J Algebr Comb (2010) 31: 411–454
DOI 10.1007/s10801-009-0194-z

(q,t)-analogues and $GL_n(\mathbb{F}_q)$

Victor Reiner · Dennis Stanton

Received: 14 October 2008 / Accepted: 17 June 2009 / Published online: 8 July 2009

Abstract We start with a (q,t)-generalization of a binomial coefficient. It can be viewed as a polynomial in t that depends upon an integer q, with combinatorial interpretations when q is a positive integer, and algebraic interpretations when q is the order of a finite field. These (q,t)-binomial coefficients and their interpretations generalize further in two directions, one relating to column-strict tableaux and Macdonald's "7^{th} variation" of Schur functions, the other relating to permutation statistics and Hilbert series from the invariant theory of $GL_n(\mathbb{F}_q)$.

Keywords q-binomial · q-multinomial · Finite field · Gaussian coefficient · Invariant theory · Coxeter complex · Tits building · Steinberg character · Principal specialization

1 Introduction, definition and main results

1.1 Definition

The usual *q-binomial coefficient* may be defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k \cdot (q;q)_{n-k}} \tag{1.1}$$

To Anders Björner on his 60^{th} birthday.

Authors supported by NSF grants DMS-0601010 and DMS-0503660, respectively.

V. Reiner (✉) · D. Stanton
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
e-mail: reiner@math.umn.edu

D. Stanton
e-mail: stanton@math.umn.edu

where $(x;q)_n := (1-q^0x)(1-q^1x)\cdots(1-q^{n-1}x)$. It is a central object in combinatorics, with many algebraic and geometric interpretations. We recall below some of these interpretations and informally explain how they generalize to our main object of study, the *(q,t)-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \frac{n!_{q,t}}{k!_{q,t} \cdot (n-k)!_{q,t^{q^k}}}, \tag{1.2}$$

where $n!_{q,t} := (1-t^{q^n-1})(1-t^{q^n-q})(1-t^{q^n-q^2})\cdots(1-t^{q^n-q^{n-1}})$.

If q is a positive integer greater than 1, the (q,t)-binomial coefficient will be shown in Section 4 to be a polynomial in t with nonnegative coefficients. It is not hard to see that it specializes to the q-binomial coefficient in two limiting cases:

$$\lim_{t\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \begin{bmatrix} n \\ k \end{bmatrix}_{q} \quad \text{and} \quad \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_{q,t^{\frac{1}{q-1}}} = \begin{bmatrix} n \\ k \end{bmatrix}_{t}. \tag{1.3}$$

Warnings: This (q,t)-binomial coefficient is *not* a polynomial in q. The parameters q and t here play very different roles, unlike the symmetric role played by the variables q and t in the theory of Macdonald polynomials (see e.g. [9, Chap. VI]). Also note that, unlike the q-binomial coefficient, the (q,t)-binomial coefficient is *not* symmetric in k and $n-k$.

1.2 A word about philosophy

Throughout this paper, there will be (q,t)-versions of various combinatorial numbers having some meaning associated with the symmetric group $\mathfrak{S}_n$. These (q,t)-numbers will have two specializations to the *same* q-version or t-version, as in (1.3). The limit as $t\to 1$ will generally give a q-version that *counts* some objects associated to $GL_n(\mathbb{F}_q)$ when q is a prime power. The $q\to 1$ limit will generally give the *Hilbert series*, in the variable t, for some graded vector space associated with the invariant theory or representation theory of $\mathfrak{S}_n$. The unspecialized (q,t)-version will generally be such a Hilbert series, in the variable t, associated with $GL_n(\mathbb{F}_q)$ when q is a prime power.

One reason for our interest in such Hilbert series interpretations is that they often give generating functions in t with interesting properties. One such property is the *cyclic sieving phenomenon* [14] interpreting the specialization at t equal to a n^{th} root-of-unity for the $\mathfrak{S}_n$ Hilbert series, and the specialization at t equal to a $(q^n-1)^{th}$ root-of-unity for the $GL_n(\mathbb{F}_q)$ Hilbert series. As an example [14, §9], the q-binomial coefficient in (1.1), when q is specialized to a root-of-unity of order d dividing n, counts the number of k-element subsets of $\{1,2,\ldots,n\}$ stable under the action of any power of the n-cycle $c=(1,2,\ldots,n)$ that shares the same multiplicative order d. Analogously, the (q,t)-binomial coefficient in (1.2), when t is specialized to a root-of-unity of order d dividing q^n-1, counts the number of k-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}$ stable under multiplication by any element of $\mathbb{F}_{q^n}^\times$ that shares the same multiplicative order d.

1.3 Partitions inside a rectangle, and subspaces

The binomial coefficient $\binom{n}{k}$ counts k-element subsets of a set with n elements, but also counts integer partitions λ whose Ferrers diagram fits inside a $k \times (n-k)$ rectangle. The usual q-binomial coefficient q-counts the same set of partitions:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda} q^{|\lambda|} \tag{1.4}$$

where $|\lambda|$ denotes the number partitioned by λ. It will be shown in Section 5 that the (q,t)-binomial has a similar combinatorial interpretation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{\lambda} \mathrm{wt}(\lambda; q, t) \tag{1.5}$$

where the sum runs over the same partitions λ as in (1.4). Here $\mathrm{wt}(\lambda; q, t)$ has simple product expressions, showing that for integers $q \geq 2$ it is a polynomial in t with nonnegative coefficients, and that

$$\lim_{t \to 1} \mathrm{wt}(\lambda; q, t) = q^{|\lambda|} \quad \text{and} \quad \lim_{q \to 1} \mathrm{wt}(\lambda; q, t^{\frac{1}{q-1}}) = t^{|\lambda|}. \tag{1.6}$$

When q is a prime power, the q-binomial coefficient also counts the k-dimensional $\mathbb{F}_q$-subspaces U of an n-dimensional $\mathbb{F}_q$-vector space V. It will be shown in Section 5.3 that (1.5) may be re-interpreted as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{U} t^{s(U)} \tag{1.7}$$

where the summation runs over all such k-dimensional subspaces U of V, and $s(U)$ is a nonnegative integer depending upon U.

1.4 Principal specialization of Schur functions

The usual q-binomial coefficient has a known interpretation (see e.g. [9, Exer. I.2.3], [20, §7.8]) as the *principal specialization* of a *Schur function* $s_\lambda(x_1, x_2, \ldots, x_N)$, in the special case where the partition $\lambda = (k)$ has only one part:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = s_{(k)}(1, q, q^2, \ldots, q^{n-k}). \tag{1.8}$$

It will be shown in Section 6 that the (q,t)-binomial coefficient is a special case of a different sort of principal specialization. In [10], Macdonald defined several variations on Schur functions, and his 7^{th} variation is a family of Schur polynomials $\mathrm{S}_\lambda(x_1, x_2, \ldots, x_n)$ that lie in $\mathbb{F}_q[x_1, \ldots, x_n]$, and are invariant under the action of $G = GL_n(\mathbb{F}_q)$. It turns out that the principal specializations $\mathrm{S}_\lambda(1, t, t^2, \ldots, t^{n-1})$ of his polynomials can be lifted in a natural way from $\mathbb{F}_q[t]$ to $\mathbb{Z}[t]$, giving a family of polynomials we will denote $\mathbf{S}_\lambda(1, t, t^2, \ldots, t^{n-1})$. When λ has a single part (k), one has

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \mathbf{S}_{(k)}(1, t, t^2, \ldots, t^{n-k}). \tag{1.9}$$

Recall also that Schur functions have a combinatorial interpretation in terms of tableaux, and hence so do their principal specializations:

$$s_\lambda(1, q, q^2, \ldots, q^n) = \sum_T q^{\sum_i T_i} \tag{1.10}$$

where T runs over all *(reverse) column-strict-tableau* T of shape λ with entries in $\{0, 1, \ldots, n\}$. It will be shown in Section 6.3 that

$$\mathbf{S}_\lambda(1, t, t^2, \ldots, t^n) = \sum_T \mathrm{wt}(T; q, t). \tag{1.11}$$

Here T runs over the same set of tableaux as in (1.10), and $\mathrm{wt}(T; q, t)$ has a simple product expression, showing that for integers $q \geq 2$ it is a polynomial in t having nonnegative coefficients, and that

$$\lim_{t \to 1} \mathrm{wt}(T; q, t) = q^{\sum_i T_i} \quad \text{and} \quad \lim_{q \to 1} \mathrm{wt}(T; q, t^{\frac{1}{q-1}}) = t^{\sum_i T_i}. \tag{1.12}$$

1.5 Hilbert series

As mentioned above, when q is a prime power, the q-binomial coefficient in (1.1) counts the points in the *Grassmannian* over the finite field $\mathbb{F}_q$, that is, the homogeneous space G/P_k where $G := GL_n(\mathbb{F}_q)$ and P_k is the parabolic subgroup stabilizing a typical k-dimensional $\mathbb{F}_q$-subspace in $\mathbb{F}_q^n$. This is related to its alternate interpretation as the *Hilbert series* for a graded ring arising in the invariant theory of the symmetric group $W = \mathfrak{S}_n$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{t} = \mathrm{Hilb}(\mathbb{Z}[\mathbf{x}]^{W_k}/(\mathbb{Z}[\mathbf{x}]^W_+), t). \tag{1.13}$$

Here $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[x_1, \ldots, x_n]$ is the polynomial algebra, with the usual action of $W = \mathfrak{S}_n$ and its *parabolic* or *Young subgroup* $W_k := \mathfrak{S}_k \times \mathfrak{S}_{n-k}$, and $(\mathbb{Z}[\mathbf{x}]^W_+)$ denotes the ideal within the ring of W_k-invariants generated by the W-invariants $\mathbb{Z}[\mathbf{x}]^W_+$ having no constant term. The original motivation for our definition of the (q, t)-binomial came from its analogous interpretation in [14, §9] as a Hilbert series:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^{P_k}/(\mathbb{F}_q[\mathbf{x}]^G_+), t), \tag{1.14}$$

where here the polynomial algebra $\mathbb{F}_q[\mathbf{x}] := \mathbb{F}_q[x_1, \ldots, x_n]$ carries the usual action of $G = GL_n(\mathbb{F}_q)$ and its parabolic subgroup P_k.

1.6 Multinomial coefficients

The above invariant theory interpretations extend naturally from binomial to *multinomial* coefficients. Given an ordered composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of n into nonnegative parts, one has the *multinomial coefficient* $\binom{n}{\alpha}$ counting cosets W/W_α where $W = \mathfrak{S}_n$ and W_α is a parabolic/Young subgroup. Alternatively, one can view W as a Coxeter system with the adjacent transpositions as generators, and this multinomial coefficient counts the minimum-length coset representatives for W/W_α. These representatives w are characterized by the property that the composition α refines the *descent composition* $\beta(w)$ that lists the lengths of the maximal increasing consecutive subsequences of the sequence $w = (w(1), w(2), \dots, w(n))$.

Generalizing the multinomial coefficient is the usual q*-multinomial coefficient* which we recall in Section 7. It is a polynomial in q with nonnegative integer coefficients, that for prime powers q counts points in the finite *partial flag manifold* G/P_α; here $G = GL_n(\mathbb{F}_q)$ and P_α is a parabolic subgroup. One also has these two interpretations, one algebraic, one combinatorial:

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_t = \text{Hilb}(\mathbb{Z}[\mathbf{x}]^{W_\alpha}/(\mathbb{Z}[\mathbf{x}]^W_+)\ , t) = \sum_{\substack{w \in W: \\ \alpha \text{ refines } \beta(w)}} t^{\ell(w)}, \tag{1.15}$$

where $\ell(w)$ denotes the *number of inversions* (or *Coxeter group length*) of w. We will consider in Section 7 a (q,t)*-multinomial coefficient* with two interpretations q-analogous to (1.15):

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \text{Hilb}(\mathbb{F}_q[\mathbf{x}]^{P_\alpha}/(\mathbb{F}_q[\mathbf{x}]^G_+)\ , t) = \sum_{\substack{w \in W: \\ \alpha \text{ refines } \beta(w)}} \text{wt}(w; q, t). \tag{1.16}$$

Here $\text{wt}(w; q, t)$ has a product expression, showing that for integers $q \geq 2$ it is a polynomial in t with nonnegative coefficients, and that

$$\lim_{t \to 1} \text{wt}(w; q, t) = q^{\ell(w)} \quad \text{and} \quad \lim_{q \to 1} \text{wt}(w; q, t^{\frac{1}{q-1}}) = t^{\ell(w)}. \tag{1.17}$$

1.7 Homology representations and ribbon numbers

The previous interpretations of multinomial coefficients are closely related, via *inclusion-exclusion*, to what we will call the *ribbon*, q*-ribbon*, and (q,t)*-ribbon numbers* for a composition α of n:

$$\begin{aligned} r_\alpha &:= |\{w \in W : \alpha = \beta(w)\}|, \\ r_\alpha(q) &:= \sum_{\substack{w \in W: \\ \alpha = \beta(w)}} q^{\ell(w)}, \\ r_\alpha(q,t) &:= \sum_{\substack{w \in W: \\ \alpha = \beta(w)}} \text{wt}(w; q, t). \end{aligned}$$

It is known that the ribbon number r_α has an expression as a determinant involving factorials, going back to MacMahon. Stanley gave an analogous determinantal expression for the q-ribbon number $r_\alpha(q)$ involving q-factorials. Section 9 discusses an analogous determinantal expression for the (q,t)-ribbon number, involving (q,t)-factorials.

