Journal of Algebraic Combinatorics 2 (1993), 111-121

Combinatorics of Maximal Minors

DAVID BERNSTEIN AND ANDREI ZELEVINSKY*
Department of Mathematics, Northeastern University, Boston, MA 02115

Received December 16, 1991; Revised November 10, 1992

Abstract. We continue the study of the Newton polytope $\prod_{m,n}$ of the product of all maximal minors of an $m \times n$-matrix of indeterminates. The vertices of $\prod_{m,n}$ are encoded by coherent matching fields $\Lambda = (\Lambda_\sigma)$, where σ runs over all m-element subsets of columns, and each Λ_σ is a bijection $\sigma \to [m]$. We show that coherent matching fields satisfy some axioms analogous to the basis exchange axiom in the matroid theory. Their analysis implies that maximal minors form a universal Gröbner basis for the ideal generated by them in the polynomial ring. We study also another way of encoding vertices of $\prod_{m,n}$ for $m \leq n$ by means of "generalized permutations", which are bijections between $(n-m+1)$-element subsets of columns and $(n-m+1)$-element submultisets of rows.

Keywords: matching field, Newton polytope, maximal minor

1. Main results

In this paper we continue the study of the Newton polytope $\prod_{m,n}$ of the product of all maximal minors of an $m \times n$ matrix of indeterminates, which had begun in [1]. This study has several algebraic-geometric and analytic motivations and applications, which were discussed in [1]. But the results and methods in this paper are mostly combinatorial, and the proofs use only a bit of convex geometry. Here we prove some of the conjectures made in [1] including Conjecture 5.7 (Theorem 1 below). As shown in [1, §7], this implies the following important property of maximal minors.

THEOREM 0. **[1, Conjecture 7.1].** *The set of all maximal minors of a generic $m \times n$ matrix $X = (x_{ij})$ is a universal Gröbner basis for the ideal generated by them in the polynomial ring* $\mathbf{C}[x_{ij}]$.

This paper is essentially self-contained and can be read independently of [1]. To state our main results we reiterate some terminology and notation from [1]. We fix two integers m and n with $2 \leq m \leq n$. Let $\mathbf{R}^{m \times n}$ be the space of real $m \times n$ matrices. We abbreviate $[n] := \{1, 2, \ldots, n\}$. Throughout the whole paper

*Partially supported by the NSF (DMS-9104867).

we identify and denote by the same symbol a subset $\Omega \subset [m] \times [n]$ of matrix indices and the corresponding indicator matrix $\sum_{(i,j)\in\Omega} E_{ij}$, where the E_{ij} are matrix units.

For every m–element subset $\sigma \in [n]$ we call a *matching with support* σ any bijection $\Lambda_\sigma : \sigma \to [m]$. By slight abuse of notation, we use the same symbol Λ_σ for the graph of a matching $\{(i, j) \in [m] \times [n] : j \in \sigma, i = \Lambda_\sigma(j)\}$ (and also for its indicator matrix). A *matching field of format* $m \times n$ is a choice of a matching Λ_σ for each m–element subset $\sigma \subset [n]$. Given any matching field $\Lambda = (\Lambda_\sigma)$, we let $v(\Lambda) = \sum_\sigma \Lambda_\sigma$ denote the sum of its indicator matrices. The polytope $\prod_{m,n}$ can be combinatorially defined as the convex hull in $\mathbf{R}^{m\times n}$ of the matrices $v(\Lambda)$ for all matching fields Λ (see[1, §1]).

A matching field Λ is called *coherent* if $v(\Lambda)$ is a vertex of the polytope $\prod_{m,n}$. By [1, §1], coherent matching fields can be described as follows. Define the scalar product on $\mathbf{R}^{m\times n}$ by $\langle U, V\rangle = \sum_{i,j} u_{ij}v_{ij}$. Then a matching field $\Lambda = (\Lambda_\sigma)$ is coherent if and only if there exists a real matrix $\psi = (\psi_{ij})$ such that $\langle \psi, \Lambda_\sigma\rangle > \langle \psi, \Lambda'_\sigma\rangle$ for every σ and every matching $\Lambda'_\sigma : \sigma \to [m]$ different from Λ_σ. In this case we say that ψ *supports* Λ. The set of matrices ψ which support Λ is the *normal cone* of $\prod_{m,n}$ at the vertex $v(\Lambda)$.

