for $i = k_0$.

Suppose $m_{i_0 j_1} > 0$. If $m_{k_0 j} > 0$ for $j_0 < j < j_1$; or if $m_{k_0 j_0} > 0$ and $(k_0, j_1) \notin \Delta$; then we may apply move (ii) to the rectangle $[k_0, i_0] \times [j, j_1]$ $[k_0, i_0] \times [j_0, j_1]$, and finish as in case (ii): $(M, \Delta) \overset{\text{(ii)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$. Otherwise, if $m_{k_0 j_0} > 0$, $m_{k_0 j} = 0$ for $j_0 < j < j_1$, and $(k_0, j_1) \in \Delta$; then apply move (i) to $[k_0, i_0]\times[j_0, j_1]$ to get $\tilde{\Delta} = [\Delta \cup \{(i_0, j_1)\}]$, yielding $(M, \Delta) \overset{\text{(i)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

Now suppose $m_{i_0 j_1} = 0$. If $m_{k_0 j} > 0$ for $j_0 < j < j_1$; then we may apply move (ii) to the rectangle $[k_0, i_1]\times[j, j_1]$: $(M, \Delta) \overset{\text{(ii)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$. If $m_{k_0 j_1} \in \Delta$, then we may apply move (iii) to $[i_0, i_1]\times[j_0, j_1]$, getting $(M, \Delta) \overset{\text{(iii)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

Suppose none of the above cases hold. Then $m_{i_0 j_1} = 0$, $m_{k_0 j_1} \notin \Delta$, and $m_{k_0 j} = 0$ for $j_0 < j < j_1$. Then we finally have the conditions to apply move (iv)(b) to the rectangle $[k_0, j_0]\times[j_0, j_1]$, and finish by again noting that $\tilde{M} \overset{\text{rk}}{\leq} M'$, $\tilde{\Delta} = \Delta \leq \Delta'$, so that $(M, \Delta) \overset{\text{(iv)}}{<} (\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

Case (v)(a). Assume $(i_1, j_1) \in \Delta$. Then perform move (v) on the rectangle $[i_0, i_1]\times[j_0, j_1]$ to obtain $(M, \Delta) \overset{\text{(v)}}{<} (\tilde{M}, \tilde{\Delta})$. We clearly have $\tilde{M} \overset{\text{rk}}{\leq} M'$ as well as $\tilde{\Delta} \leq \Delta \leq \Delta'$. Hence $(\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{\leq} (M', \Delta')$.

We have now proved the Lemma assuming $\Delta \leq \Delta'$. Next assume $\Delta \not\leq \Delta'$.

Case (v)(b). Consider any (i, j) with $(i+1, j+1) \leq \Delta$ but $(i+1, j+1) \not\leq \Delta'$. Then we have $r_{ij} + 0 = r_{\langle ij\rangle} \geq r'_{\langle ij\rangle} = r'_{ij} + 1$, so that $r_{ij} > r'_{ij}$.

Now, choose some $(k_1, l_1) \in \Delta$ with $(k_1, l_1) \not\leq \Delta'$; and choose $(k_0, l_0) < (k_1, l_1)$ so that the rectangle $[k_0, k_1]\times[l_0, l_1]$ is as large as possible with $r_{ij} > r'_{ij}$ for $(i, j) \in [k_0, k_1-1]\times[l_0, l_1-1]$. (The above remarks show that $k_0 < k_1$, $l_0 < l_1$.)

Claim: There exists $(i_0, j_0) \in [k_0, k_1-1]\times[l_0, l_1-1]$ with $m_{i_0 j_0} > 0$. Otherwise, we would have $m_{ij} = 0$ inside this rectangle. By the maximality, there exists (i, j) in the rectangle with $r_{i,l_0-1} = r'_{i,l_0-1}$ and $r_{k_0-1,j} = r'_{k_0-1,j}$. Then

$$r_{k_0-1,l_0-1} = r_{k_0-1,j} + r_{i,l_0-1} - r_{ij} < r'_{k_0-1,j} + r'_{i,l_0-1} - r'_{ij} \leq r'_{k_0-1,l_0-1}.$$

This contradicts $r_{ij} \geq r'_{ij}$, establishing the claim.