The ribbon number r_α can also be interpreted as the rank of the only non-vanishing homology group in the α-rank-selected subcomplex of the *Coxeter complex* for $W = \mathfrak{S}_n$. This homology carries an interesting and well-studied $\mathbb{Z}W$-module structure that we will call χ^α. This leads (see Theorem 10.4 and Remark 10.5) to the homological/algebraic interpretation

$$r_\alpha(t) = \text{Hilb}(M/\mathbb{Z}[\mathbf{x}]_+^W M\ , t) \tag{1.18}$$

where M is the graded $\mathbb{Z}[\mathbf{x}]^W$-module $\text{Hom}_{\mathbb{Z}W}(\chi^\alpha, \mathbb{Z}[\mathbf{x}])$ which is the W-intertwiner space between the homology W-representation χ^α and the polynomial ring $\mathbb{Z}[\mathbf{x}]$.

For prime powers q, Björner reinterpreted $r_\alpha(q)$ as the rank of the homology in the α-rank-selected subcomplex of the *Tits building* for $G = GL_n(\mathbb{F}_q)$. Section 10 then explains the analogous homological/algebraic interpretation:

$$r_\alpha(q,t) = \text{Hilb}(M/\mathbb{F}_q[\mathbf{x}]_+^G M\ , t) \tag{1.19}$$

where M is the graded $\mathbb{F}_q[\mathbf{x}]^G$-module $\text{Hom}_{\mathbb{F}_q G}(\chi_q^\alpha, \mathbb{F}_q[\mathbf{x}])$ which is the G-intertwiner space between the homology G-representation χ_q^α on the α-rank-selected subcomplex of the Tits building and the polynomial ring $\mathbb{F}_q[\mathbf{x}]$. This last interpretation generalizes work of Kuhn and Mitchell [8], who dealt with the case where $\alpha = 1^n := (1, 1, \dots, 1)$ in order to determine the composition multiplicities of the *Steinberg character* of $GL_n(\mathbb{F}_q)$ within each graded component of the polynomial algebra $\mathbb{F}_q[\mathbf{x}]$.

1.8 The coincidence for hooks

In an important special case, there is a coincidence between the principal specializations $\mathbf{S}_\lambda(1, t, t^2, \dots)$, and the (q,t)-ribbon numbers $r_\alpha(q,t)$. Specifically, when $\lambda = (m, 1^k)$ (a *hook*) and $\alpha = (1^k, m)$ (the *reverse hook*), we will show in Section 11 that for $n \geq k$ one has

$$\mathbf{S}_\lambda(1, t, t^2, \dots, t^n) = \begin{bmatrix} m+n \\ n-k \end{bmatrix}_{q,t} r_\alpha(q, t^{q^{n-k}}). \tag{1.20}$$

In particular, when $k = 0$, both sides revert to the (q,t)-multinomial $\begin{bmatrix} m+n \\ n \end{bmatrix}_{q,t}$, and when $n = k$, one has the coincidence $\mathbf{S}_\lambda(1, t, t^2, \dots, t^k) = r_\alpha(q,t)$, generalizing the case $\alpha = 1^k$ studied by Kuhn and Mitchell [8].

2 Making sense of the formal limits

Recall the definition of the (q,t)-binomial from (1.2):

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \frac{n!_{q,t}}{k!_{q,t} \cdot (n-k)!_{q,t^{q^k}}} = \prod_{i=1}^{k} \frac{1 - t^{q^n - q^{i-1}}}{1 - t^{q^k - q^{i-1}}}, \tag{2.1}$$

where $n!_{q,t} := (1 - t^{q^n - 1})(1 - t^{q^n - q})(1 - t^{q^n - q^2}) \cdots (1 - t^{q^n - q^{n-1}})$. We pause to define here carefully the ring where such generating functions involving q and t live, and how the various formal limits in the Introduction will make sense.

Let

$$\mathbf{t} := (\ldots, t^{q^{-2}}, t^{q^{-1}}, t, t^{q^1}, t^{q^2}, \ldots)$$

be a doubly-infinite sequence of *algebraically independent* indeterminates, and let

$$\hat{\mathbb{Q}}(\mathbf{t}) := \mathbb{Q}(\ldots, t^{q^{-2}}, t^{q^{-1}}, t, t^{q^1}, t^{q^2}, \ldots)$$

denote the field of rational functions in these indeterminates. We emphasize that q here is *not* an integer, and there is *no relation* between the different variables t^{q^r}. However, they are related by the *Frobenius operator* φ acting invertibly via

$$\begin{aligned} \hat{\mathbb{Q}}(\mathbf{t}) &\xrightarrow{\varphi} \hat{\mathbb{Q}}(\mathbf{t}) \\ t^{q^r} &\overset{\varphi}{\longmapsto} t^{q^{r+1}}. \end{aligned}$$

We sometimes abbreviate this by saying $f(t) \overset{\varphi}{\longmapsto} f(t^q)$ and $f(t) \overset{\varphi^{-1}}{\longmapsto} f(t^{\frac{1}{q}})$.

Most generating functions considered in this paper are contained in the subfield $\hat{\mathbb{Q}}(\mathbf{t})_0$ of $\hat{\mathbb{Q}}(\mathbf{t})$ generated by all quotients $\frac{t^{q^r}}{t^{q^s}} =: t^{q^r - q^s}$ of the indeterminates. For example,

$$n!_{q,t} = \left(1 - \frac{t^{q^n}}{t}\right)\left(1 - \frac{t^{q^n}}{t^{q^1}}\right) \cdots \left(1 - \frac{t^{q^n}}{t^{q^{n-1}}}\right)$$

lies in this subfield $\hat{\mathbb{Q}}(\mathbf{t})_0$, and hence so does the (q,t)-binomial coefficient.

Because the t^{q^r} are algebraically independent, the homomorphism $\hat{\mathbb{Q}}(\mathbf{t}) \to \mathbb{Q}(t)$ that sends $t^{q^r} \mapsto t^r$ is well-defined, and restricts to a homomorphism

$$\begin{aligned} \hat{\mathbb{Q}}(\mathbf{t})_0 &\longrightarrow \mathbb{Q}(t) \\ t^{q^r - q^s} = \frac{t^{q^r}}{t^{q^s}} &\longmapsto t^{r-s}. \end{aligned} \tag{2.2}$$

It is this homomorphism which makes sense of the $q \to 1$ formal limits that appeared in the Introduction: when we write $\lim_{q\to 1}[F(q,t)]_{t \mapsto t^{\frac{1}{q-1}}}$ for some element $F(q,t) \in \hat{\mathbb{Q}}(t)$, we mean by this the element in $\mathbb{Q}(t)$ which is the image of $F(q,t)$ under the homomorphism in (2.2).

To make sense of the $t \to 1$ formal limits, choose a positive integer $q \geq 2$. The field of rational functions $\mathbb{Q}(t)$ in a single variable t lies at the bottom of a tower of field extensions

$$\mathbb{Q}(t) \subset \mathbb{Q}(t^{q^{-1}}) \subset \mathbb{Q}(t^{q^{-2}}) \subset \cdots$$

obtained by adjoining a q^{th} root at each stage. The union $\bigcup_{r \geq 0} \mathbb{Q}(t^{q^{-r}})$ is a ring with a specialization homomorphism from the ring $\hat{\mathbb{Q}}(\mathbf{t})$ defined above:

$$\begin{aligned} \hat{\mathbb{Q}}(\mathbf{t}) &\longrightarrow \bigcup_{r \geq 0} \mathbb{Q}(t^{q^{-r}}) \\ t^{q^r} &\longmapsto t^{q^r}. \end{aligned} \tag{2.3}$$

Note that in (2.3), the symbol "t^{q^r}" has two different meanings: on the left it is one of the doubly-indexed family of indeterminates, and on the right it is the $(q^r)^{th}$ power of the variable t. Many of our results will assert that various generating functions $F(q,t)$ in $\hat{\mathbb{Q}}(\mathbf{t})$, when specialized as in (2.3), have image lying in the subring $\mathbb{Z}[t] \subset \bigcup_{r \geq 0} \mathbb{Q}(t^{q^{-r}})$; this is what is meant when we say $F(q,t)$ in $\hat{\mathbb{Q}}(\mathbf{t})$ "is a polynomial in t for integers $q \geq 2$". In this situation, we will write $\lim_{t \to 1} F(q,t)$ for the result after applying the further evaluation homomorphism $\mathbb{Z}[t] \to \mathbb{Z}$ that sends t to 1.

In Section 6, we shall be interested in working with n variables $x_1, \ldots, x_n$, rather than just the single variable t. To this end, define $\hat{\mathbb{Q}}(\mathbf{x})$ to be the n-fold tensor product $\hat{\mathbb{Q}}(\mathbf{t}) \otimes \cdots \otimes \hat{\mathbb{Q}}(\mathbf{t})$, renaming the variable t as x_i in the n^{th} tensor factor. Here the Frobenius automorphism φ acts by $(\varphi f)(x_i) = f(x_1^q, \ldots, x_n^q)$. There are various specialization homomorphisms from $\hat{\mathbb{Q}}(\mathbf{x}) \to \hat{\mathbb{Q}}(\mathbf{t})$, but the one that will be of interest here is the *principal specialization* $\hat{\mathbb{Q}}(\mathbf{x}) \to \hat{\mathbb{Q}}(\mathbf{t})$ sending $x_i^{q^r} \longmapsto (t^{q^r})^{i-1}$, and abbreviated by $f(x_1, \ldots, x_n) \longmapsto f(1, t, t^2, \ldots, t^{n-1})$.

3 Why call it a "binomial coefficient"?

We give two reasons for the name "(q,t)-binomial coefficient".

3.1 Binomial theorem

The elementary symmetric function $e_r(x_1, \ldots, x_n)$ can be defined by the identity $\prod_{i=1}^n (y + x_i) = \sum_{s=0}^n y^s e_{n-s}(\mathbf{x})$ in $\mathbb{Z}[y, x_1, \ldots, x_n]$. When specialized to $x_i = t^{i-1}$ this gives the following version of the *t-binomial theorem*:

$$\prod_{i=1}^n (y + t^{i-1}) = \sum_{s=0}^n y^s e_{n-s}(1, t, t^2, \ldots, t^{n-1}) = \sum_{s=0}^n y^s \begin{bmatrix} n \\ s \end{bmatrix}_t t^{\binom{n-s}{2}}. \tag{3.1}$$

On the other hand, a special case of Macdonald's 7th variation on Schur functions, to be discussed in Section 6, are polynomials $E_r(x_1, \ldots, x_n)$ which can be defined by the following identity in in $\mathbb{F}_q[x_1, \dots, x_n, y]$ (see [9, Chap. I, §2, Exer. 26, 27], [10, §7]):

$$\prod_{\ell(\mathbf{x}) \in (\mathbb{F}_q^n)^*} (y + \ell(\mathbf{x})) = \sum_{s=0}^{n} y^{q^s} E_{n-s}(\mathbf{x}). \tag{3.2}$$

Here the product runs over all $\mathbb{F}_q$-linear functionals $\ell(x_1, \dots, x_n)$ on $\mathbb{F}_q^n$. We will later prove a formula (6.7) for the specialization $x_i = t^{i-1}$ in $E_r(\mathbf{x})$, from which (3.2) gives the following identity valid in $\mathbb{F}_q[t, y]$:

$$\prod_{\ell(\mathbf{x}) \in (\mathbb{F}_q^n)^*} (y + \ell(1, t, t^2, \dots, t^{n-1})) = \sum_{s=0}^{n} y^{q^s} \begin{bmatrix} n \\ s \end{bmatrix}_{q,t} \prod_{j=1}^{n-s} \frac{t^{q^n} - t^{q^{n-j}}}{t^{q^{s+j}} - t^{q^s}}. \tag{3.3}$$

Although (3.3) does not have a rigorous limit as q approaches 1, it can viewed as a q-analogue of the t-binomial theorem (3.1).