It was shown in [1, Proposition 2.2] that every coherent matching field satisfies the following *linkage axiom:*

For every $i \in [m]$ and every $(m + 1)$–element subset $\tau \subset [n]$ there exist two distinct $j, j' \in \tau$ such that the matchings $\Lambda_{\tau\setminus j}$ and $\Lambda_{\tau\setminus j'}$ agree outside the ith row, i.e.,

$$\Lambda_{\tau\setminus j} - E_{ij'} = \Lambda_{\tau\setminus j'} - E_{ij}. \tag{1}$$

Following [1, §5] we associate to every matching field $\Lambda = (\Lambda_\sigma)$ and every subset $\rho \subset [n]$ of cardinality $n - m + 1$ a mapping $\Omega_\rho : \rho \to [m]$ by the following rule:

$$\Omega_\rho(j) := \Lambda_{\overline{\rho}\cup\{j\}}(j), \tag{2}$$

where $\overline{\rho} := [n]\setminus\rho$. As in the case of matchings, we use the same symbol Ω_ρ for its graph $\{(i, j) \in [m] \times [n] : j \in \rho, i = \Omega_\rho(j)\}$ (and also for its indicator matrix).

We are now in a position to state our first main result. We say that a subset $\Sigma \subset [m] \times [n]$ is *transversal* to a matching field $\Lambda = (\Lambda_\sigma)$ if $\Sigma \cap \Lambda_\sigma \neq \emptyset$ for all σ.

THEOREM 1. **[1, Conjecture 5.7].** *Let $\Lambda = (\Lambda_\sigma)$ be a matching field of format $m \times n$ satisfying the linkage axiom. Then a subset $\Sigma \subset [m] \times [n]$ is transversal to Λ if and only if $\Sigma \supset \Omega_\rho$ for some $(n - m + 1)$–element subset $\rho \subset [n]$.*

As indicated in the introduction to [1], the linkage axiom is an analog to the basis exchange axiom in the theory of matroids. In the course of the proof of Theorem 1 we sharpen this analogy by presenting two other "matroid-like" characterizations of matching fields which satisfy the linkage axiom.

THEOREM 2. *The following conditions on a matching field $\Lambda = (\Lambda_\sigma)$ are equivalent:*

(a) *Λ satisfies the linkage axiom.*
(b) *For every two distinct m–element subsets $\sigma, \sigma' \subset [n]$ there exists $j' \in \sigma' \backslash \sigma$ with the following property: if $\Lambda_{\sigma'}(j') = i$ and $j = \Lambda_\sigma^{-1}(i)$ then the matchings $\Lambda_{\sigma \backslash j \cup \{j'\}}$ and Λ_σ agree outside the ith row, i.e.,*

$$\Lambda_{\sigma \backslash j \cup \{j'\}} - E_{ij'} = \Lambda_\sigma - E_{ij}. \tag{3}$$

(c) *For every two distinct m–element subsets σ, σ' of $[n]$ having a common column j_0 such that $\Lambda_\sigma(j_0) \neq \Lambda_{\sigma'}(j_0)$ there exists an m–element subset $\sigma'' \subset \sigma \cup \sigma' \backslash j_0$ such that $\Lambda_{\sigma''} \subset \Lambda_\sigma \cup \Lambda_{\sigma'}$.*

Theorem 2 will allow us to prove the following converse of Theorem 1.

THEOREM 3. *Let $\Lambda = (\Lambda_\sigma)$ be a matching field of format $m \times n$. Suppose that every minimal with respect to inclusion subset of $[m] \times [n]$ transversal to Λ coincides with some Ω_ρ. Then Λ satisfies the linkage axiom.*

Theorems 1–3 will be proven in §2.

From now on we set $p := n - m + 1$. If $p = 1$ i.e., $m = n$, then a matching field Λ is simply a bijection between the set of columns and the set of rows of a square matrix. It was suggested in [1, §6], that the vertices of $\prod_{m,n}$ for the general m and n can be encoded by some bijections between the sets of "generalized columns" and "generalized rows" of a rectangular matrix. To be more precise, let C be the set of all p–element subsets of $[n]$, and R the set of all nonnegative integer vectors $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\sum_i \alpha_i = p$. Clearly, C and R have the same cardinality $\binom{n}{p} = \binom{n}{m-1}$. Every matching field Λ gives rise to a mapping $w = w_\Lambda : C \to R$ defined by

$$w(\rho)_i := card(\Omega_\rho^{-1}(i)). \tag{4}$$

THEOREM 4. *If Λ is a coherent matching field then $w_\Lambda : C \to R$ is a bijection.*

This result establishes Conjecture 6.11 from [1] for coherent matching fields.

THEOREM 5. *A coherent matching field Λ is uniquely determined by the bijection $w_\Lambda : C \to R$.*

We will give two different proofs of Theorem 5. The first one shows that for every matching field Λ (not necessarily coherent) the point $v(\Lambda) := \sum_\sigma \Lambda_\sigma \in \prod_{m,n}$ is recovered from the mapping $w_\Lambda : C \to R$ as follows:

We identify a subset $\rho \subset [n]$ with its indicator vector $(\rho_1, \ldots, \rho_n)$, where $\rho_j = 1$ if $j \in \rho$, and $\rho_j = 0$ if $j \notin \rho$. For every $\alpha \in R$, $\rho \in C$ we denote by $\alpha \cdot \rho$ an $m \times n$ matrix whose (i, j)–th entry is equal to $\alpha_i \rho_j$. We associate to a mapping $w : C \to R$ the matrix

$$M(w) := \sum_{\rho \in C} w(\rho) \cdot \rho. \tag{5}$$

Let $\mathbf{1}_{m \times n}$ denote an $m \times n$ matrix with all entries equal to 1.

