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Semiboolean SQS-skeins

ANDREAS J. GUELZOW
Department of Mathematics and Computer Science, Brandon University, Brandon, MB R7A 6A9, Canada

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Abstract. We will present a counter example to the conjecture that the class of boolean SQS-skeins is defined by the equation $q(x, u, q(y, u, z)) = q(q(x, u, y), u, z)$. The SQS-skeins satisfying this equation will be seen to be exactly those SQS-skeins that correspond to Steiner quadruple systems whose derived Steiner triple systems are all projective geometries.

Keywords: Steiner quadruple system, Steiner triple system, SQS-skein, semiboolean, derived Steiner triple system, projective geometry

1. Introduction

An *SQS-skein*, which is also called *Steiner Ternar, idempotent totally symmetric 3-quasigroup,* or *Steiner 3-quasigroup,* is an algebra $\langle S; q\rangle$ of type (3) satisfying the equations:

$$\begin{aligned} q(x, x, y) &= y \\ q(x, y, z) &= q(x, z, y) \\ q(x, y, z) &= q(y, z, x) \text{ and} \\ q(x, y, q(x, y, z)) &= z \end{aligned}$$

SQS-skeins arise as a coordinatization of Steiner quadruple systems (see [5]) and have been extensively studied by Armanious in [1]. It is known that the smallest nontrivial subvariety is the class of all boolean SQS-skeins. An SQS-skein $\langle S; q\rangle$ is called *boolean* if there exists a boolean group $\langle S; +, 0\rangle$ such that $q(x, y, z) = x + y + z$.

In [7], [8], and [9] it is stated without proof that the class of all boolean SQS-skeins is characterized by the equation

$$q(x, u, q(y, u, z)) = q(q(x, u, y), u, z). \tag{1}$$

We will show that this is incorrect. Obviously, an SQS-skein is boolean if and only if it satisfies

$$q(x, u, q(y, u, z)) = q(x, y, z). \tag{2}$$

(Equation (2) corresponds to the associative law in the boolean group.) Every boolean SQS-skein must therefore also satisfy (1). In none of the three papers

[7], [8], and [9] has the converse been shown; in fact, in [1] Armanious used (2) to define boolean SQS-skeins instead of (1) and stated that he was unable to prove or disprove the existence of a nonboolean SQS-skein satisfying (1).

We will now construct an SQS-skein $\mathfrak{H}_{16} = \langle H; q\rangle$ that satisfies (1), but not (2).

2. An example

Let H be a four-dimensional vector space over GF(2) and let q be the ternary operation on H given by:

$$q\left(\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix},\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix}\right)=\begin{pmatrix}x_1+y_1+z_1\\x_2+y_2+z_2\\x_3+y_3+z_3\\x_4+y_4+z_4+\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}\end{pmatrix}$$

It is straightforward to verify that $\langle H; q\rangle$ is indeed an SQS-skein. (Note that only one of the defining equations requires some work.) It does not satisfy (2) since:

$$q\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},q\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right)\right)=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

$$\neq\begin{pmatrix}1\\1\\0\\0\end{pmatrix}=q\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right)$$

It is also easy to check that (1) holds. We have (omitting a few steps):

$$q(x,u,q(y,u,z))=\begin{pmatrix}x_1+y_1+z_1\\x_2+y_2+z_2\\x_3+y_3+z_3\\x_4+y_4+z_4+\begin{vmatrix}x_1+z_1&y_1&u_1\\x_2+z_2&y_2&u_2\\x_3+z_3&y_3&u_3\end{vmatrix}+\begin{vmatrix}x_1&u_1&z_1\\x_2&u_2&z_2\\x_3&u_3&z_3\end{vmatrix}\end{pmatrix}=q(q(x,u,y),u,z))$$

This example, which we will in future refer to as $\mathfrak{H}_{16}$, justifies the introduction of a new term.

Definition. An SQS-skein $\langle S; q\rangle$ is called *semiboolean* if it satisfies the equation:

$$q(x, u, q(y, u, z)) = q(q(x, u, y), u, z)$$

We will see in the next section that the term *semiboolean* is appropriate.