3.2 Binomial convolution

Consider three algebras of generating functions. The first two are the power series rings $\mathbb{Q}[[y]]$, $\mathbb{Q}(q)[[y]]$ in a single variable y with coefficients in the field $\mathbb{Q}$ or in the rational function field $\mathbb{Q}(q)$. The third is an associative but noncommutative algebra, isomorphic as $\hat{\mathbb{Q}}(\mathbf{t})$-vector space to $\hat{\mathbb{Q}}(\mathbf{t})[[y]]$ so that it has a $\hat{\mathbb{Q}}(\mathbf{t})$-basis $\{1, y, y^2, \dots\}$, but with its multiplication twisted[1] as follows:

$$f(t)y^k \cdot g(t)y^\ell := f(t)\, g(t^{q^k})\, y^{k+\ell}.$$

Each of these three algebras, has a *divided power* basis $\{y^{(n)}\}_{n \geq 0}$ as a vector space over $\mathbb{Q}$ (resp. $\mathbb{Q}(q)$, $\hat{\mathbb{Q}}(\mathbf{t})$), defined by

$$y^{(n)} := \frac{y^n}{n!} \left(\text{resp. } \frac{y^n}{(q;q)_n}, \quad \frac{y^n}{n!_{q,t}} \right).$$

One can readily check that the various binomials give the structure constants for multiplication in this basis:

$$y^{(k)} y^{(\ell)} = \binom{k+\ell}{k} y^{(k+\ell)} \left(\text{resp. } \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q y^{(k+\ell)}, \quad \begin{bmatrix} k+\ell \\ k \end{bmatrix}_{q,t} y^{(k+\ell)} \right).$$

Therefore one has a *binomial convolution formula*: the product $A(y)B(y)$ of two exponential generating functions $A(y) := \sum_{k \geq 0} a_k\, y^{(k)}$, $B(y) := \sum_{\ell \geq 0} b_\ell\, y^{(\ell)}$ has its coefficient of $y^{(n)}$ given by

$$\sum_{k+\ell=n} \binom{n}{k} a_k\, b_\ell \left(\text{resp. } \sum_{k+\ell=n} \begin{bmatrix} n \\ k \end{bmatrix}_q a_k(q)\, b_\ell(q), \quad \sum_{k+\ell=n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} a_k(t)\, b_\ell(t^{q^k}) \right).$$

[1] This is the *twisted semigroup algebra* for the mulitplicative semigroup $\{1, y, y^2, \dots\}$, in which the semigroup acts on the coefficients $\hat{\mathbb{Q}}(\mathbf{t})$ by letting the generator y act as φ.

4 Pascal relations

Our starting point will be the (q,t)-analogue of the two q-Pascal relations for the q-binomial; see e.g. [7, Prop. 6.1], [19, §1.3]:

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q . \end{aligned} \tag{4.1}$$

Proposition 4.1 *If* $0 \le k \le n$,

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q,t^q} + t^{q^k-1} \frac{k!_{q,t^q}}{k!_{q,t}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q,t^q} \\ \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} &= t^{q^n-q^k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q,t^q} + \frac{k!_{q,t^q}}{k!_{q,t}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q,t^q} . \end{aligned} \tag{4.2}$$

Proof Both relations are straightforward to check; we check here only the second. We will make frequent use of the fact that

$$n!_{q,t} = (1 - t^{q^n-1}) \cdot (n-1)!_{q,t^q}. \tag{4.3}$$

Starting with the right side of the second relation in (4.2), one checks

$$\begin{aligned} & t^{q^n-q^k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q,t^q} + \frac{k!_{q,t^q}}{k!_{q,t}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q,t^q} \\ &= t^{q^n-q^k} \frac{(n-1)!_{q,t^q}}{(k-1)!_{q,t^q}(n-k)!_{q,t^{q^k}}} + \frac{k!_{q,t^q}}{k!_{q,t}} \cdot \frac{(n-1)!_{q,t^q}}{k!_{q,t^q}(n-k-1)!_{q,t^{q^{k+1}}}} \\ &= \frac{(n-1)!_{q,t^q}}{(k-1)!_{q,t^q}(n-k-1)!_{q,t^{q^{k+1}}}} \left(\frac{t^{q^n-q^k}}{1-t^{q^n-q^k}} + \frac{1}{1-t^{q^k-1}} \right) \\ &= \frac{(n-1)!_{q,t^q}}{(k-1)!_{q,t^q}(n-k-1)!_{q,t^{q^{k+1}}}} \cdot \frac{1-t^{q^n-1}}{(1-t^{q^n-q^k})(1-t^{q^k-1})} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \end{aligned}$$

□

Note that the quotient $\frac{k!_{q,t^q}}{k!_{q,t}}$ appearing in both (q,t)-Pascal relations (4.2) factors as follows:

$$\frac{k!_{q,t^q}}{k!_{q,t}} = [q]_{t^{q^k-1}} [q]_{t^{q^k-q}} \cdots [q]_{t^{q^k-q^{k-1}}} \tag{4.4}$$

where $[N]_t := \frac{1-t^N}{1-t} = 1 + t + t^2 + \cdots + t^{N-1}$. Consequently, if q is a positive integer, this quotient is a polynomial in t with nonnegative coefficients.

Corollary 4.2 *If $q \geq 2$ is an integer, then $\begin{bmatrix} n \\ k \end{bmatrix}_{q,t}$ is a polynomial in t with nonnegative coefficients.*

Proof This follows by induction on n from either of the (q,t)-Pascal relations, using (4.4). □

Remark 4.3 Note that taking the formal limit as $t \to 1$ in *either* of the (q,t)-Pascal relations (4.2) leads to the *same* q-Pascal relation, namely the first one in (4.1). On the other hand, replacing t with $t^{\frac{1}{q-1}}$ and then taking the other formal limit as $q \to 1$ in the two different (q,t)-Pascal relations (4.2) leads to the *two different* q-Pascal relations in (4.1).

5 Combinatorial interpretations

Our goal here is to explain the combinatorial interpretation for the (q,t)-binomial described in (1.5):

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{\lambda} \mathrm{wt}(\lambda; q, t)$$

where the sum runs over partitions λ whose Ferrers diagram fits inside a $k \times (n-k)$ rectangle, that is, λ has at most k parts, each of size at most $n-k$.

5.1 A product formula for $\mathrm{wt}(\lambda; q, t)$

The weight $\mathrm{wt}(\lambda; q, t)$ actually depends upon the number of rows k of the rectangle in which λ is confined, or alternatively, upon the number of parts of λ if one counts parts of size 0. To emphasize this dependence upon k, we redefine the notation $\mathrm{wt}(\lambda; q, t) := \mathrm{wt}(\lambda, k; q, t)$. This weight will be defined as a product over the cells x in the Ferrers diagram for λ of a contribution $\mathrm{wt}(x, \lambda, k; q, t)$ for each cell. To this end, first define an exponent

$$e_k(x) := q^{\mathrm{r}(x)+\mathrm{d}(x)} - q^{\mathrm{d}(x)}$$

where $\mathrm{r}(x)$ is the index of the row containing x, and $\mathrm{d}(x)$ is the *taxicab* (or *Manhattan*, or L^1-) distance from x to the bottom left cell of the rectangle, that is, the cell in the k^{th} row and first column. Alternatively, $\mathrm{d}(x) = \mathrm{content}(x) + k - 1$, where the content of a cell x is $j - i$ if x lies in row i and column j. Then

$$\begin{aligned} \mathrm{wt}(x, \lambda, k; q, t) &:= \begin{cases} t^{e_k(x)}[q]_{t^{e_k(x)}} & \text{if } x \text{ is the bottom cell of } \lambda \text{ in its column.} \\ [q]_{t^{e_k(x)}} & \text{otherwise.} \end{cases} \\ \mathrm{wt}(\lambda, k; q, t) &:= \prod_{x \in \lambda} \mathrm{wt}(x, \lambda, k; q, t). \end{aligned} \tag{5.1}$$

Example 5.1 Let $n=10$ and $k=4$, so $n-k=6$. Let $\lambda=(4,3,1,0)$ inside the 4×6 rectangle. The cells x of the Ferrers diagram for λ are labelled with the value $d(x)$ below

$$\begin{array}{cccccc} 3 & 4 & 5 & 6 & \cdot & \cdot \\ 2 & 3 & 4 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

and $\mathrm{wt}(\lambda,k;q,t)$ is the product of the corresponding factors $\mathrm{wt}(x,\lambda,k;q,t)$ shown below:

$$\begin{array}{rrrr} [q]_{t^{q^4-q^3}} & [q]_{t^{q^5-q^4}} & [q]_{t^{q^6-q^5}} & t^{q^7-q^6}[q]_{t^{q^7-q^6}} \\ [q]_{t^{q^4-q^2}} & t^{q^5-q^3}[q]_{t^{q^5-q^3}} & t^{q^6-q^4}[q]_{t^{q^6-q^4}} & \cdot \\ t^{q^4-q^1}[q]_{t^{q^4-q^1}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

One can alternatively define $\mathrm{wt}(\lambda,k;q,t)$ recursively. Say that λ fitting inside a $k\times(n-k)$ rectangle has a *full first column* if λ has k nonzero parts; in this case, let $\hat{\lambda}$ denote the partition inside a $k\times(n-k-1)$ rectangle obtained by removing this full first column from λ.

Proposition 5.2 *One can characterize* $\mathrm{wt}(\lambda,k;q,t)$ *by the recurrence*

$$\mathrm{wt}(\lambda,k;q,t)=\begin{cases} t^{q^k-1}\frac{k!_{q,t^q}}{k!_{q,t}}\,\mathrm{wt}(\hat{\lambda},k;q,t^q) & \text{if } \lambda \text{ has a full first column},\\ \mathrm{wt}(\lambda,k-1;q,t^q) & \text{otherwise},\end{cases}$$

together with the initial condition $\mathrm{wt}(\varnothing,k;q,t):=1$.

Proof This is straightforward from the definition (5.1) and equation (4.4). □

Theorem 5.3 *With the* (q,t)*-binomial defined as in* (1.2) *and* $\mathrm{wt}(\lambda,k;q,t)$ *defined as above, one has*

$$\begin{bmatrix} n\\ k\end{bmatrix}_{q,t}=\sum_{\lambda}\mathrm{wt}(\lambda,k;q,t)$$

where the sum ranges over all partitions λ *whose Ferrers diagram fits inside a* $k\times(n-k)$ *rectangle.*

Furthermore,

$$\lim_{t\to 1}\mathrm{wt}(\lambda,k;q,t)=q^{|\lambda|}\quad\text{and}\quad\lim_{q\to 1}\mathrm{wt}(\lambda,k;q,t^{\frac{1}{q-1}})=t^{|\lambda|}. \tag{5.2}$$

Note that this gives a second proof of Corollary 4.2.

Proof The first assertion of the theorem then follows by induction on n, comparing the first (q,t)-Pascal relation in (4.2) with Proposition 5.2; classify the terms in the sum $\sum_\lambda \mathrm{wt}(\lambda, k; q, t)$ according to whether λ does not or does have a full first column.

The limit evaluations in the theorem also follow by induction on n, using the following basic limits:

$$\begin{aligned} \lim_{t\to 1}[q]_{t^{q^{r+d}-q^d}} &= q, & \lim_{t\to 1} t^{q^{r+d}-q^d} &= 1 \\ \lim_{q\to 1}\left([q]_{t^{q^{r+d}-q^d}}\right)_{t\mapsto t^{\frac{1}{q-1}}} &= 1, & \lim_{q\to 1}\left(t^{q^{r+d}-q^d}\right)_{t\mapsto t^{\frac{1}{q-1}}} &= t^r, \end{aligned} \tag{5.3}$$

and hence

$$\begin{aligned} \lim_{t\to 1}\frac{k!_{q,t^q}}{k!_{q,t}} &= q^k, \\ \lim_{q\to 1}\left(\frac{k!_{q,t^q}}{k!_{q,t}}\right)_{t\mapsto t^{\frac{1}{q-1}}} &= t^k. \end{aligned} \tag{5.4}$$

□

In the remainder of this section, we reformulate Theorem 5.3 in two other ways, each with their own advantages.

5.2 A different partition interpretation

First, note that (2.1) implies that for fixed $k \geq 0$ one has

$$\lim_{n\to\infty}\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \prod_{i=1}^{k}\frac{1}{1-t^{q^k-q^{k-i}}}.$$

For integers $q \geq 2$, this is the generating function in t counting integer partitions μ whose part sizes are restricted to the set $\{q^k-1, q^k-q, \ldots, q^k-q^{k-1}\}$. One might ask whether there is an analogous result for the case where n is finite, perhaps by imposing some restriction on the multiplicities of the above parts in μ. This turns out indeed to be the case, as we now explain.