THEOREM 6. *Let Λ be an arbitrary matching field (not necessarily coherent), and $w_\Lambda : C \to R$ be the corresponding mapping. Then*

$$v(\Lambda) = M(w_\Lambda) - \binom{n-1}{m} \mathbf{1}_{m \times n}. \tag{6}$$

In the second proof of Theorem 5 we assume that Λ is coherent and show that all the sets Ω_ρ can be recovered by the bijection $w_\Lambda : C \to R$. The set R is a set of all integer points of a "thick" simplex with vertices $pe_1, pe_2, \ldots, pe_m$, where $e_1, \ldots, e_m$ are standard basis vectors in $\mathbf{R}^m$. We make R a graph with α and α' joined by an edge if and only if $\alpha - \alpha' = e_i - e_{i'}$ for some indices $i \neq i'$. By a *path* from α to α' we mean a chain $(\alpha(0) = \alpha, \alpha(1), \ldots, \alpha(d) = \alpha')$ of minimal possible length such that $\alpha(k-1)$ and $\alpha(k)$ are joined by an edge for $k = 1, \ldots, d$; here d is the *distance* between α and α' in R.

THEOREM 7. *Let $w = w_\Lambda : C \to R$ be a bijection corresponding to a coherent matching field Λ. Let $\rho \in C$, $i \in [m]$, and let $(\alpha(0), \alpha(1), \ldots, \alpha(d))$ be an arbitrary path from $w(\rho)$ to pe_i in R. Then*

$$\Omega_\rho^{-1}(i) = \cap_{k=0}^{d} w^{-1}(\alpha(k)). \tag{7}$$

Theorem 7 provides some necessary conditions for a bijection $w : C \to R$ to be of the form w_Λ for a coherent matching field Λ. Namely, the RHS of (7) must be independent of the choice of a path $(\alpha(0), \alpha(1), \ldots, \alpha(d))$, and the subsets $\Omega_\rho^{-1}(i)$ defined by means of (7) must satisfy

$$card(\Omega_\rho^{-1}(i)) = w(\rho)_i, \; \Omega_\rho^{-1}(i) \cap \Omega_\rho^{-1}(i') = \emptyset \tag{8}$$

for all $\rho \in C$ and distinct $i, i' \in [m]$. It turns out that these necessary conditions are sufficient in the case $n = m + 1$. More precisely, we have the following theorem.

THEOREM 8. *Suppose that $n = m + 1$, i.e., $p = 2$. Then the following conditions on a bijection $u : R \to C$ are equivalent :*

(a) A bijection u has the form w_Λ^{-1} for some coherent matching field Λ of format $m \times (m+1)$.

(b) For every two distinct $i, i' \in [m]$

$$card(u(2e_i) \cap u(e_i + e_{i'})) = 1, \; u(2e_i) \cap u(e_i + e_{i'}) \cap u(2e_{i'}) = \emptyset. \tag{9}$$

It would be interesting to investigate how far the conditions (8) are from being sufficient for general m and n.

As a by-product of our proof of Theorem 4 we will derive the following alternative description of the polytope $\prod_{m,n}$. For each nonnegative integer vector $\beta = (\beta_1, \ldots, \beta_m)$ with sum n let Ξ_β denote the polytope of nonnegative $m \times n$ matrices having row sums $\beta_1, \ldots, \beta_m$ and all column sums equal to 1.

THEOREM 9. *The polytope $\prod_{m,n}$ coincides with the Minkowski sum $\sum_\beta \Xi_\beta$, the summation over all integer vectors $\beta = (\beta_1, \ldots, \beta_m)$ with $\sum_i \beta_i = n$ and all $\beta_i \geq 1$.*

Theorems 4–9 will be proven in §3.

2. Proof of Theorems 1–3

Proof of Theorem 2. We will use two equivalent versions of the linkage axiom established in [1].

LEMMA 10. **[1, Theorem 2.4 and Proposition 2.13]**. *The following conditions on a matching field $\Lambda = (\Lambda_\sigma)$ are equivalent:*

- *(a) Λ satisfies the linkage axiom.*
- *(a′) For every $(m+1)$-element subset $\tau \subset [n]$ there exists a tree T with the set of vertices τ and the set of edges labeled bijectively by $[m]$, such that for every $j_0 \in \tau$ the matching $\Lambda_{\tau\setminus j_0}$ sends each $j \in \tau\setminus j_0$ to an index i such that the unique path from j to j_0 in T starts with the edge labeled by i.*
- *(a″) For every $(m+1)$-element subset $\tau \subset [n]$ and any three different elements j_1, j_2, $j_3 \in \tau$: if $\Lambda_{\tau\setminus j_1}(j_2) \neq \Lambda_{\tau\setminus j_3}(j_2)$ then $\Lambda_{\tau\setminus j_1}(j_3) = \Lambda_{\tau\setminus j_2}(j_3)$.*

To prove Theorem 2 we will establish the implications $(a') \Rightarrow (b) \Rightarrow (c) \Rightarrow (a'')$.