3. Properties

The following lemma is a straightforward consequence of the defining equations. It justifies the choice of the term *semiboolean*.

LEMMA 1. *If* $\langle S; q\rangle$ *is a semiboolean SQS-skein, then for every* $0 \in S$ *the algebra* $\langle S; +, 0\rangle$ *with* $x + y = q(x, y, 0)$ *is a boolean group.*

An immediate consequence is:

COROLLARY 2. *If* $\langle S; q\rangle$ *is a finite semiboolean SQS-skein then* $|S| = 2^r$ *for some nonnegative integer r.*

It is well known that an SQS-skeins is boolean if and only if it is of nilpotence class 1. (A general definition of the concept of nilpotence can be found in [4].) Since $\mathfrak{H}_{16}$ is not boolean, it can therefore not be of nilpotence class 1. We will show that it is of nilpotence class 2. For this purpose we require the following fact (for a proof of a more general statement see [4]):

LEMMA 3. *Let* $\mathfrak{V}$ *be a permutable variety with Mal'cev term* $p(x, y, z)$, *let* $\langle A, \Omega\rangle = \mathfrak{A}$ *be an algebra in* $\mathfrak{V}$ *and let* $\zeta(\mathfrak{A})$ *denote the center of* $\mathfrak{A}$. *Then* $a\,\zeta(\mathfrak{A})\,b$ *if and only if*

$$f(p(r_1(a,b),r_1(b,b),c_1),\ldots,p(r_n(a,b),r_n(b,b),c_n)) = p(f(r_1(a,b),\ldots,r_n(a,b)),f(r_1(b,b),\ldots,r_n(b,b)),f(\mathbf{c}))$$

for all $f \in \Omega$, *all* $\mathbf{c} = (c_1, \ldots, c_n) \in A^n$ *(n being the arity of f) and all binary term functions* $r_1(x, y), \ldots, r_n(x, y)$.

For SQS-skeins this lemma becomes:

COROLLARY 4. *Let* $\mathfrak{S} = \langle S; q\rangle$ *be an SQS-skein. Then* $a\,\zeta(\mathfrak{S})\,b$ *if and only if for all* $c_1, c_2, c_3 \in S$:

$$q(q(a, b, c_1), c_2, c_3) = q(a, b, q(c_1, c_2, c_3))$$

Proof. Since SQS-skeins have only the two term binary functions $r_1(x, y) = x$ and $r_2(x, y) = y$ and the ternary operation q itself is a Mal'cev polynomial, Lemma 3 implies that $a\,\zeta(\mathfrak{S})\,b$ if and only if the following three statements hold:

$$q(q(a,b,c_1),c_2,c_3) = q(a,b,q(c_1,c_2,c_3)) \quad \text{for all } c_1,c_2,c_3 \in S \tag{3}$$

$$q(q(a,b,c_1),q(a,b,c_2),c_3) = q(c_1,c_2,c_3) \quad \text{for all } c_1,c_2,c_3 \in S \tag{4}$$

$$q(q(a,b,c_1),q(a,b,c_2),q(a,b,c_3)) = q(a,b,q(c_1,c_2,c_3)) \quad \text{for all } c_1,c_2,c_3 \in S \tag{5}$$

It is straightforward to verify that (3) implies (4) and (5). □

Let us now consider the center of $\mathfrak{H}_{16}$. By Corollary 4 it is easy to verify that

$$\begin{pmatrix} w_1 \\ \vdots \\ w_4 \end{pmatrix} \in \left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\right]_{\zeta(\mathfrak{H}_{16})}$$

if and only if $w_1 = w_2 = w_3 = 0$, i.e., the center of $\mathfrak{H}_{16}$ is the kernel of the projection onto the first three components. Since the image of this projection is obviously boolean, we have shown that $\mathfrak{H}_{16}$ is nilpotent of class 2.

Since our example is semiboolean and nilpotent (of class 2), we are faced with the two questions:

(1) Is every semiboolean SQS-skein nilpotent?
(2) Is every SQS-skein of nilpotence class 2 also semiboolean?