Let us say a rectangle $[i, k]\times[j, l]$ has property (R) if:

$$\text{(R)} \quad \begin{array}{l} m_{ij}, m_{kl} > 0, \quad (k, l) \in \Delta, \quad (k, l) \not\leq \Delta', \\ r_{st} > r'_{ij} \quad \text{for } (s, t) \in [i, k-1]\times[j, l-1], \end{array}$$

Our argument above shows that we may choose (i_0, j_0) so that $[i_0, k_1]\times[j_0, l_1]$ has property (R). Next choose points $(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)$, as many as possible, such that: the rectangles $[i_0, i_s]\times[j_0, j_s]$ all have property (R), and

$$\begin{gathered} i_1 < i_2 < \cdots < i_t, \quad j_1 > j_2 > \cdots > j_t, \\ [i_s, i_{s+1}-1]\times[j_{s+1}, j_s-1] \cap \Delta = \emptyset \\ [i_s, i_{s+1}-1]\times[j_{s+1}, j_s-1] \cap \Delta' = \emptyset \end{gathered}$$

for all s. That is, $(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)$ are consecutive elements of Δ (listed from northeast to southwest), not separated by intermediate elements of Δ'. By moving (i_0, j_0) southeast if necessary, and keeping only the rectangles with $(i_0, j_0) \leq (i_s, j_s)$, we may assume, in addition to all the above, that $m_{ij} = 0$ for all $(i, j) \neq (i_0, j_0)$ in the region

$$S := \bigcup_{s=1}^{t} [i_0, i_s-1]\times[j_0, j_s-1],$$

the union of all our rectangles.

Now apply move (v) to the region $[i_0, i_1]\times[j_0, j_t]$ of (M, Δ), which contains all the $[i_0, i_s]\times[j_0, j_s]$ as subrectangles, and obtain $(M, \Delta) \overset{(v)}{<} (\tilde{M}, \tilde{\Delta})$. Then $\tilde{r}_{ij} = r_{ij}$ outside S, and $\tilde{r}_{ij} = r_{ij} - 1 \geq r'_{ij}$ inside S. Hence $\tilde{r}_{ij} \geq r'_{ij}$ for all (i, j).

Further, for $(i, j) \notin S$, we have $\delta_{ij}(\tilde{\Delta}) = \delta_{ij}(\Delta)$, so clearly $\tilde{r}_{\langle ij\rangle} = r_{\langle ij\rangle} \geq r'_{\langle ij\rangle}$.

Finally, consider $(i, j) \in S$. The definition of $\tilde{\Delta}$ and S ensures that $\delta_{ij}(\tilde{\Delta}) = 0 \Rightarrow \delta_{ij}(\Delta') = 0$; if this were not so, we could enlarge the list $(i_1, j_1), \ldots, (i_t, j_t)$ while keeping (i_0, j_0) fixed. Hence for $(i, j) \in S$, we have:

$$\tilde{r}_{\langle ij\rangle} = \tilde{r}_{ij} + \delta_{ij}(\tilde{\Delta}) \geq r'_{ij} + \delta_{ij}(\Delta') = r'_{\langle ij\rangle}.$$

That is, $\tilde{r}_{\langle ij\rangle} \geq r'_{\langle ij\rangle}$ for all (i, j), and $(\tilde{M}, \tilde{\Delta}) \overset{\mathrm{rk}}{\leq} (M', \Delta')$.

This concludes the proof of Lemma 4, and hence of the Rank and Move Theorems.

6. Minimality

In this section, we prove the Minimality Theorem for triple flags. In the case of full flags $\mathbf{b} = \mathbf{c} = (1^n)$ considered in the Introduction, this follows from the fact that each simple move corresponds to a codimension-one containment of orbits $F_{M,\Delta} \subset \bar{F}_{M',\Delta'}$. However, this is not true in general. We give an alternative purely combinatorial argument.

By the Move Theorem, only the moves (i)–(v) are candidates for covers of our Bruhat order: we must show that each such move is indeed a minimal relation.

We will denote the Bruhat order on decorated matrices simply by $(M, \Delta) \leq (M', \Delta')$. We retain the notations $m_{ij}, m'_{ij}, \tilde{m}_{ij}$ for the matrix entries of $M, M', \tilde{M}$; and $r_{ij}, r'_{ij}, \tilde{r}_{ij}$ for the rank-numbers $r_{ij}(M), r_{ij}(M'), r_{ij}(\tilde{M})$.