Proof of $(a') \Rightarrow (b)$. Let $\Lambda = (\Lambda_\sigma)$ be a matching field satisfying (a').

LEMMA 11. **[1, Proposition 5.6]**. *Each subset Ω_ρ in (2) is transversal to Λ.*

In order to prove that Λ satisfies (b) we choose two distinct m–element subsets $\sigma, \sigma' \subset [n]$. We have to show that there exists $j' \in \sigma'\setminus\sigma$ which satisfies (3). Choose arbitrary $j_0 \in \sigma\setminus\sigma'$ and consider the $(n-m+1)$–element set $\rho := \overline{\sigma} \cup \{j_0\}$, where $\overline{\sigma} = [n]\setminus\sigma$. By Lemma 11, $\Omega_\rho \cap \Lambda_{\sigma'} \neq \emptyset$. Choose a point $(i, j') \in \Omega_\rho \cap \Lambda_{\sigma'}$. Now take $\tau := \sigma \cup \{j'\}$ and consider the tree T provided by (a'). By definition of Ω_ρ, we have $\Lambda_{\tau\setminus j_0}(j') = i$; therefore, the edge i in T passes through the vertex j'. Let j be the second end of this edge. Again using (a') we see that $\Lambda_{\tau\setminus j'}(j) = i$, and that $\Lambda_{\tau\setminus j}$ and $\Lambda_{\tau\setminus j'}$ agree with each other outside the ith row. But this is exactly the property (3) because $\tau \backslash j' = \sigma$ and $\tau \backslash j = \sigma \backslash j \cup \{j'\}$.

Proof of (*b*) $\Rightarrow$ (*c*). We proceed by induction on $p(\sigma, \sigma') := card(\Lambda_{\sigma'} \backslash \Lambda_\sigma)$. Clearly, $p(\sigma, \sigma') \geq 2$, so we start with the case when $p(\sigma, \sigma') = 2$. Then the set $\tau := \sigma \cup \sigma'$ consists of $m + 1$ elements, and we have $\sigma = \tau \backslash j', \sigma' = \tau \backslash j$ for some $j, j' \in \tau$, and $\Lambda_\sigma(k) = \Lambda_{\sigma'}(k)$ for $k \in \tau \backslash \{j_0, j, j'\}$. Applying the property (b) to the subsets σ and σ' and taking into account that in this case $\sigma' \backslash \sigma$ consists of one element j', we obtain that $\Lambda_{\tau \backslash j_0}(j') = \Lambda_{\sigma'}(j')$, and $\Lambda_{\tau \backslash j_0}(k) = \Lambda_\sigma(k)$ for $k \in \sigma \backslash j_0$. Therefore, the subset $\sigma'' := \tau \backslash j_0$ satisfies (c), as desired.

Now suppose that $p(\sigma, \sigma') \geq 3$, and assume that (c) holds for all pairs σ_0, σ' with $p(\sigma_0, \sigma') < p(\sigma, \sigma')$. Apply (b) to σ and σ', and let $j' \in \sigma' \backslash \sigma, j \in \sigma, i \in [m]$ be the elements satisfying (3). Set $\sigma_0 := \sigma \backslash j \cup \{j'\}$. By (3), $\Lambda_{\sigma_0}(j') = i = \Lambda_{\sigma'}(j')$, and $\Lambda_{\sigma_0}(k) = \Lambda_\sigma(k)$ for $k \in \sigma \backslash j$. This implies, in particular, that $\Lambda_{\sigma_0} \subset \Lambda_\sigma \cup \Lambda_{\sigma'}$. If $j = j_0$ then the subset $\sigma'' = \sigma_0$ satisfies (c), and we are done; so assume that $j \neq j_0$. By construction, $p(\sigma_0, \sigma') < p(\sigma, \sigma')$, so by inductive assumption we can find $\sigma'' \subset \sigma_0 \cup \sigma' \backslash j_0$ such that $\Lambda_{\sigma''} \subset \Lambda_{\sigma_0} \cup \Lambda_{\sigma'}$. But then $\Lambda_{\sigma''} \subset \Lambda_\sigma \cup \Lambda_{\sigma'}$, and we are done.