Definition 5.4 Given λ a partition with at most k nonzero parts, set $\lambda_{k+1} = 0$, and define for $i = 1, 2, \ldots, k$

$$\delta_i(\lambda) := \sum_{j=\lambda_{i+1}}^{\lambda_i - 1} q^j = q^{\lambda_{i+1}}[\lambda_i - \lambda_{i+1}]_q.$$

For an integer $q \geq 2$, say that a partition μ is *q-compatible with* λ if the parts of μ are restricted to the set $\{q^k-1, q^k-q, \ldots, q^k-q^{k-1}\}$, and for each $i = 1, 2, \ldots, k$, the multiplicity of the part $q^k - q^{k-i}$ lies in the semi-open interval $[\delta_i(\lambda), \delta_i(\lambda) + q^{\lambda_i})$.

For example, if $n=5$, $k=2$ and $\lambda=31$, the partitions μ which are q-compatible with λ are of the form $\mu=(q^2-q)^{m_1}(q^2-1)^{m_2}$ where

$$q^1+q^2 \le m_1 \le q^1+q^2+q^3-1, \quad 1 \le m_2 \le q.$$

It turns out that μ determines λ in this situation:

Proposition 5.5 *Given a partition μ with parts restricted to the set $\{q^k-q^{k-i}\}_{i=1}^k$, there is at most one partition λ having k nonzero parts or less which can be q-compatible with μ.*

Proof Recall that $\lambda_{k+1}=0$. Then for $i=k,k-1,\dots,2,1$, check that the multiplicity m_i of the part q^k-q^{k-i} in μ determines λ_i uniquely, by downward induction on i. □

Theorem 5.6

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{\lambda} \quad \prod_{i=1}^{k} (t^{q^k-q^{k-i}})^{\delta_i(\lambda)} \; [q^{\lambda_i}]_{t^{q^k-q^{k-i}}} \tag{5.5}$$

where the sum ranges over all partitions λ fitting inside a $k \times (n-k)$ rectangle.

Consequently, for integers $q \ge 2$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{\mu} t^{|\mu|} \tag{5.6}$$

where the sum runs over partitions μ which are q-compatible with a λ that fits inside a $k \times (n-k)$ rectangle.

Proof Equation (5.6) follows immediately from equation (5.5). The latter follows from Theorem 5.3 if one can show that

$$\mathrm{wt}(\lambda,k;q,t) = \prod_{i=1}^{k} (t^{q^k-q^{k-i}})^{\delta_i(\lambda)} \; [q^{\lambda_i}]_{t^{q^k-q^{k-i}}}.$$

This would follow from showing that for each row $i=1,2,\dots,k$ in λ, one has

$$\prod_{\substack{x\in\lambda:\\ \mathrm{r}(x)=i}} \mathrm{wt}(x,\lambda,k;q,t) = (t^{q^k-q^{k-i}})^{\delta_i(\lambda)} \; [q^{\lambda_i}]_{t^{q^k-q^{k-i}}}. \tag{5.7}$$

This is not hard. The left side of (5.7) is easily checked to equal

$$(t^{q^k-q^{k-i}})^{\delta_i(\lambda)} [q]_{t^{q^0(q^k-q^{k-i})}} [q]_{t^{q^1(q^k-q^{k-i})}} \cdots [q]_{t^{q^{\lambda_i-1}(q^k-q^{k-i})}}.$$

Repeated apply to this expression the identity

$$[M]_Q \cdot [N]_{Q^M} = [MN]_Q,$$

with $Q = t^{q^k - q^{k-i}}$ and $N = q$ each time, and the result is the right side of (5.7). □

Note that by setting $t = 1$ in (5.6), one obtains a new interpretation for the usual q-binomial coefficient, as the cardinality of a set of integer partitions, rather than as a generating function in q:

Corollary 5.7 *For integers $q \geq 2$, the the integer $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of partitions q-compatible with a λ that fits inside a $k \times (n-k)$ rectangle.*

5.3 A subspace interpretation

The q-binomial coefficient is the number of k-spaces of an n-space over a finite field with q elements; see e.g. [7, §7], [19, Prop. 1.3.18]. We give in Theorem 5.9 a statistic on these subspaces whose generating function in t is the (q, t)-binomial coefficient.

Fix a basis for the n-dimensional space V over $\mathbb{F}_q$. Relative to this basis, any k-dimensional subspace U of V is the row-space of a unique matrix $k \times n$ matrix A over $\mathbb{F}_q$ in *row-reduced echelon form*. This matrix A will have exactly k pivot columns, and the sparsity pattern for the (possibly) nonzero entries in its nonpivot columns have an obvious bijection to the cells x in a partition λ inside a $k \times (n-k)$ rectangle; call these $|\lambda|$ entries a_{ij} of A the *parametrization* entries for U.

Example 5.8 If $n = 10$ and $k = 4$ one possible such row-reduced echelon form could be

$$U = \begin{bmatrix} 0 & * & 0 & * & * & 0 & * & 1 & 0 & 0 \\ 0 & * & 0 & * & * & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where each $*$ is a parametrization entry representing an element of the field $\mathbb{F}_q$. The associated partition $\lambda = (4, 3, 1, 0)$ fits inside a 4×6 rectangle, and is the one considered in Example 5.1 above:

$$\begin{matrix} * & * & * & * & \cdot & \cdot \\ * & * & * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

To define the statistic $s(U)$, first fix two bijections ϕ_0, ϕ_1:

$$\mathbb{F}_q \stackrel{\phi_0}{\to} \{0, 1, \ldots, q-1\}$$

$$\mathbb{F}_q \stackrel{\phi_1}{\to} \{1, 2, \ldots, q\}.$$

For each parametrization entry a_{ij} of A, define the integer value $\mathrm{v}_U(a_{ij})$ to be $\phi_1(a_{ij})$ or $\phi_0(a_{ij})$, depending on whether or not (i, j) is the lowest parametrization entry in its column of A. Then define

$$\mathrm{d}_U(i,j) := k - i + j - |\{\text{pivot columns left of } j\}| - 1$$
$$s(U) := \sum_{(i,j)} \mathrm{v}_U(a_{ij})(q^{i+\mathrm{d}_U(i,j)} - q^{\mathrm{d}_U(i,j)})$$

where the summation has (i,j) ranging over all parametrization positions in the row-reduced echelon form A for U.

Theorem 5.9 *If q is a prime power then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_U t^{s(U)}, \tag{5.8}$$

in which the summation runs over all k-dimensional $\mathbb{F}_q$ subspaces U of the n-dimensional $\mathbb{F}_q$-vector space V.

Proof This is simply a reformulation of Theorem 5.3. When the parametrization entry a_{ij} of A corresponds bijectively to the cell x of λ, then one has $i = \mathrm{r}(x)$, $\mathrm{d}_U(i,j) = \mathrm{d}(x)$, and (i,j) is the lowest parametrization entry in its column of A if and only if x lies at the bottom of its column of λ. From this and the definition (5.1), it easily follows that $\mathrm{wt}(\lambda,k;q,t) = \sum_U t^{s(U)}$ as U ranges over all subspaces whose echelon form corresponds to λ. □

6 Generalization 1: principally specialized Schur functions

We review here some of the definition and properties of Macdonald's 7*th variation on Schur functions*, and then lift them from polynomials with $\mathbb{F}_q$ coefficients to elements of the ring $\hat{\mathbb{Q}}(\mathbf{x})$. We then show (Corollary 6.7) that, for integers $q \geq 2$, the principal specializations of these rational functions are actually polynomials in a variable t with nonnegative integer coefficients, generalizing the (q,t)-binomials.

6.1 Lifting Macdonald's finite field Schur polynomials

Macdonald's 7^{th} variation on Schur functions from [10, §7] are elements of $\mathbb{F}_q[\mathbf{x}] := \mathbb{F}_q[x_1,\ldots,x_n]$ defined as follows. For each $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, first define anti-symmetric polynomials

$$A_\alpha(\mathbf{x}) := \det(x_i^{q^{\alpha_j}})_{i,j=1}^n.$$

Let $\delta_n := (n-1, n-2, \ldots, 1, 0)$. Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, define

$$\mathrm{S}_\lambda(\mathbf{x}) := \frac{A_{\lambda+\delta_n}(\mathbf{x})}{A_{\delta_n}(\mathbf{x})},$$

which is a priori only a rational function in $\mathbb{F}_q(\mathbf{x})$, but which Macdonald shows is actually a polynomial lying in $\mathbb{F}_q[\mathbf{x}]$. Since both $A_{\lambda+\delta_n}(\mathbf{x})$ and $A_{\delta_n}(\mathbf{x})$ are antisymmetric polynomials, their quotient $\mathrm{S}_\lambda(\mathbf{x})$ is a symmetric polynomial in the x_i. Macdonald shows that $\mathrm{S}_\lambda(\mathbf{x})$ enjoys the stronger property of being invariant under the entire general linear group $G = GL_n(\mathbb{F}_q)$.

Definition 6.1 Recall from Section 2 that $\hat{\mathbb{Q}}(\mathbf{x}) = \hat{\mathbb{Q}}(\mathbf{t})^{\otimes n}$ where $\hat{\mathbb{Q}}(\mathbf{t})$ was defined there, with Frobenius automorphism φ acting by $(\varphi f)(x_i) = f(x_1^q, \dots, x_n^q)$. First define

$$\mathbf{A}_\alpha(\mathbf{x}) := \det(x_i^{q^{\alpha_j}})_{i,j=1}^n$$

regarded as an element of $\hat{\mathbb{Q}}(\mathbf{x})$, and then define

$$\mathbf{S}_\lambda(\mathbf{x}) := \frac{\mathbf{A}_{\lambda+\delta_n}(\mathbf{x})}{\mathbf{A}_{\delta_n}(\mathbf{x})}$$

in $\hat{\mathbb{Q}}(\mathbf{x})$. Even after specializing to integers $q \geq 2$ this will turn out *not* to be a polynomial in general. We will be interested later in its *principal specialization*, lying in $\hat{\mathbb{Q}}(\mathbf{t})$:

$$\begin{aligned}
\mathbf{S}_\lambda(1, t, t^2, \dots, t^{n-1}) &= \frac{\det\left((t^{i-1})^{q^{\lambda_j+n-j}}\right)_{i,j=1}^n}{\det\left((t^{i-1})^{q^{n-j}}\right)_{i,j=1}^n} \\
&= \frac{\det\left((t^{q^{\lambda_j+n-j}})^{i-1}\right)_{i,j=1}^n}{\det\left((t^{q^{n-j}})^{i-1}\right)_{i,j=1}^n} \\
&= \prod_{0 \leq i < j \leq n-1} \frac{t^{q^{\lambda_{n-j}+j}} - t^{q^{\lambda_{n-i}+i}}}{t^{q^j} - t^{q^i}}.
\end{aligned} \tag{6.1}$$

Here the last equality used the Vandermonde determinant formula.

Define the analogue of the complete homogeneous symmetric function h_r by

$$\mathbf{H}_r(\mathbf{x}) = \mathbf{S}_{(r)}(\mathbf{x}) \tag{6.2}$$

for $r \geq 0$, and $\mathbf{H}_r(\mathbf{x}) = 0$ for $r < 0$.

Theorem 6.2 *For integers $n \geq k \geq 0$,*

$$\mathbf{H}_{(n-k)}(1, t, \cdots, t^k) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t}.$$

Proof When $\lambda = (\lambda_0, \dots, \lambda_k) = (n-k, 0, \dots, 0)$, the last product in (6.1)

$$\mathbf{S}_\lambda(1, t, \dots, t^k) = \prod_{0 \leq i < j \leq k} \frac{t^{q^{\lambda_{k+1-j}+j}} - t^{q^{\lambda_{k+1-i}+i}}}{t^{q^j} - t^{q^i}}$$

will have the numerator exactly matching the denominator in all of its factors except those with $j = k$. This leaves only these factors:

$$\prod_{0 \le i < k} \frac{t^{q^{(n-k)+k}} - t^{q^{0+i}}}{t^{q^k} - t^{q^i}} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t}.$$

□

6.2 Jacobi-Trudi formulae

Macdonald also proved Jacobi-Trudi-style determinantal formulae for his polynomials S_λ, allowing him to generalize them to skew shapes λ/μ. We review/adapt his proof here so that it lifts to $\hat{\mathbb{Q}}(\mathbf{x})$, giving the same formula for $\mathbf{S}_\lambda(\mathbf{x})$.