Given a decorated matrix (M, Δ), suppose we perform some simple move on the block $[i_0, i_1]\times[j_0, j_1]$, obtaining (M', Δ'). We say a position $(i, j) \in [1, q]\times[1, r]$ is *M-inactive* with respect to the move $(M, \Delta) < (M', \Delta')$ if:

$$r_{ij} = r'_{ij},\ r_{i-1,j} = r'_{i-1,j},\ r_{i,j-1} = r'_{i,j-1},\ r_{i-1,j-1} = r'_{i-1,j-1}.$$

Otherwise (i, j) is M-*active*. Similarly, (i, j) is Δ-*inactive* if:

$$r_{\langle i-1,j-1\rangle} = r'_{\langle i-1,j-1\rangle},\ r_{\langle i-1,j\rangle} = r'_{\langle i-1,j\rangle},\ r_{\langle i,j-1\rangle} = r'_{\langle i,j-1\rangle};$$

and otherwise (i, j) is Δ-*active*. The following result implies that we may obtain (M', Δ') from (M, Δ) by changing the entries of M only at the M-active positions, and the elements of Δ only at the M-active and Δ-active positions, leaving M and Δ unchanged at all inactive positions.

Lemma 8 *If (i, j) is M-inactive, then $m_{ij} = m'_{ij}$. If (i, j) is M-inactive and Δ-inactive, then $(i, j) \in \Delta \Leftrightarrow (i, j) \in \Delta'$.*

Proof: For the first statement, $m_{ij} = r_{ij} - r_{i-1,j} - r_{i,j-1} + r_{i-1,j-1}$. For the second statement, note that $(i, j) \in \Delta$ exactly when $(i, j) \leq \Delta$ and $(i+1, j), (i, j+1) \not\leq \Delta$. □

Now suppose we have a possible couterexample to the Minimality Theorem, a single move relation $(M, \Delta) \overset{\text{mv}}{<} (M', \Delta')$, with an intervening element $(\tilde{M}, \tilde{\Delta})$ in the Bruhat order:

$$(M, \Delta) < (\tilde{M}, \tilde{\Delta}) \leq (M', \Delta').$$

We must show that $(\tilde{M}, \tilde{\Delta}) = (M', \Delta')$.

Since rank numbers must decrease at an active position, we have: if (i, j) is M-active for the move $(M, \Delta) < (\tilde{M}, \tilde{\Delta})$, then (i, j) is M-active for the move $(M, \Delta) < (M', \Delta')$; and similarly for Δ-active positions. That is, the move $(M, \Delta) \overset{\text{mv}}{<} (\tilde{M}, \tilde{\Delta})$ may act only at the active positions of $(M, \Delta) < (M', \Delta')$.

We now need only inspect the active positions of each possible move $(M, \Delta) \overset{\text{mv}}{<} (M', \Delta')$, and verify that there is no other move $(M, \Delta) \overset{\text{mv}}{<} (\tilde{M}, \tilde{\Delta})$ which can be performed on the given active positions and which satisfies $(\tilde{M}, \tilde{\Delta}) \overset{\text{rk}}{<} (M', \Delta')$.

(i) If $(M, \Delta) \overset{\text{(i)}}{<} (M', \Delta')$, then there are no M-active positions, so that $\tilde{M} = M$. Hence $\Delta < \tilde{\Delta} < \Delta'$, but this is clearly impossible since $m_{ij} = 0$ for all (i, j) with $(i, j) \not\leq \Delta$, $(i, j) \leq \Delta'$, $(i, j) \neq (i_1, j_1)$.

(ii) The active positions form the block $[i_0, i_1] \times [j_0, j_1]$. If Δ is disjoint from this block, then only other moves of type (ii) are possible, but there are none such.

If $(i_0, j_0) \in \Delta$, $m_{i_0 j_0} > 1$, we must also consider type (i) applied to $\{i_0\} \times [j_0, j_1]$ or $[i_0, i_1] \times \{j_0\}$ or $[i_1, i_2] \times [j_0, j_1]$, each of which leads to $(M, \Delta) \overset{\text{(i)}}{<} (\tilde{M}, \tilde{\Delta}) \not< (M', \Delta')$, since $\tilde{r}_{\langle i_0-1, j_0\rangle} = r_{i_0-1, j_0} < r_{i_0-1, j_0} + 1 = r'_{\langle i_0-1, j_0\rangle}$. Also possible is type (iii)(a) applied to $[i_0, i_1] \times [j_0, j_1]$, but then $(\tilde{M}, \tilde{\Delta}) \not< (M', \Delta')$, as above. We apply the transposed arguments for (iii)(b).

Similarly for the cases $(i_0, j_1) \in \Delta$ and $(i_1, j_0) \in \Delta$.