To complete the proof of Theorem 2 it remains to observe that the property (a'') is a special case of (c) for $\sigma = \tau \backslash j_1, \sigma' = \tau \backslash j_3, j_0 = j_2$. □

Proof of Theorem 1. Let $\Lambda = (\Lambda_\sigma)$ be a matching field which satisfies the linkage axiom, and hence, by Theorem 2, also the property (c). Taking into account Lemma 11 we have only to prove the following statement: If $\Sigma \subset [m] \times [n]$ is transversal to Λ then $\Sigma \supset \Omega_\rho$ for some $(n - m + 1)$-element subset $\rho \subset [n]$. Without loss of generality we can assume that $(m, n) \in \Sigma$ and Σ is minimal transversal to Λ with respect to inclusion. By minimality of Σ, there exists an m-element subset $\sigma \subset [n]$ such that

$$\Sigma \cap \Lambda_\sigma = \{(m, n)\}. \tag{10}$$

Consider the restriction of Λ to $[m] \times [n-1]$, i.e., the matching field Λ' of format $m \times (n-1)$ formed by all matchings $\Lambda_{\sigma'}$ with $\sigma' \subset [n-1]$. Let $(\Omega_{\rho'})$ be the corresponding family of subsets of $[m] \times [n-1]$ constructed from Λ' by means of (2); here ρ' runs over $(n-m)$-element subsets of $[n-1]$. Let $\Sigma' = \Sigma \cap ([m] \times [n-1])$. Then Σ' is transversal to Λ'. Using induction on n we can assume that $\Sigma' \supset \Omega_{\rho'}$ for some $\rho' \subset [n-1]$ (as a first inductive step we can take $n = m$, in which case our statement becomes tautological).

Set $\Omega := \Omega_{\rho'} \cup \{(m, n)\}$. We claim that Ω is transversal to Λ. By Lemma 11, Ω is transversal to $\Lambda_{\sigma'}$ for all $\sigma' \subset [n-1]$. It remains to prove that $\Omega \cap \Lambda_{\sigma'} \neq \emptyset$ for each σ' containing n and such that $\Lambda_{\sigma'}(n) \neq m$. To show this we apply the property (c) from Theorem 2 to σ, σ' and $j_0 = n$. We obtain that there exists $\sigma'' \subset ([n-1] \cap \sigma) \cup \sigma'$ such that $\Lambda_{\sigma''} \subset \Lambda_\sigma \cup \Lambda_{\sigma'}$. We have already seen that $\Omega \cap \Lambda_{\sigma''} \neq \emptyset$, therefore

$$\Omega \cap ((([m] \times [n-1]) \cap \Lambda_\sigma) \cup \Lambda_{\sigma'}) \neq \emptyset.$$

But $\Omega \cap (([m] \times [n-1]) \cap \Lambda_\sigma) = \emptyset$ by (10), so $\Omega \cap \Lambda_{\sigma'} \neq \emptyset$, as claimed.

It remains to show that $\Omega = \Omega_\rho$ for $\rho := \rho' \cup \{n\}$. But this is clear because for every $j \in \rho$ the matching $\Lambda_{\bar{\rho} \cup \{j\}}$ can intersect Ω only in the element $\Lambda_{\bar{\rho} \cup \{j\}}(j)$ (cf. [1, Lemma 5.4]). Theorem 1 is proven. □

Proof of Theorem 3. Suppose that a matching field Λ does not satisfy the linkage axiom. Then the property (c) of Theorem 2 also fails, i.e., there exist two m-element subsets σ and σ' of $[n]$ and an index $j \in \sigma \cap \sigma'$ such that $\Lambda_\sigma(j) \neq \Lambda_{\sigma'}(j)$, and that there is no matching $\Lambda_{\sigma''}$ contained in $\Lambda_\sigma \cup \Lambda_{\sigma'} \backslash ([m] \times \{j\})$. Let $i = \Lambda_\sigma(j)$, $i' = \Lambda_{\sigma'}(j)$, and

$$\Sigma := ([m] \times [n]) \backslash (\Lambda_\sigma \cup \Lambda_{\sigma'}) \cup ([m] \times \{j\}).$$

Our choice of σ and σ' implies that Σ is transversal to Λ. Choose a subset $\Sigma_0 \subset \Sigma$ which is minimal with respect to inclusion transversal to Λ. Since $\Sigma \cap \Lambda_\sigma = \{(i, j)\}$, $\Sigma \cap \Lambda_{\sigma'} = \{(i', j)\}$, it follows that Σ_0 contains both $\{(i, j)\}$ and $\{(i', j)\}$. Therefore, Σ_0 is not one of the subsets Ω_ρ, which proves Theorem 3. □

3. Proof of Theorems 4–9

Proof of Theorem 4. We fix a coherent matching field $\Lambda = (\Lambda_\sigma)$; a matrix $\psi = (\psi_{ij})$ supporting Λ; and a vector $\alpha = (\alpha_1, \dots, \alpha_m) \in R$. Since $card(R) = card(C)$, to prove our theorem it is enough to construct $\rho \in C$ such that $w_\Lambda(\rho) = \alpha$.

