Journal of Algebraic Combinatorics 2 (1993), 123–124

On the Sperner Capacity of the Cyclic Triangle

AART BLOKHUIS
Math and Computing Science, Eindhoven Univ. of Techn., P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Received October 27, 1992; Revised January 13, 1993

Abstract. Using a single trick it is shown that the Sperner capacity of the cyclic triangle equals log 2.

Keywords: information theory, directed graph, Sperner theorem, Shannon capacity

This paper should be considered as an appendix to [1], and we refer to this paper for most definitions and explanatory remarks. We restrict ourselves to the proof of the following statement:

THEOREM. *The size of a subset S of $\{0, 1, 2\}^n$ with the property that for every distinct pair of vectors $x = (x_i)$, $y = (y_i) \in S$, we have $x_j - y_j \equiv 1 \pmod 3$ for some index j, is less than or equal to 2^n.*

Proof. Identify $\{0, 1, 2\}$ with $GF(3)$, the field with three elements. For each vector $y \in GF(3)^n$ consider the polynomial

$$F_y(X) = (X_1 - y_1 - 1)(X_2 - y_2 - 1)\dots(X_n - y_n - 1).$$

We have $F_x(x) = (-1)^n$ is nonzero, for every $x \in GF(3)^n$, but if x, y are different elements from the subset S, then $F_y(x) = 0$, so the polynomials F_y, $y \in S$ form an independent set in the vector space V of polynomials in the variables $X_1, \dots, X_n$, of degree at most 1 in each separate variable. It follows that $|S| \leq \dim(V) = 2^n$.

Note that the constant 2 in the result is best possible since the collection S of all (0, 1)-vectors of length n and weight $\lfloor n/2 \rfloor$ satisfies the conditions of the theorem and has size $\binom{n}{\lfloor n/2 \rfloor}$.

Essentially the same proof can be used to prove the following generalization, that can also be found in [1]:

THEOREM. *Let D be a set of d residue classes modulo p. The size of a subset S of $\{0, 1, \dots, p-1\}^n$ with the property that for every distinct pair of vectors $x = (x_i)$, $y = (y_i) \in S$, we have $x_j - y_j \pmod p \in D$ for some index j, is less than or equal to $(d+1)^n$.*

Proof. Identify $\{0, 1, p-1\}$ with $GF(p)$. For each vector $y \in GF(k)^n$ consider the polynomial

$$F_y(X) = \prod_{i \in D} (X_1 - y_1 - i)(X_2 - y_2 - i) \dots (X_n - y_n - i).$$

We have again that $F_x(x)$ is nonzero, for every $x \in GF(p)^n$, but if x, y are different elements from the subset S, then $F_y(x) = 0$, so the polynomials F_y, $y \in S$ form an independent set in the vector space V of polynomials in the variables $X_1, \dots, X_n$, of degree at most d in each separate variable. It follows that $|S| \leq \dim(V) = (d + 1)^n$.

Reference

1. A.R. Calderbank, P. Frankl, R.L. Graham, W.-C. Li, and L.A. Shepp, "The Sperner Capacity of Linear and Nonlinear Codes for the Cyclic Triangle," *J. Algebraic Combin.* **2** (1993), 31–48.