Theorem 6.3 (*Jacobi-Trudi*) *For any partition* $\lambda = (\lambda_1, \ldots, \lambda_n)$

$$\mathbf{S}_\lambda(\mathbf{x}) = det(\varphi^{1-j}\,\mathbf{H}_{\lambda_i - i + j}(\mathbf{x})_{i,j=1}^n)$$

Proof (cf. [10, §7]) For convenience of notation, we omit the variable set $\mathbf{x}$ from the notation for $\mathbf{H}$, $\mathbf{A}$, $\mathbf{S}_\lambda$ etc. The definition of $\mathbf{H}_r$ says that

$$\mathbf{H}_r := \mathbf{S}_{(r)} = \mathbf{A}_{\delta_n}^{-1} \cdot \det \begin{bmatrix} x_1^{q^{n+r-1}} & x_1^{q^{n-2}} & x_1^{q^{n-3}} & \cdots & x_1^{q^1} & x_1 \\ x_2^{q^{n+r-1}} & x_2^{q^{n-2}} & x_2^{q^{n-3}} & \cdots & x_2^{q^1} & x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{q^{n+r-1}} & x_n^{q^{n-2}} & x_n^{q^{n-3}} & \cdots & x_n^{q^1} & x_n \end{bmatrix} \tag{6.3}$$

Expanding the determinant along the first column shows that

$$\mathbf{H}_r(\mathbf{x}) = \sum_{k=1}^n \varphi^{n+r-1}(x_k) \cdot u_k \tag{6.4}$$

where u_k in $\hat{\mathbb{Q}}(\mathbf{x})$ do not depend on r. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, write down equation (6.4) for $r = \alpha_i - n + j$ for $i = 1, 2, \ldots, n$, and apply φ^{1-j} for $j = 1, 2, \ldots, n$, giving the $n \times n$ system of equations

$$\varphi^{1-j}\,\mathbf{H}_{\alpha_i - n + j} = \sum_{k=1}^n \varphi^{\alpha_i}(x_k) \cdot \varphi^{1-j} u_k.$$

Reinterpret this as a matrix equality:

$$\left(\varphi^{1-j}\,\mathbf{H}_{\alpha_i - n + j}\right)_{i,j=1}^n = \left(\varphi^{\alpha_i}(x_k)\right)_{i,k=1}^n \cdot \left(\varphi^{1-j} u_k\right)_{k,j=1}^n$$

Taking the determinant of both sides yields

$$\det\left(\varphi^{1-j}\,\mathbf{H}_{\alpha_i - n + j}\right)_{i,j=1}^n = \mathbf{A}_\alpha \cdot B \tag{6.5}$$

where $B := \det\left(\varphi^{1-j} u_k\right)_{k,j=1}^n$. One pins down the value of B by choosing $\alpha = \delta_n$ in (6.5): then $\alpha_i - n + j = j - i$, so the left side becomes the determinant of an upper unitriangular matrix, giving

$$1 = \mathbf{A}_{\delta_n} B$$

and hence $B = \mathbf{A}_{\delta_n}^{-1}$. Thus (6.5) becomes

$$\det\left(\varphi^{1-j}\,\mathbf{H}_{\alpha_i - n + j}\right)_{i,j=1}^n = \frac{\mathbf{A}_\alpha}{\mathbf{A}_{\delta_n}}.$$

Now taking $\alpha = \lambda + \delta_n$ yields the theorem. □

Theorem 6.3 shows that $\mathbf{S}_\lambda(\mathbf{x})$ is the special case when $\mu = \varnothing$ of the following "skew" construction.

Definition 6.4 Given two partitions

$$\lambda = (\lambda_1, \ldots, \lambda_\ell)$$
$$\mu = (\mu_1, \ldots, \mu_\ell)$$

with $\mu_i \le \lambda_i$ for $i = 1, 2, \ldots, \ell$, the set difference of their Ferrers diagrams λ/μ is called a *skew shape*. Define

$$\mathbf{S}_{\lambda/\mu}(\mathbf{x}) := \det\left(\varphi^{\mu_j - (j-1)}\,\mathbf{H}_{\lambda_i - \mu_j - i + j}\right)_{i,j=1,2,\ldots,\ell} \tag{6.6}$$

In the special case where $\mu = \varnothing$ and λ is a single row of size r, note that $\mathbf{S}_{(r)}(\mathbf{x}) = \mathbf{H}_r(\mathbf{x})$, consistent with (6.2).

In the special case where $\mu = \varnothing$ and $\lambda = 1^r$ is a single column of size r, define

$$\mathbf{E}_r(\mathbf{x}) := \mathbf{S}_{(1^r)}(\mathbf{x}),$$

and set $\mathbf{E}_r(\mathbf{x}) := 0$ for $r < 0$. The following analogue of Theorem 6.2 is proven analogously:

$$\mathbf{E}_r(1, t, t^2, \ldots, t^{n-1}) = \begin{bmatrix} n \\ n-r \end{bmatrix}_{q,t} \prod_{i=1}^{r} \frac{t^{q^n} - t^{q^{n-r+i-1}}}{t^{q^{n-r+i}} - t^{q^{n-r}}}. \tag{6.7}$$

In [10, §9] Macdonald explains how in a more general setting, one can write down a dual Jacobi-Trudi determinantal formula for $\mathbf{S}_{\lambda/\mu}(\mathbf{x})$, involving the *conjugate* or *transpose* partitions λ', μ' and the $\mathbf{E}_r$ instead of the $\mathbf{H}_r$.

Proposition 6.5

$$\mathbf{S}_{\lambda/\mu}(\mathbf{x}) := \det\left(\varphi^{-\mu'_j + j - 1}\,\mathbf{E}_{\lambda'_i - \mu'_j - i + j}(\mathbf{x})\right)_{i,j=1,2,\ldots,\ell}.$$

Proof We sketch Macdonald's proof from [10, §9] for completeness, and again, omit the variables $\mathbf{x}$ from the notation for convenience. We first prove that, for any interval I in $\mathbb{Z}$, the two matrices

$$H_I := \left(\varphi^{1-j}\,\mathbf{H}_{j-i}\right)_{i,j\in I}$$

$$E_I := \left((-1)^{j-i}\varphi^{-i}\,\mathbf{E}_{j-i}\right)_{i,j\in I}$$

are inverses of each other. Since both H_I, E_I are upper unitriangular, this will follow if one checks that for $i,k \in I$ with $i < k$ one has

$$\sum_{j=i}^{k}\varphi^{1-j}\,\mathbf{H}_{j-i}\cdot(-1)^{k-j}\varphi^{-j}\,\mathbf{E}_{k-j} = 0,$$

or equivalently, after applying φ^i and re-indexing $k - i =: r$,

$$\sum_{j=0}^{r}(-1)^j\varphi^{1-j}\,\mathbf{H}_j\cdot\varphi^{-j}\,\mathbf{E}_{r-j} = 0.$$

But this is exactly what one gets from expanding along its top row the determinant in this definition:

$$\mathbf{E}_r = \det\left(\phi^{1-j}\,\mathbf{H}_{1-i+j}\right)_{i,j=1}^{r}.$$

Once one knows that H_I, E_I are inverses of each other, and since they have determinant 1 by their unitriangularity, it follows that each minor subdeterminant of H_I equals the complementary cofactor of the transpose of E_I. One can check that for each λ/μ there is a choice of the interval I and the appropriate subdeterminant so that this equality is the one asserted in the proposition; see [9, Chap. 1, eqn. (2.9)]. □

Our goal in the next section is to describe a combinatorial interpretation for $\mathbf{S}_\lambda(1, t, \cdots, t^n)$ in terms of tableaux, which will imply that for integers $q \geq 2$ this polynomial in t has nonnegative coefficients.

6.3 Nonintersecting lattice paths, and the weight of a tableau

The usual skew Schur function $s_{\lambda/\mu}(\mathbf{x})$ has the following well-known combinatorial interpretation (see e.g. [10, §I.5], [20, §7.10]):

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \sum_T \mathbf{x}^T \tag{6.8}$$

where T runs over all reverse column-strict tableaux of shape λ/μ with entries in $\{1,2,\ldots,n\}$. Here a *reverse column-strict tableaux* T is an assignment of an entry to each cell of the skew shape λ/μ in such a way that the entries decrease weakly left-to-right in each row, and decrease strictly from top-to-bottom in each column. The monomial $\mathbf{x}^T := \prod_i x_{T_i}$ as i ranges over the cells of λ/μ.

There is a well-known combinatorial proof of this formula (see [19, Theorem 2.7.1] and [20, §7.16]) due to Gessel and Viennot. This proof begins with the Jacobi-Trudi determinantal expression for $s_{\lambda/\mu}$, reinterprets this as a signed sum over tuples of lattice paths, cancels this sum down to the nonintersecting tuples of lattice paths, and then shows how these biject with the tableaux.

Note that (6.8) implies the following interpretation for the principal specialization of $s_{\lambda/\mu}$:

$$s_{\lambda/\mu}(1, t, \dots, t^k) = \sum_T t^{\sum_i T_i}$$

where T ranges over the reverse column-strict tableaux of shape λ/μ with entries in $\{0, 1, \dots, k\}$. Our goal is to generalize this to $\mathbf{S}_{\lambda/\mu}(1, t, \dots, t^k)$, using the Gessel-Viennot proof mentioned above.

We begin by recalling how lattice paths biject with partitions and tableaux, in order to put the appropriate weight on the tuples of lattice paths. We start with the easy bijections between these three objects:

(i) Partitions ν inside a $k \times r$ rectangle.
(ii) Lattice paths P taking unit steps north (N) and east (E) from (x, y) to $(x + r, y + k)$.
(iii) Reverse column-strict tableaux of the single row shape (r) and entries in $\{0, 1, \dots, k\}$.

The bijection between (i) and (ii) sends the lattice path P to the Ferrers diagram $\nu(P)$ (in English notation) having P as its outer boundary and northwest corner at $(x, y + k)$. The bijection between (ii) and (iii) sends the lattice path P to the tableau whose entries give the depths below the line $y = k$ of the horizontal steps in the path P. For example, if $k = 4, r = 5$ then the partition $\nu = (4, 4, 1, 0)$ corresponds to the path P whose unit steps form the sequence $(N, E, N, E, E, E, N, N, E)$, which corresponds to the single-row tableau $T = 32220$; see Figure 1(a).

Given any skew shape λ/μ, the bijection between (ii) and (iii) above generalizes to one between these two sets:

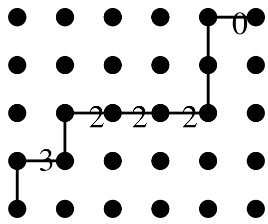
- All ℓ-tuples $(P_1, \dots, P_\ell)$ of lattice paths, where P_i goes from $(\mu_i - (i - 1), 0)$ to $(\lambda_i - (i - 1), k)$, and no pair P_i, P_j of paths touches, that is, the paths are *nonintersecting*.
- Reverse column strict tableaux of shape λ/μ with entries in $\{0, 1, \dots, k\}$.

Here the bijection has the i^{th} row of the tableaux giving the depths below the line $y = k$ of the horizontal steps in the i^{th} path P_i. For example, if $k = 4$ and $\ell = 3$ with $\lambda/\mu = (8, 6, 5)/(3, 2, 0)$, then the tableau T of shape λ/μ having entries in $\{0, 1, 2, 3, 4\}$ given by

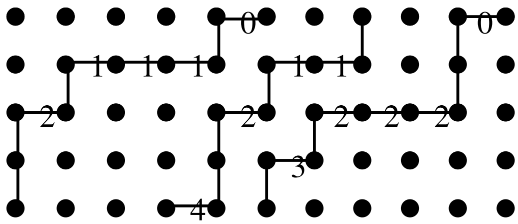
$$T = \begin{array}{cccccccc} \cdot & \cdot & \cdot & 3 & 2 & 2 & 2 & 0 \\ \cdot & \cdot & 4 & 2 & 1 & 1 & & \\ 2 & 1 & 1 & 1 & 0 & & & \end{array}$$

Fig. 1 (a) The lattice path corresponding to the partition $\nu = (4, 4, 1, 0)$ inside a 4×5 rectangle, or to the single-row tableau $T = 32220$ with entries in $\{0, 1, 2, 3, 4\}$. (b) From right to left, the 3-tuple (P_1, P_2, P_3) of nonintersecting lattice paths corresponding to the tableau T described in the text.

(a)



(b)



corresponds to the 3-tuple (P_1, P_2, P_3) where

P_1 is the path from $(3, 0)$ to $(8, 4)$ with steps $(N, E, N, E, E, E, N, N, E)$

P_2 is the path from $(1, 0)$ to $(5, 4)$ with steps (E, N, N, E, N, E, E, N)

P_3 is the path from $(-2, 0)$ to $(3, 4)$ with steps $(N, N, E, N, E, E, E, N, E)$

as depicted from right to left in Figure 1(b).