For each nonnegative integer vector $\beta = (\beta_1, \dots, \beta_m)$ with sum n, let Ξ_β denote the polytope of nonnegative $m \times n$ matrices having row sums $\beta_1, \dots, \beta_m$ and all column sums equal to 1. Let Γ_β be the vertex of Ξ_β supported by ψ; we assume ψ to be sufficiently generic so that Γ_β is unique. Clearly, each Γ_β is a (0,1)-matrix, and we identify it with its support which is a subset of $[m] \times [n]$, and also with the mapping $[n] \to [m]$ whose graph is this subset.

For each $i \in [m]$ let $\beta(i)$ denote the vector

$$\beta(i) = (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_{i-1} + 1, \alpha_i, \alpha_{i+1} + 1, \dots, \alpha_m + 1). \tag{11}$$

LEMMA 12. *For every two distinct indices $i, i' = 1, \dots, m$ the ith row of $\Gamma_{\beta(i)}$ is contained in the ith row of $\Gamma_{\beta(i')}$.*

Proof. We associate to i and i' an edge-colored oriented graph $G_{i,i'}$ with the vertex set $[m]$ and the color set $[n]$ as follows: There is an edge from i_1 to i_2 colored by j whenever $i_1 \neq i_2$, $(i_1, j) \in \Gamma_{\beta(i)}$ and $(i_2, j) \in \Gamma_{\beta(i')}$. Lemma 12 is an immediate consequence of the following lemma.

LEMMA 13. *The graph $G_{i,i'}$ is an oriented chain from i' to i.*

Proof. First we show that $G_{i,i'}$ has no oriented cycles. Suppose there is an oriented cycle

$$i_1 \xrightarrow{j_1} i_2 \xrightarrow{j_2} \dots \xrightarrow{j_{p-1}} i_p \xrightarrow{j_p} i_1.$$

Consider two subsets $S = \{(i_1, j_1), (i_2, j_2), \dots, (i_p, j_p)\}$ and $S' = \{(i_2, j_1), (i_3, j_2), \dots, (i_p, j_{p-1}), (i_1, j_p)\}$ of $[m] \times [n]$. These two subsets occupy the same rows and columns, and we have $S \subset \Gamma_{\beta(i)}$, $S' \subset \Gamma_{\beta(i')}$. But this contradicts the fact that both $\Gamma_{\beta(i)}$ and $\Gamma_{\beta(i')}$ are supported by the same linear functional ψ. Indeed, without loss of generality we can assume that $\langle \psi, S \rangle < \langle \psi, S' \rangle$. But then $\Gamma_{\beta(i)} + S' - S$ is a point in the polytope $\Xi_{\beta(i)}$ with $\langle \psi, \Gamma_{\beta(i)} + S' - S \rangle > \langle \psi, \Gamma_{\beta(i)} \rangle$, which contradicts our choice of $\Gamma_{\beta(i)}$.

We define the *valency* of a vertex i_0 in $G_{i,i'}$ as the number of edges going out of i_0 minus the number of edges going into i_0. Since the graph describes passing from $\Gamma_{\beta(i)}$ to $\Gamma_{\beta(i')}$, all the vertices have valency 0, except i having valency -1 and i' having valency 1. Since $G_{i,i'}$ has no oriented cycles, it can be only a chain from i' to i. This completes the proof of Lemmas 12 and 13. □

Now let $\rho(i) := \Gamma^{-1}_{\beta(i)}(i)$ be the set of columns occupied by the ith row of $\Gamma_{\beta(i)}$, and let $\rho = \cup_{i=1}^{m} \rho(i)$. By Lemma 12, the subsets $\rho(i)$ are mutually disjoint. Since $card(\rho(i)) = \alpha_i$, it follows that $card(\rho) = p$, i.e., $\rho \in C$.

Put $\Omega := \cup_{i=1}^{m}(\{i\} \times \rho(i))$. To prove Theorem 4 it suffices to show that $\Omega = \Omega_\rho$, which implies that $w_A(\rho) = \alpha$.

LEMMA 14. *There exists a tree T with the vertex set $[m]$ whose edges are labeled bijectively by $\overline{\rho} := [n] \backslash \rho$, such that:*

(a) For every $i \in [m]$

$$\Gamma_{\beta(i)} = \Omega \cup \{(i', j) \in [m] \times \overline{\rho} : i' \xrightarrow{j} i\}.$$

(b) For every $j_0 \in \rho_i$

$$A_{\overline{\rho} \cup \{j_0\}} = \{(i, j_0)\} \cup \{(i', j) \in [m] \times \overline{\rho} : i' \xrightarrow{j} i\}; \tag{12}$$

here the notation "$i' \xrightarrow{j} i$" means that the unique path from i' to i in T starts with the edge labeled by j.

Proof.