(iii)(a) The M-active positions are the block $[i_0, i_1] \times [j_0, j_1]$, and the Δ-active positions are those $(i, j) \in [i_0, i_1] \times [1, j_1]$ with $(i, j) \not\leq \Delta$. The only other possible moves are of type (i). If we apply move (i) to any block inside $[i_0, i_1] \times [1, j_0 - 1]$, then we must have $\Delta < \tilde{\Delta} < \Delta'$, which is impossible. If we apply (i) to $[i_0, i_1] \times [j_0, j_1]$, then $(M, \Delta) \overset{(i)}{<} (\tilde{M}, \tilde{\Delta}) \not< (M', \Delta')$ as before. Transpose for (iii)(b).

(iv)(a) The M-active positions are $\bar{R} = [i_0, i_1] \times [j_0, j_1] \cup [i_2, i_1] \times [j_2, j_1]$, whereas the Δ-active positions are $(i, j) \in [i_2, i_1] \times [j_0, j_1]$ with $(i, j) \not\leq \Delta$. The other possible moves are (iii) applied to $[i_2, i_1] \times [j_0, j_2]$ or to $[i_2, i_0] \times [j_0, j_1]$; or (i) applied to these same blocks. All of these give $(\tilde{M}, \tilde{\Delta}) \not< (M', \Delta')$.

(iv)(b) The active positions are $[i_0, i_1] \times [j_0, j_1]$. One possible move is (i) applied to $[i_2, i_1] \times [j_0, j_1]$. In this case $\tilde{r}_{\langle i_2, j_0 - 1\rangle} = r_{\langle i_2, j_0 - 1\rangle} - 1 < r_{\langle i_2, j_0 - 1\rangle} = r'_{\langle i_2, j_0 - 1\rangle}$, so $(\tilde{M}, \tilde{\Delta}) \not\leq (M', \Delta')$.

Another possibility is (iii)(a) applied to $[i_2, i_1] \times [j_0, j_1]$. Again $\tilde{r}_{\langle i_0 - 1, j_0\rangle} < r'_{\langle i_0 - 1, j_0\rangle}$ and $(\tilde{M}, \tilde{\Delta}) \not\leq (M', \Delta')$. Similarly for (iii)(b) applied to $[i_2, i_1] \times [j_0, j_1]$.

(iv)(c) Transpose of (iv)(b).

(v). The active positions are: $\bigcup_{s=1}^{t} [i_0, i_s] \times [j_0, j_s]$. One possible move is (ii) applied to some block $[i_0, i_s] \times [j_0, j]$ for $j_{s+1} \leq j \leq j_s$. Then $\tilde{r}_{\langle i_0 j_0\rangle} = r_{\langle i_0 j_0\rangle} - 1 < r_{\langle i_0 j_0\rangle} = r'_{\langle i_0 j_0\rangle}$, so $(\tilde{M}, \tilde{\Delta}) \not\leq (M', \Delta')$. Similarly for (ii) applied to a block $[i_0, i] \times [j_0, j_s]$.

The only other possible move is (v) applied to some smaller block $[i_0, i_l] \times [j_0, j_m]$, where $[l, m] \subset [1, t]$ (strict inclusion). Then $\tilde{r}_{\langle i_{l-1} j_{m+1}\rangle} = r_{\langle i_{l-1} j_{m+1}\rangle} - 1 + 1 < r_{\langle i_{l-1} j_{m+1}\rangle} + 1 = r'_{\langle i_{l-1} j_{m+1}\rangle}$, because $(i_0, j_0) < (i_{l-1}, j_{m+1})$. Hence $(\tilde{M}, \tilde{\Delta}) \not\leq (M', \Delta')$.

Minimality is thus proved.

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This work grew out of a project with Jerzy Weyman and Andrei Zelevinsky. The author is indebted to them for suggesting the topic, as well as numerous helpful suggestions.

Notes

1. This fact was also noted by Brion [2]. More generally, our work [5] uses the theory of quiver representations to classify *all* products of partial flag varieties having finitely many orbits of GL_n, and see also [6] for the case of the symplectic and orthogonal groups. Our case of two flags and a line is the simplest of these finite-orbit cases beyond the double flag varieties. Even for the other triple flag varieties of GL_n, the combinatorial complexity of the degenerations seems to grow formidably.
2. That is, F' degenerates to F if we can find a continuous family of configurations $(A(\tau), B_\bullet(\tau), C_\bullet(\tau))$ indexed by a parameter $\tau \in \mathbb{C}$, such that the configurations for $\tau \neq 0$ are all of type F', but for $\tau = 0$ we enter type F.
3. Note that these limits are guaranteed to exist by the properness of $\mathbb{P}^{n-1} \times \text{Flag}(\mathbf{b}) \times \text{Flag}(\mathbf{c})$. We do not need this general fact, however, since we explicitly identify the limit $(B_\bullet(0), C_\bullet(0))$.

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