Given such a tableau T corresponding to the tuple of paths $(P_1, \dots, P_\ell)$, and letting $\nu(P_i)$ be the partition within a rectangle of width k that has P_i as its outer boundary as before, define

$$\mathrm{wt}(T; q, t) := \prod_{i=1}^{\ell} \varphi^{\mu_i - (i-1)} \, \mathrm{wt}(\nu(P_i), k; q, t). \tag{6.9}$$

In the next proof, we will use the fact that for a lattice path P and a cell x of $\nu(P)$, the distance $\mathrm{d}_P(x)$ which appears in the formula (5.1) for $\mathrm{wt}(x, \nu(P), k; q, t)$ is the taxicab distance from the starting point of path P to the cell x.

Theorem 6.6

$$\mathbf{S}_{\lambda/\mu}(1, t, \cdots, t^k) = \sum_T \mathrm{wt}(T; q, t)$$

where the sum ranges over all reverse column-strict tableaux T of shape λ/μ with entries in $\{0, 1, \dots, k\}$.

Furthermore,

$$\lim_{t \to 1} \mathrm{wt}(T; q, t) = q^{\sum_i T_i} \quad \text{and} \quad \lim_{q \to 1} \mathrm{wt}(T; q, t^{\frac{1}{q-1}}) = t^{\sum_i T_i}. \tag{6.10}$$

Proof We recapitulate and adapt the usual Gessel-Viennot proof alluded to above, being careful to ensure that the weights behave correctly in this context.

Starting with the definition (6.6), and using Theorems 6.2 and 5.3 to replace occurrences of $\mathbf{H}_r$, one has

$$\begin{aligned}\mathbf{S}_{\lambda/\mu}(1,t,\cdots,t^k) &= \det(\varphi^{\mu_j-(j-1)}\,\mathbf{H}_{\lambda_i-\mu_j-i+j})_{i,j=1,2,\ldots,\ell} \\ &= \sum_{w\in\mathfrak{S}_\ell}\operatorname{sgn}(w)\prod_{i=1}^{\ell}\varphi^{\mu_{w(i)}-(w(i)-1)}\,\mathbf{H}_{\lambda_i-\mu_{w(i)}-i+w(i)}(1,t,\ldots,t^k) \\ &= \sum_{(w,(P_1,\ldots,P_\ell))}\operatorname{sgn}(w)\prod_{i=1}^{\ell}\varphi^{\mu_{w(i)}-(w(i)-1)}\,\mathrm{wt}(\nu(P_i),k;q,t).\end{aligned}$$

The last summation runs over pairs $(w,(P_1,\ldots,P_\ell))$, where w is a permutation in $\mathfrak{S}_\ell$, and $(P_1,\ldots,P_\ell)$ is an ℓ-tuple of lattice paths where P_i goes from $(\mu_{w(i)}-(w(i)-1),0)$ to $(\lambda_i-(i-1),k)$.

One wants to cancel all the terms in the last sum above that have at least one pair (P_i,P_j) of intersecting lattice paths. In particular, this occurs if w is not the identity permutation. Therefore all terms remaining will have w equal to the identity, and the paths $(P_1,\ldots,P_\ell)$ nonintersecting. Note that these leftover paths have the correct weight $\mathrm{wt}(T;q,t)$ for their corresponding tableau T, as given in the theorem.

The Gessel-Viennot cancellation argument involves tail-swapping. One cancels a term $(w,(P_1,\ldots,P_\ell))$ with another term having equal weight and opposite sign. One finds this other term, say by choosing the southeasternmost intersection point p for any pair of the paths, and choosing the lexicographically smallest pair of indices $i<j$ of paths P_i,P_j which touch at p. Then replace w by $w':=w\cdot(i,j)$, and replace the pair of paths (P_i,P_j) with the pair of paths (P'_j,P'_i), in which P'_i (resp. P'_j) follows the path P_i up until it reaches p, but then follows P_j (resp. P_i) from that point onward.

One must check that this replacement does not change the weight, that is,

$$\begin{aligned}&\varphi^{\mu_{w(i)}-(w(i)-1)}\,\mathrm{wt}(\nu(P_i),k;q,t)\cdot\varphi^{\mu_{w(j)}-(w(j)-1)}\,\mathrm{wt}(\nu(P_j),k;q,t) \\ &=\varphi^{\mu_{w'(i)}-(w'(i)-1)}\,\mathrm{wt}(\nu(P'_j),k;q,t)\cdot\varphi^{\mu_{w'(j)}-(w'(j)-1)}\,\mathrm{wt}(\nu(P'_i),k;q,t) \qquad (6.11)\\ &\quad=\varphi^{\mu_{w(i)}-(w(i)-1)}\,\mathrm{wt}(\nu(P'_i),k;q,t)\cdot\varphi^{\mu_{w(j)}-(w(j)-1)}\,\mathrm{wt}(\nu(P'_j),k;q,t).\end{aligned}$$

This follows by comparing how each cell x of $\nu(P_i)$ (or of $\nu(P_j)$) contributes a factor of the form $\varphi^m\,\mathrm{wt}(x,\nu(P),k;q,t)$ to the products on the left and right sides of (6.11). There are two cases. If x is a cell of either $\nu(P_i)$ or $\nu(P_j)$ lying to the left of the point p, then it will contribute the same factor on both sides. If x is a cell of $\nu(P_i)$ lying to the right of p, then it contributes

$$\varphi^{\mu_{w(i)}-(w(i)-1)}\,\mathrm{wt}(x,\nu(P_i),k;q,t) \text{ on the leftmost side of (6.11) and}$$

$$\varphi^{\mu_{w(j)}-(w(j)-1)}\,\mathrm{wt}(x,\nu(P'_j),k;q,t) \text{ on the rightmost side of (6.11).}$$

However, we claim there is an equality

$$\varphi^{\mu_{w(i)}-(w(i)-1)}\,\mathrm{wt}(x,\nu(P_i),k;q,t)=\varphi^{\mu_{w(j)}-(w(j)-1)}\,\mathrm{wt}(x,\nu(P'_j),k;q,t).$$

Definition (5.1) shows that the exponent $e_k(x):=q^{\mathrm{r}(x)+\mathrm{d}_P(x)}-q^{\mathrm{d}_P(x)}$ determining the powers of t appearing in $\mathrm{wt}(x,\nu(P),k;q,t)$ will be computed with the *same* row index $\mathrm{r}(x)$, whether one considers x inside $\nu(P_i)$ or inside $\nu(P'_j)$. However, the distance $\mathrm{d}_{P_i}(x)$ from x to the start of P_i is $\mu_{w(j)}-\mu_{w(i)}-w(i)+w(j)$ *less* than its distance $\mathrm{d}_{P'_j}(x)$ to the start of P'_j. This difference is exactly compensated by the difference in the Frobenius power which will be applied: $\varphi^{\mu_{w(i)}-(w(i)-1)}$ versus $\varphi^{\mu_{w(j)}-(w(j)-1)}$. The argument is similar for a cell of $\nu(P_j)$ lying to the right of p. Thus the two terms have the same weight, and opposite signs: $\mathrm{sgn}(w')=-\mathrm{sgn}(w)$.

One can check that this bijection between terms is an involution, and hence it provides the necessary cancellation. □

Corollary 6.7 *If $q\geq 2$ is an integer, then $\mathbf{S}_\lambda(1,t,\cdots,t^k)$ is a polynomial in t with nonnegative coefficients.*

Proof Use Theorem 6.6. We wish to show that when $\mu=\varnothing$, for every column-strict tableau T of shape $\lambda=\lambda/\mu$, every factor in the product formula for $\mathrm{wt}(T;q,t)$ given by (6.9) is a polynomial in t with nonnegative coefficients for $q\geq 2$ an integer.

The key point is that $\mu=\varnothing$ means the sequence of nonintersecting lattice paths $(P_1,P_2,\ldots,P_\ell)$ corresponding to T will start in the *consecutive* positions

$$(0,0),(-1,0),(-2,0),\ldots,(-\ell+1,0).$$

For each i, this forces the path P_i to start with at least $i-1$ vertical steps, in order to avoid intersecting the next path P_{i-1}. Hence the corresponding partition shape $\nu(P_i)$ will be missing at least $i-1$ boxes in its first column. Consequently, in the product formula for $\mathrm{wt}(T;q,t)$ given by (6.9), one can rewrite each factor

$$\begin{aligned}\varphi^{\mu_i-(i-1)}\,\mathrm{wt}(\nu(P_i),k;q,t)&=\varphi^{-(i-1)}\,\mathrm{wt}(\nu(P_i),k;q,t)\\&=\mathrm{wt}(\nu(P_i),k-(i-1);q,t)\end{aligned}$$

where the last equality uses $i-1$ times repeatedly the second case of the recurrence in Proposition 5.2. The formula for $\mathrm{wt}(\nu(P_i),k-(i-1);q,t)$ given in (5.1) then shows that it is a polynomial in t with nonnegative coefficients whenever $q\geq 2$ is an integer. □

Example 6.8 The two skew shapes

$$(2,1)=(2,1)/(0,0)=\begin{matrix}\times&\times\\\times&\end{matrix}\qquad\text{and}\qquad(2,2)/(1,0)=\begin{matrix}\cdot&\times\\\times&\times\end{matrix}$$

have the same *ordinary* Schur functions, and hence the same principal specializations

$$s_{(2,1)}(1,q)=s_{(2,2)/(1,0)}(1,q)=q^1+q^2$$

corresponding to either the two reverse column-strict tableaux

$$T_1 = \begin{matrix} 1 & 0 \\ 0 & \end{matrix} \qquad T_2 = \begin{matrix} 1 & 1 \\ 0 & \end{matrix} \qquad \text{of shape } (2, 1)$$

or the tableaux

$$T_1' = \begin{matrix} \cdot & 1 \\ 0 & 0 \end{matrix} \qquad T_2' = \begin{matrix} \cdot & 1 \\ 1 & 0 \end{matrix} \qquad \text{of shape } (2,2)/(1,0).$$

We compare here what the preceding results say for

$$\begin{aligned} \mathbf{S}_{(2,1)/(0,0)}(1,t) = \mathbf{S}_{(2,1)}(1,t) &= \det \begin{bmatrix} \varphi^0\,\mathbf{H}_2(1,t) & \varphi^{-1}\,\mathbf{H}_3(1,t) \\ \varphi^0\,\mathbf{H}_0(1,t) & \varphi^{-1}\,\mathbf{H}_1(1,t) \end{bmatrix} \\ &= [1+q+q^2]_{t^{q-1}} \cdot [1+q]_{t^{\frac{q-1}{q}}} \\ &\quad - [1+q+q^2+q^3]_{t^{\frac{q-1}{q}}} \\ &= t+t^2+t^3+t^4+t^5+t^6 \text{ when } q=2, \end{aligned}$$

versus

$$\begin{aligned} \mathbf{S}_{(2,2)/(1,0)}(1,t) &= \det \begin{bmatrix} \varphi^1\,\mathbf{H}_1(1,t) & \varphi^{-1}\,\mathbf{H}_3(1,t) \\ \varphi^1\,\mathbf{H}_0(1,t) & \varphi^{-1}\,\mathbf{H}_2(1,t) \end{bmatrix} \\ &= [1+q]_{t^{q(q-1)}} \cdot [1+q+q^2]_{t^{\frac{q-1}{q}}} - [1+q+q^2+q^3]_{t^{\frac{q-1}{q}}} \\ &= t^2+t^{\frac{5}{2}}+t^3+t^4+t^{\frac{9}{2}}+t^5 \text{ when } q=2. \end{aligned}$$

Note that $\mathbf{S}_{(2,1)}(1,t) \neq \mathbf{S}_{(2,2)/(1,0)}(1,t)$. Both are elements of $\hat{\mathbb{Q}}(\mathbf{t})$ and can be rewritten as weighted sums over tableaux, according to Theorem 6.6:

$$\begin{aligned} \mathbf{S}_{(2,1)/(0,0)}(1,t) &= \mathrm{wt}(T_1;q,t) + \mathrm{wt}(T_2;q,t) \\ &= t^{q-1}[q]_{t^{q-1}} + t^{q-1}[q]_{t^{q-1}} \cdot t^{q^2-q}[q]_{t^{q^2-q}} \\ &= (t+t^2) + (t^3+t^4+t^5+t^6) \text{ when } q=2, \\ &\quad \text{versus} \\ \mathbf{S}_{(2,2)/(1,0)}(1,t) &= \mathrm{wt}(T_1';q,t) + \mathrm{wt}(T_2';q,t) \\ &= t^{q^2-q}[q]_{t^{q^2-q}} + t^{q^2-q}[q]_{t^{q^2-q}} \cdot t^{1-\frac{1}{q}}[q]_{t^{1-\frac{1}{q}}} \\ &= (t^2+t^4) + (t^{\frac{5}{2}}+t^3+t^{\frac{9}{2}}+t^5) \text{ when } q=2. \end{aligned}$$

Lastly, note that that $\mathbf{S}_{(2,1)}(1,t)$ is a polynomial in t (with nonnegative coefficients) for integers $q \geq 2$, as predicted by Proposition 6.5, but this is not true for the skew example $\mathbf{S}_{(2,2)/(1,0)}(1,t)$.