(a) By Lemma 12, $\Omega \subset \Gamma_{\beta(i)}$ for each $i \in [m]$, and the difference $\Gamma_{\beta(i)} \backslash \Omega$ is the graph of a bijection $A'_{[m] \backslash i} : [m] \backslash i \to \overline{\rho}$. Passing to transpose matrices, we represent the family of bijections $(\Lambda'_{[m]\setminus i}), i = 1, \ldots, m$ as a matching field of format $(m-1) \times m$ with the row set $\overline{\rho}$ and the column set $[m]$. This matching field is coherent because it is supported by the transpose of ψ. Therefore, it can be described as in Lemma 10 (a'), which is exactly our statement.

(b) Put $\sigma = \overline{\rho} \cup \{j_0\}$. Clearly, the RHS of (12) is (the graph of) a matching $\Lambda'_\sigma : \sigma \to [m]$. By (a), $\Lambda'_\sigma \subset \Gamma_{\beta(i)}$. It follows that $\Lambda'_\sigma = \Lambda_\sigma$; otherwise, $\Gamma_{\beta(i)} - \Lambda'_\sigma + \Lambda_\sigma$ would be a point of the polytope $\Xi_{\beta(i)}$ having a bigger value of the supporting form ψ than $\Gamma_{\beta(i)}$. □

Comparing (12) with (2), we see that if $j_0 \in \rho(i)$ then $\Omega_\rho(j_0) = i$. Therefore, $\Omega = \Omega_\rho$, which completes the proof of Theorem 4. □

Proof of Theorem 9. Let ψ be a generic linear form on the space of matrices. Let $\Lambda = (\Lambda_\sigma)$ be a coherent matching field supported by ψ, and for each nonnegative integer vector $\beta = (\beta_1, \ldots, \beta_m)$ with $\sum_i \beta_i = n$ and all $\beta_i \geq 1$, let Γ_β be the vertex of Ξ_β supported by ψ. It is enough to show that the vertex $v(\Lambda) := \sum_\sigma \Lambda_\sigma$ of $\prod_{m,n}$ coincides with the vertex $\sum_\beta \Gamma_\beta$ of $\sum_\beta \Xi_\beta$. This is a consequence of the following lemma.

LEMMA 15. *For every $(i, j) \in [m] \times [n]$ there exists a bijection between the set of all m-element subsets $\sigma \subset [n]$ such that $j \in \sigma$ and $\Lambda_\sigma(j) = i$, and the set of all β as above such that $(i, j) \in \Gamma_\beta$.*

Proof. Choose $\sigma \ni j$ such that $\Lambda_\sigma(j) = i$. Take $\rho = \overline{\sigma} \cup \{j\} \in C$ and define $\alpha = w_\Lambda(\rho) \in R$. Let $\beta = \beta(i)$ be a vector associated to α by means of (11). Using the description of Γ_β and Λ_σ given by Lemma 14, we obtain the following relation:

$$\Gamma_\beta = \Lambda_\sigma + \Omega_\rho - E_{ij}. \tag{13}$$

This implies that $(i, j) \in \Gamma_\beta$, and hence the mapping $\sigma \mapsto \beta$ is a desired bijection. Lemma 15 and Theorem 9 are proven. □

Proof of Theorems 5 and 6. As usual, we will identify subsets of $[m] \times [n]$ with their indicator matrices. By definition,

$$v(\Lambda) = \sum_\sigma \Lambda_\sigma = \sum_\sigma \sum_{j \in \sigma} \{(\Lambda_\sigma(j), j)\}.$$

Let us represent each one-element set $\{(\Lambda_\sigma(j), j)\}$ in the sum as a difference $\{\Lambda_\sigma(j)\} \times (\overline{\sigma} \cup \{j\}) - \{\Lambda_\sigma(j)\} \times \overline{\sigma}$. We can write then $v(\Lambda) = M_1 - M_2$, where

$$M_1 = \sum_\sigma \sum_{j \in \sigma} \{\Lambda_\sigma(j)\} \times (\overline{\sigma} \cup \{j\}),\ M_2 = \sum_\sigma \sum_{j \in \sigma} \{\Lambda_\sigma(j)\} \times \overline{\sigma}.$$

We will show that $M_1 = M(w_\Lambda)$, $M_2 = \binom{n-1}{m} \mathbf{1}_{m\times n}$.

If we put $\rho = \overline{\sigma} \cup \{j\}$, then by (2) we have $\Lambda_\sigma(j) = \Omega_\rho(j)$. Therefore, the summation for M_1 can be rewritten as

$$M_1 = \sum_{\rho\in C} \sum_{j\in\rho} \{\Omega_\rho(j)\} \times \rho. \tag{14}$$

Remembering the definition of w_Λ we see that the inner sum in (14) is $w_\Lambda(\rho)\cdot\rho$, so $M_1 = M(w_\Lambda)$, as desired.