While the first question is still open, the answer to the second question is negative. We can construct a 16-element SQS-skein $\mathfrak{U}_{16} = \langle A; q\rangle$ that is nilpotent of class 2 but not semiboolean:

Let $A = \mathrm{GF}(2)^4$ and q be a ternary operation A defined by:

$$q\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \\ x_3 + y_3 + z_3 \\ x_4 + y_4 + z_4 + x_1 y_1 z_1 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ 1 & 1 & 1 \end{vmatrix} \end{pmatrix}$$

It is again easy to verify that $\mathfrak{U}_{16} = \langle A; q\rangle$ is an SQS-skein and of nilpotent class at most 2. $\mathfrak{U}_{16}$ is not semiboolean (and therefore not boolean) since:

$$q\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, q\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right)\right) = q\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\neq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = q\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right)$$

$$= q\left(q\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right), \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right)$$

0 1 2 3	0 4 9 D	1 2 9 A	1 A D E	2 7 A F	3 A E F	5 9 B F
0 1 4 5	0 4 A E	1 3 4 E	1 B C E	2 9 C F	4 5 6 F	5 A B C
0 1 6 7	0 4 B F	1 3 5 F	1 B D F	2 9 D E	4 5 7 E	5 C E F
0 1 8 9	0 5 8 D	1 3 6 C	2 3 4 D	2 B C D	4 5 8 9	6 7 8 9
0 1 A B	0 5 9 C	1 3 7 D	2 3 5 C	2 B E F	4 5 C D	6 7 E F
0 1 C D	0 5 A F	1 3 8 A	2 3 6 F	3 4 5 A	4 6 7 D	6 9 A D
0 1 E F	0 5 B E	1 3 9 B	2 3 7 E	3 4 6 9	4 6 8 A	6 9 B C
0 2 4 6	0 6 8 E	1 4 6 B	2 3 8 9	3 4 8 F	4 6 C E	6 A B F
0 2 5 7	0 6 9 F	1 4 7 A	2 3 A B	3 4 B C	4 7 8 B	6 C D F
0 2 8 A	0 6 A C	1 4 8 D	2 4 5 B	3 5 7 9	4 7 C F	7 9 A C
0 2 9 B	0 6 B D	1 4 9 C	2 4 7 9	3 5 8 E	4 9 A F	7 9 B D
0 2 C E	0 7 8 F	1 5 6 A	2 4 8 E	3 5 B D	4 9 B E	7 A B E
0 2 D F	0 7 9 E	1 5 7 B	2 4 A C	3 6 7 A	4 A B D	7 C D E
0 3 4 7	0 7 A D	1 5 8 C	2 5 6 9	3 6 8 D	4 D E F	8 9 A B
0 3 5 6	0 7 B C	1 5 9 D	2 5 8 F	3 6 B E	5 6 7 C	8 9 C D
0 3 8 B	1 2 4 F	1 6 8 F	2 5 A D	3 7 8 C	5 6 8 B	8 9 E F
0 3 9 A	1 2 5 E	1 6 9 E	2 6 7 B	3 7 B F	5 6 D E	8 A C E
0 3 C F	1 2 6 D	1 7 8 E	2 6 8 C	3 9 C E	5 7 8 A	8 A D F
0 3 D E	1 2 7 C	1 7 9 F	2 6 A E	3 9 D F	5 7 D F	8 B C F
0 4 8 C	1 2 8 B	1 A C F	2 7 8 D	3 A C D	5 9 A E	8 B D E