7 Generalization 2: multinomial coefficients

We explore here a different generalization of the (q,t)-binomial coefficient, this time to a multinomial coefficient that appears naturally within the invariant theory of $GL_n(\mathbb{F}_q)$.

7.1 Definition

Given a composition $\alpha=(\alpha_1,\dots,\alpha_\ell)$ of n, define its partial sums $\sigma_s:=\alpha_1+\alpha_2+\cdots+\alpha_s$, so that $\sigma_0=0$ and let

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} := \begin{bmatrix} n \\ \alpha_1,\dots,\alpha_\ell \end{bmatrix}_{q,t} := \frac{n!_{q,t}}{\alpha!_{q,t}} = \frac{\prod_{i=1}^{n}(1-t^{q^n-q^{n-i}})}{\prod_{s=1}^{\ell}\prod_{i=1}^{\alpha_s}(1-t^{q^{\sigma_s}-q^{\sigma_s-i}})},$$

where

$$\alpha!_{q,t} := \alpha_1!_{q,t}\cdot\alpha_2!_{q,t^{q^{\sigma_1}}}\cdot\alpha_3!_{q,t^{q^{\sigma_2}}}\cdots\alpha_\ell!_{q,t^{q^{\sigma_{\ell-1}}}}.$$

Note its relation to the (q,t)-binomial

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \begin{bmatrix} n \\ k,n-k \end{bmatrix}_{q,t},$$

as well as these formulae:

$$\begin{aligned}\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} &= \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}_{q,t} \varphi^{\alpha_1}\begin{bmatrix} n-\alpha_1 \\ \alpha_2,\dots,\alpha_\ell \end{bmatrix}_{q,t} \\ &= \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}_{q,t} \varphi^{\sigma_1}\begin{bmatrix} n-\alpha_1 \\ \alpha_2 \end{bmatrix}_{q,t} \varphi^{\sigma_2}\begin{bmatrix} n-\alpha_1-\alpha_2 \\ \alpha_3 \end{bmatrix}_{q,t}\cdots\end{aligned} \tag{7.1}$$

Equations 7.1 along with Corollary 4.2 imply that for integers $q\geq 2$, the (q,t)-multinomial is a polynomial in t with nonnegative coefficients. As with the (q,t)-binomial, it has two limiting values given by the usual q-*multinomial coefficient*:

$$\begin{aligned}\lim_{t\to 1}\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} &= \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q} := \frac{\prod_{i=1}^{n}\left(q^n-q^{n-i}\right)}{\prod_{s=1}^{\ell}\prod_{i=1}^{\alpha_s}\left(q^{\sigma_s}-q^{\sigma_s-i}\right)} \\ \lim_{q\to 1}\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t^{\frac{1}{q-1}}} &= \begin{bmatrix} n \\ \alpha \end{bmatrix}_{t}.\end{aligned}$$

7.2 Algebraic interpretation of multinomials

We recall here two algebraic interpretations of the usual multinomial and q-multinomial that were mentioned in the Introduction, and then give the analogue for the (q,t)-multinomial.

The symmetric group $W=\mathfrak{S}_n$ acts transitively on the collection of all flags of subsets

$$\varnothing =: S_0 \subset S_1 \subset \cdots \subset S_{\ell-1} \subset S_\ell := \{1,2,\dots,n\}$$

in which $|S_i| = \sigma_i$. The stabilizer of one such flag is the *Young* or *parabolic subgroup* W_α which permutes separately the first α_1 integers, the next α_2 integers, etc. Thus the coset space W/W_α is identified with the collection of these flags, and hence has cardinality $[W : W_\alpha] = \binom{n}{\alpha}$. When q is a prime power, the q-multinomial analogously gives the cardinality of the finite partial flag manifold G/P_α where the group $G = GL_n(\mathbb{F}_q)$ and P_α is the parabolic subgroup that stabilizes one of the flags of $\mathbb{F}_q$-subspaces

$$\{0\} =: V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell := \mathbb{F}_q^n$$

in which $\dim_{\mathbb{F}_q} V_i = \sigma_i$.

On the other hand, one also has parallel Hilbert series interpretations arising from the invariant theory of these groups acting on appropriate polynomial algebras. In the case of the (q,t)-multinomial this is where its definition arose initially in work of the authors with D. White [14, §9].

Let $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[x_1, \dots, x_n]$ carry its usual action of $W = \mathfrak{S}_n$ by permutations of the variables, and let $\mathbb{F}_q[\mathbf{x}] := \mathbb{F}_q[x_1, \dots, x_n]$ carry its usual action of $G = GL_n(\mathbb{F}_q)$ by linear substitution of variables. The *fundamental theorem of symmetric functions* states that the invariant subring $\mathbb{Z}[\mathbf{x}]^W$ is a polynomial algebra generated by the elementary symmetric functions $e_1(\mathbf{x}), \dots, e_n(\mathbf{x})$. A well-known theorem of Dickson (see e.g. [1, §8.1]) asserts that the invariant subring $\mathbb{F}_q[\mathbf{x}]^G$ is a polynomial algebra; its generators can be chosen to be the *Dickson polynomials*, which are the same as Macdonald's polynomials $E_1(\mathbf{x}), \dots, E_n(\mathbf{x})$ discussed in Section 6 above. It is also not hard to see that, for any composition α of n, the invariant subring $\mathbb{Z}[\mathbf{x}]^{W_\alpha}$ is a polynomial algebra, whose generators may be chosen as the elementary symmetric functions in the first α_1 variables, then those in the next α_2 variables, etc. The following result of Mui [12] (see also Hewett [4]) is less obvious.

Theorem 7.1 *For every composition α of n, the parabolic subgroup P_α has invariant subring $\mathbb{F}_q[\mathbf{x}]^{P_\alpha}$ isomorphic to a polynomial algebra $\mathbb{F}_q[f_1, \dots, f_n]$. Furthermore, the generators $f_1, \dots, f_n$ may be chosen homogeneous with degrees $q^{\sigma_s} - q^{\sigma_s - i}$ for $s = 1, \dots, \ell$ and $i = 1, \dots, \alpha_s$.*

Corollary 7.2 *For every composition α of n,*

$$\begin{aligned} \begin{bmatrix} n \\ \alpha \end{bmatrix}_t &= \frac{\mathrm{Hilb}(\mathbb{Z}[\mathbf{x}]^{W_\alpha}, t)}{\mathrm{Hilb}(\mathbb{Z}[\mathbf{x}]^{W}, t)} \\ \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} &= \frac{\mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^{P_\alpha}, t)}{\mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^{G}, t)}. \end{aligned} \tag{7.2}$$

Furthermore, $\mathbb{Z}[\mathbf{x}]^{W_\alpha}, \mathbb{F}_q[\mathbf{x}]^{P_\alpha}$ are free as modules over over $\mathbb{Z}[\mathbf{x}]^W, \mathbb{F}_q[\mathbf{x}]^{GL_n}$, respectively, and hence

$$\begin{aligned} \begin{bmatrix} n \\ \alpha \end{bmatrix}_t &= \mathrm{Hilb}(\mathbb{Z}[\mathbf{x}]^{W_\alpha}/(\mathbb{Z}[\mathbf{x}]^W_+), t) \\ \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} &= \mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^{P_\alpha}/(\mathbb{F}_q[\mathbf{x}]^G_+), t). \end{aligned} \tag{7.3}$$

Proof The Hilbert series for a graded polynomial algebra with generators in degrees $d_1, \dots, d_n$ is $\prod_{i=1}^n \frac{1}{1-t^{d_i}}$. Hence the multinomial expressions (7.2) for the quotient of Hilbert series follows in each case from consideration of the degrees of the generators for $\mathbb{Z}[\mathbf{x}]^W, \mathbb{Z}[\mathbf{x}]^{W_\alpha}, \mathbb{F}_q[\mathbf{x}]^G, \mathbb{F}_q[\mathbf{x}]^{P_\alpha}$.

For the re-interpretations in (7.3), one needs a little invariant theory and commutative algebra, such as can be found in the book by Benson [1] or the survey by Stanley [18]. When two nested finite groups $H \subset G$ act on a Noetherian ring R, the invariant subring R^H is finitely generated as a module over R^G. If R^G is a polynomial subalgebra, this means its generators will form a system of parameters for R^H. When R^H is also polynomial, it is Cohen-Macaulay, and hence a free module over the polynomial subalgebra generated by any system of parameters. In this situation, when the rings and group actions are all graded, a free basis for R^H as an R^G-module can be obtained by lifting any basis for $R^H/(R^G_+)$ as a module over the ring R^G/R^G_+ (which equals $\mathbb{F}_q$ or $\mathbb{Z}$ in our setting). Therefore

$$\frac{\mathrm{Hilb}(R^H, t)}{\mathrm{Hilb}(R^G, t)} = \mathrm{Hilb}(R^H/(R^G_+), t).$$

□

Remark 7.3 It should be clear that the above Hilbert series interpretation of the (q,t)-multinomial generalizes the "$q = 1$" interpretation of the t-multinomial. It turns out that it also generalizes the interpretation of the q-multinomial as $[G : P_\alpha] = |G/P_\alpha|$ when $t = 1$, for the following reason: when two nested finite subgroups $H \subset G \subset GL_n(k)$ act on $k[\mathbf{x}]$, one always has (see [1, §2.5])

$$\lim_{t \to 1} \frac{\mathrm{Hilb}(k[\mathbf{x}]^H, t)}{\mathrm{Hilb}(k[\mathbf{x}]^G, t)} = [G : H].$$

8 The (q,t)-analogues of $q^{\ell(w)}$

Corollary 7.2 interprets the (q,t)-multinomial algebraically when one specializes q to be a prime power. Our goal here is a combinatorial interpretation valid in general, generalizing Theorem 5.3.

Given a composition α of n, recall that $W = \mathfrak{S}_n$ has a parabolic subgroup W_α whose index $[W : W_\alpha]$ is given by the multinomial coefficient $\binom{n}{\alpha}$. Viewing W as a Coxeter group with the adjacent transpositions $(i, i+1)$ as its usual set of Coxeter generators, one has its *length function* $\ell(w)$ defined as the minimum length of w as a product of these generators. This is well-known to be the *inversion number*, counting pairs $1 \le i < j \le n$ with $w(i) > w(j)$.

There are distinguished *minimum-length coset representatives* W^α for W/W_α: a permutation w lies in W^α if and only if α refines its *descent composition* $\beta(w) = (\beta_1, \dots, \beta_\ell)$. Recall that $\beta(w)$ is the composition of n defined by the property that

$$w(i) > w(i+1) \text{ if and only if } i \in \{\beta_1, \beta_1 + \beta_2, \dots, \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}\}.$$

Rephrased, $\beta(w)$ lists the lengths of the maximal increasing consecutive subsequences of $w=(w(1),w(2),\ldots,w(n))$.

It is well-known (see [19, Prop. 1.3.17]) that

$$\begin{aligned}\binom{n}{\alpha} &= |W^\alpha| \\ \begin{bmatrix} n \\ \alpha \end{bmatrix}_q &= \sum_{w\in W^\alpha} q^{\ell(w)}.\end{aligned} \tag{8.1}$$

We wish to similarly express the (q,t)-multinomial as a sum over w in W^α of a weight $\mathrm{wt}(w;q,t)$, simultaneously generalizing (8.1) and Theorem 5.6. The weight $\mathrm{wt}(w;q,t)$ will be defined recursively in a way that generalizes the defining recurrence for $\mathrm{wt}(\lambda,k;q,t)$ in Proposition 5.2.

Recall that when $\alpha=(k,n-k)$ has only two parts, there is a bijection $\lambda\leftrightarrow u_\lambda$ between partitions λ inside a $k\times(n-k)$ rectangle and the minimum length coset representatives u_λ in W^α, determined by

$$u_\lambda(i)=\lambda_{k+1-i}+i-1 \text{ for } i=1,2,\ldots,k.$$

Now given any permutation w in $W=\mathfrak{S}_n$, if k is defined by $w^{-1}(1)=k+1$, then taking $\alpha=(k,1,n-k-1)$, one can uniquely express

$$w=u_\lambda\cdot a\cdot e\cdot b$$

with

$$\ell(w)=\ell(u_\lambda)+\ell(a)+\ell(b)$$

where

- $u_\lambda\in W^{(k,n-k)}=\mathfrak{S}_n^{(k,n-k)}$,
- $a\in\mathfrak{S}_{\{1,2,\ldots,k\}}\cong\mathfrak{S}_k$,
- $e\in\mathfrak{S}_{\{k+1\}}\cong\mathfrak{S}_1$ (so e is the identity permutation of $\{k+1\}$),
- $b\in\mathfrak{S}_{\{k+2,k+3,\ldots,n\}}\cong\mathfrak{S}_{n-k-1}$.