As for M_2, substituting $i = \Lambda_\sigma(j)$ we can rewrite the summation as

$$M_2 = \sum_{i,\sigma} \{i\} \times \overline{\sigma}.$$

Therefore, for every i and j the (i, j)-th entry of M_2 is equal to the number of $(n-m)$-element subsets $\overline{\sigma} \subset [n]$ which contain j, i.e., to $\binom{n-1}{n-m-1} = \binom{n-1}{m}$. This completes the proof of Theorem 6. To prove Theorem 5 it remains to observe that a coherent matching field Λ is uniquely determined by the vertex $v(\Lambda)$. □

Proof of Theorem 7. Let $\alpha = (\alpha_1, \ldots, \alpha_m) = w_\Lambda(\rho)$. Then the distance d from α to pe_i is equal to $p - \alpha_i$, and the first vector in a path from α to pe_i has the form $\alpha(1) = \alpha + e_i - e_{i'}$ for some $i' \neq i$. Let $\rho' = w_\Lambda^{-1}(\alpha(1))$. By induction on d, to prove Theorem 7 it is enough to show that

$$\Omega_\rho^{-1}(i) = \rho \cap \Omega_{\rho'}^{-1}(i). \tag{15}$$

Let $\beta(1), \ldots, \beta(m)$ be a family of vectors associated to α by means of (11). By (13), $\Omega_\rho^{-1}(i)$ is the set of columns occupied by the ith row of $\Gamma_{\beta(i)}$. Now apply the same statement to a family $\beta'(1), \ldots, \beta'(m)$ associated by the same rule (11) to the vector $\alpha(1)$. By definition, $\beta'(i) = \beta(i')$, so we obtain that $\Omega_{\rho'}^{-1}(i)$ is the set of columns occupied by the ith row of $\Gamma_{\beta(i')}$. Now (15) follows at once from the description of $\Gamma_{\beta(i)}$ and $\Gamma_{\beta(i')}$ given by Lemma 14. Theorem 7 is proven. □

Proof of Theorem 8. The implication $(a) \Rightarrow (b)$ is clear because for $n = m+1$, the condition (9) is simply another way of saying (8). So we suppose that a bijection $u : R \to C$ satisfies (9), and we wish to show that $u = w_\Lambda^{-1}$ for some coherent matching field Λ of format $m \times (m+1)$.

Consider the edge-colored graph T with the set of vertices $[m+1]$ and the set of colors $[m]$, where two vertices j and j' are joined by an edge colored with i whenever $u(2e_i) = \{j, j'\}$. Clearly, T has m edges, each color appearing exactly once.

LEMMA 16. *Suppose that $s \geq 2$, and $j_1, j_2, \ldots, j_s$ are distinct elements of $[m+1]$ such that j_k and j_{k+1} are joined in T by an edge colored with i_k for $k = 1, \ldots, s-1$. Then*

$$u(e_{i_k} + e_{i_{k'}}) = \{j_k, j_{k'+1}\} \tag{16}$$

for all $1 \leq k \leq k' \leq s - 1$.

Proof of Lemma 16. We proceed by induction on s. For $s = 2$ the equality (16) is simply the definition of the graph T, and for $s = 3$ it follows at once from (9). So we can assume that $s \geq 4$, and that (16) holds for all pairs (k, k') except $(1, s-1)$. By (9), the two-element set $u(e_{i_1} + e_{i_{s-1}})$ has nonempty intersection with each of the sets $u(2e_{i_1}) = \{j_1, j_2\}$ and $u(2e_{i_{s-1}}) = \{j_{s-1}, j_s\}$. Therefore, $u(e_{i_1} + e_{i_{s-1}})$ has the form $\{j, j'\}$, where $j \in \{j_1, j_2\}, j' \in \{j_{s-1}, j_s\}$. But our inductive assumption implies that

$$\{j_1, j_{s-1}\} = u(e_{i_1} + e_{i_{s-2}}), \{j_2, j_{s-1}\} = u(e_{i_2} + e_{i_{s-2}}), \{j_2, j_s\} = u(e_{i_2} + e_{i_{s-1}}).$$

Since u is a bijection, the only opportunity left is $u(e_{i_1} + e_{i_{s-1}}) = \{j_1, j_s\}$, as desired. □

By Lemma 16, every two vertices in T are connected by at most one chain; hence, T has no loops. Since T has $m + 1$ vertices and m edges, we conclude that T is a tree. Consider the matching field Λ of format $m \times (m + 1)$ associated to the tree T as in Lemma 10 (a'). Then Λ is coherent by [1, Theorem 2.4]. Comparing Lemma 16 with the formulas (2) and (4) above, we see that $u = w_\Lambda^{-1}$, which completes the proof of Theorem 8. □

Reference

1. B. Sturmfels and A. Zelevinsky, "Maximal minors and their leading terms," *Advances in Math*, **98** (1993), 65–112.