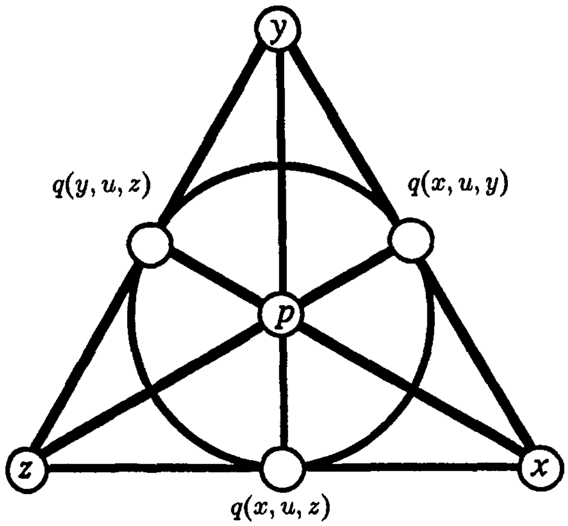
Figure 1. The Steiner quadruple system corresponding to the SQS-skein $\mathfrak{H}_{16}$.

i.e., there are non-semiboolean SQS-skeins of nilpotence class 2. Note that the Steiner quadruple system corresponding to $\mathfrak{U}_{16}$ has already been described in [2] and it can easily be obtained from the affine eight-element Steiner quadruple system using a recursive construction given in [3]. In none of these papers has the algebraic importance of $\mathfrak{U}_{16}$ been recognized.

4. Steiner quadruple systems

Given a Steiner quadruple system (P, B), we can define a ternary operation q on P by: $q(y, x, x) = q(x, y, x) = q(x, x, y) = y$ and $q(x, y, z)$ = fourth point on the block through x, y and z for all $x \neq y \neq z \neq x$ in P. The algebra $\langle P; q\rangle$ is then an SQS-skein. Vice versa, if $\langle P; q\rangle$ is an SQS-skein and B is the set of all four-element subalgebras of $\langle P; q\rangle$ then (P, B) is a Steiner quadruple system. This describes a one-to-one correspondence between Steiner quadruple systems and SQS-skeins. The system corresponding to $\mathfrak{H}_{16}$ is given in Figure 1.

It is possible to characterize the semiboolean SQS-skeins by a design-theoretic property of the corresponding Steiner quadruple system. If (P, B) is any Steiner quadruple system, $u \in P$ and $C = \{\{x, y, z\} \mid x, y, z \in P\backslash\{u\}$ and $\{x, y, z, u\} \in B\}$, then $(P\backslash\{u\}, C)$ is a Steiner triple system and it is called a *derived Steiner triple system of* (P, B). With this concept, we obtain the following theorem:



$$q(q(x, u, y), u, z) = p = q(x, u, q(y, u, z))$$

Figure 2. A subplane in a projective geometry over GF(2).

THEOREM 5. *Let $\mathfrak{S} = \langle S; q \rangle$ be an SQS-skein with the corresponding Steiner quadruple system (S, B). $\mathfrak{S}$ is semiboolean if and only if all derived Steiner triple systems of (S, B) are projective geometries over $GF(2)$.*

Proof. Suppose all derived Steiner triple systems of (S, B) are projective geometries over GF(2). Let $u, x, y, z \in S$. If $|\{u, x, y, z\}| < 4$ or $\{u, x, y, z\}$ forms a subalgebra of $\mathfrak{S}$ then $q(x, u, q(y, u, z)) = q(q(x, u, y), u, z)$ since every four-element SQS-skein is boolean. Otherwise, in the derived triple system $(S\backslash\{u\}, C)$ x, y, and z are noncollinear. The subplane generated by x, y, and z has seven elements and is shown in Figure 2. It is straightforward to verify that in fact:

$$q(x, u, q(y, u, z)) = q(q(x, u, y), u, z),$$

i.e., $\mathfrak{S}$ is semiboolean.

If $\mathfrak{S}$ is semiboolean, consider the sloop corresponding to the derived Steiner triple system $(S\backslash\{u\}, C)$. The semiboolean law implies immediately that the sloop satisfies the associative law and it is well known that the Steiner triple system corresponding to such a sloop is a projective geometry over GF(2) (see [5]). □

Note that the existence of nonboolean Steiner quadruple systems whose derived

Steiner triple systems are projective geometries over GF(2) was already known (see [10, p. 294]).

The results presented in this paper are also included in the Ph.D. thesis [6] of the author.

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