Note also that in the above factorization $w=u_\lambda aeb$, by the definition of k, one knows that λ has its first column *full* of length k. Let $\hat{\lambda}$ denote the partition inside a $k\times(n-1-k)$ rectangle obtained from λ by removing this first column, so that $u_{\hat{\lambda}}$ lies in $\mathfrak{S}_{n-1}^{(k,n-1-k)}$.

Definition 8.1 For $w\in\mathfrak{S}_n$, define $\mathrm{wt}(w;q,t)$ in $\hat{\mathbb{Q}}(\mathbf{t})$ *recursively* to be 1 if $n=1$, and otherwise if $w^{-1}(1)=k+1$ set

$$\mathrm{wt}(w;q,t):=t^{q^k-1}\frac{k!_{q,t^q}}{k!_{q,t}}\cdot\mathrm{wt}(u_{\hat{\lambda}};q,t^q)\,\mathrm{wt}(a;q,t)\,\mathrm{wt}(b;q,t^{q^{k+1}}) \tag{8.2}$$

Example 8.2 Let $n=8$ and choose

$$w=\begin{pmatrix}1&2&3&4&5&6&7&8\\5&2&7&4&1&3&8&6\end{pmatrix}$$

Then $k+1=w^{-1}(1)=5$, so that $k=4$, and the above factorization is

$$\begin{aligned} w &= u_\lambda \cdot a \cdot e \cdot b \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & \mid & 5 & \mid & 6 & 7 & 8 \\ 2 & 4 & 5 & 7 & \mid & 1 & \mid & 3 & 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 6 & 7 & 8 \\ 6 & 8 & 7 \end{pmatrix}. \end{aligned}$$

Here $\lambda=(3,2,2,1)$, so that $\hat{\lambda}=(2,1,1,0)$ and

$$u_{\hat{\lambda}} = \begin{pmatrix} 1 & 2 & 3 & 4 & \mid & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & \mid & 2 & 5 & 7 \end{pmatrix}.$$

Then the recursive definition says

$$\begin{aligned} &\mathrm{wt}(w;q,t) \\ &:= t^{q^4-1} \frac{4!_{q,t^q}}{4!_{q,t}} \cdot \mathrm{wt}(u_{\hat{\lambda}};q,t^q)\,\mathrm{wt}(a;q,t)\,\mathrm{wt}(b;q,t^{q^5}) \\ &:= t^{q^4-1}[q]_{t^{q^4-1}}[q]_{t^{q^4-q}}[q]_{t^{q^4-q^2}}[q]_{t^{q^4-q^3}} \cdot \mathrm{wt}(u_{\hat{\lambda}};q,t^q)\,\mathrm{wt}(a;q,t)\,\mathrm{wt}(b;q,t^{q^5}). \end{aligned}$$

where we regard b as an element of $\mathfrak{S}_3$.

Proposition 8.3 *Given any w in $\mathfrak{S}_n$, the weight* $\mathrm{wt}(w;q,t)$ *for integers $q\geq 2$ is a polynomial in t with nonnegative coefficients, taking the following form*

$$\mathrm{wt}(w;q,t)=t^x \prod_{i=1}^{\ell(w)} [q]_{t^{q^{y_i}-q^{z_i}}}$$

for some nonnegative integers x, y_i, z_i with $y_i > z_i$ for all i. Furthermore,

$$\lim_{t\to 1} \mathrm{wt}(w;q,t) = q^{\ell(w)} \quad \textit{and} \quad \lim_{q\to 1} \mathrm{wt}(w;q,t^{\frac{1}{q-1}}) = t^{\ell(w)}. \tag{8.3}$$

Proof For all of these assertions, induct on $\ell(w)$, using the fact that

$$\begin{aligned} \ell(w) &= \ell(u_\lambda)+\ell(a)+\ell(b) \\ &= k+\ell(u_{\hat{\lambda}})+\ell(a)+\ell(b) \end{aligned}$$

along with the recursive definition (8.2), equation (4.4), and the limits in (5.3). □

We first show that this recursively defined $\mathrm{wt}(w;q,t)$ coincides with $\mathrm{wt}(\lambda,k;q,t)$ when $w=u_\lambda$.

Proposition 8.4 *For any partition μ inside a $k\times(n-k)$ rectangle, one has* $\mathrm{wt}(u_\mu;q,t)=\mathrm{wt}(\mu,k;q,t)$. *Consequently,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{u_\mu \in \mathfrak{S}_n^{(k,n-k)}} \mathrm{wt}(u_\mu;q,t).$$

Proof One checks that $\mathrm{wt}(u_\mu; q, t)$ satisfies the same defining recursion (5.2) as $\mathrm{wt}(\mu; q, t)$. Temporarily denote

$$\mathrm{wt}(u_\mu, k; q, t) := \mathrm{wt}(u_\mu; q, t)$$

to emphasize the dependence on k. The fact that $w = u_\mu$ lies in $W^{(k,n-k)}$ implies either $w^{-1}(1) = k+1$ or 1, depending upon whether or not μ has its first column full of length k. In the former case, one can check that the recursion (8.2) gives

$$\mathrm{wt}(u_\mu, k; q, t) = t^{q^k-1} \frac{k!_{q,t^q}}{k!_{q,t}} \cdot \mathrm{wt}(u_{\hat{\mu}}, k; q, t^q)$$

and in the latter case that it gives

$$\mathrm{wt}(u_\mu, k; q, t) = \mathrm{wt}(u_{\hat{\mu}}, k-1; q, t^q),$$

as desired. Thus the equality $\mathrm{wt}(u_\mu, k; q, t) = \mathrm{wt}(\mu, k; q, t)$ follows by induction on n.

The last assertion of the proposition is simply the restatement of Theorem 5.3. □

In order to generalize Proposition 8.4 to the (q,t)-multinomial, it helps to have a *multinomial (q,t)-Pascal relation.*

Proposition 8.5 *For any composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of n, with partial sums $\sigma_s = \sum_{i=1}^{s} \alpha_s$ (and $\sigma_0 := 0$), one has*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \sum_{i=1}^{\ell} t^{q^{\sigma_{i-1}}-1} \frac{(\alpha_1, \alpha_2, \dots, \alpha_{i-1})!_{q,t^q}}{(\alpha_1, \alpha_2, \dots, \alpha_{i-1})!_{q,t}} \times \begin{bmatrix} n-1 \\ \alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_n \end{bmatrix}_{q,t^q}$$

where we recall that $\alpha!_{q,t} := \alpha_1!_{q,t} \cdot \alpha_2!_{q,t^{q^{\sigma_1}}} \cdot \alpha_3!_{q,t^{q^{\sigma_2}}} \cdots \alpha_\ell!_{q,t^{q^{\sigma_{\ell-1}}}}$.

Proof Induct on ℓ, with the base case $\ell = 2$ being the first of the two (q,t)-Pascal relations from Proposition 4.1. In the inductive step, write $\alpha = (\alpha_1, \hat{\alpha})$ where $\hat{\alpha} := (\alpha_2, \dots, \alpha_\ell)$ is a composition of $n - \alpha_1$. Beginning with (7.1), start manipulating as follows:

$$\begin{aligned}
\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} &= \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}_{q,t} \varphi^{\alpha_1} \begin{bmatrix} n-\alpha_1 \\ \hat{\alpha} \end{bmatrix}_{q,t} \\
&= \left(\begin{bmatrix} n-1 \\ \alpha_1 - 1 \end{bmatrix}_{q,t^q} + t^{q^{\alpha_1}-1} \frac{\alpha_1!_{q,t^q}}{\alpha_1!_{q,t}} \begin{bmatrix} n-1 \\ \alpha_1 \end{bmatrix}_{q,t^q} \right) \varphi^{\alpha_1} \begin{bmatrix} n-\alpha_1 \\ \hat{\alpha} \end{bmatrix}_{q,t} \\
&= \begin{bmatrix} n-1 \\ \alpha_1 - 1 \end{bmatrix}_{q,t^q} \varphi^{\alpha_1} \begin{bmatrix} n-\alpha_1 \\ \hat{\alpha} \end{bmatrix}_{q,t} \\
&\quad + t^{q^{\alpha_1}-1} \frac{\alpha_1!_{q,t^q}}{\alpha_1!_{q,t}} \begin{bmatrix} n-1 \\ \alpha_1 \end{bmatrix}_{q,t^q} \varphi^{\alpha_1} \begin{bmatrix} n-\alpha_1 \\ \hat{\alpha} \end{bmatrix}_{q,t}
\end{aligned}$$

The first summand is exactly

$$\begin{bmatrix} n-1 \\ \alpha_1 - 1, \hat{\alpha} \end{bmatrix}_{q,t^q}$$

which is the $i=1$ term in the proposition. If one applies the inductive hypothesis to $\begin{bmatrix} n-\alpha_1 \\ \hat{\alpha} \end{bmatrix}_{q,t}$ in the second summand, one obtains a sum of $\ell - 1$ terms. When multiplied by the other factors in the second summand, these give the desired remaining terms $i = 2, 3, \ldots, \ell$ in the proposition. □

Theorem 8.6 *For any composition α of n, and $W = \mathfrak{S}_n$,*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \sum_{w \in W^\alpha} \mathrm{wt}(w; q, t).$$

Proof Induct on n, with the base case $n=1$ being trivial. If $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ then one can group the terms in the sum on the right into the subsums

$$\sum_{\substack{w \in \mathfrak{S}_n^\alpha: \\ w^{-1}(1) = \sigma_{i-1}+1}} \mathrm{wt}(w; q, t) \tag{8.4}$$

for $i = 1, 2, \ldots, \ell$. Introducing the following notations

$$\begin{aligned} k &:= \sigma_{i-1} \\ \alpha' &:= (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}) \\ \alpha'' &:= (\alpha_i - 1, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_\ell) \end{aligned}$$

we wish to show that the subsum (8.4) equals the following term from the right side of Proposition 8.5:

$$t^{q^k - 1} \frac{\alpha'!_{q,t^q}}{\alpha'!_{q,t}} \begin{bmatrix} n-1 \\ \alpha', \alpha'' \end{bmatrix}_{q,t^q}. \tag{8.5}$$

Note that when $w^{-1}(1) = k+1$, the recursive definition of $\mathrm{wt}(w; q, t)$ says

$$\mathrm{wt}(w; q, t) = t^{q^k - 1} \frac{k!_{q,t^q}}{k!_{q,t}} \cdot \mathrm{wt}(u; q, t^q)\, \mathrm{wt}(a; q, t)\, \mathrm{wt}(b; q, t^{q^{k+1}})$$

where $u \in \mathfrak{S}_{n-1}^{(k, n-1-k)}$, $a \in \mathfrak{S}_k^{\alpha'}$, $b \in \mathfrak{S}_{n-1-k}^{\alpha''}$. Thus one can rewrite (8.4) as

$$\begin{aligned} & t^{q^k - 1} \frac{k!_{q,t^q}}{k!_{q,t}} \sum_{u \in \mathfrak{S}_{n-1}^{(k,n-1-k)}} \mathrm{wt}(u; q, t^q) \sum_{a \in \mathfrak{S}_n^{\alpha'}} \mathrm{wt}(a; q, t) \sum_{b \in \mathfrak{S}_n^{\alpha''}} \mathrm{wt}(b; q, t^{q^{k+1}}) \\ &= t^{q^k - 1} \frac{k!_{q,t^q}}{k!_{q,t}} \begin{bmatrix} n-1 \\ k, n-1-k \end{bmatrix}_{q,t^q} \begin{bmatrix} k \\ \alpha' \end{bmatrix}_{q,t} \begin{bmatrix} n-1-k \\ \alpha'' \end{bmatrix}_{q,t^{q^{k+1}}} \end{aligned}$$

$$= t^{q^k-1} \frac{(n-1)!_{q,t^q}}{\alpha'!_{q,t}\alpha''!_{q,t^{q^{k+1}}}}$$

$$= t^{q^k-1} \frac{\alpha'!_{q,t^q}}{\alpha'!_{q,t}} \begin{bmatrix} n-1 \\ \alpha', \alpha'' \end{bmatrix}_{q,t^q}$$

in which the first equality replaced all three sums; Proposition 8.4 was used to replace the first sum, while the inductive hypothesis was used to replace the second and third sums. □

9 Ribbon numbers and descent classes

Recall that the minimum-length coset representatives W^α for W/W_α are the permutations w in $W = \mathfrak{S}_n$ whose descent composition $\beta(w)$ is refined by α. The set of permutations w for which $\beta(w) = \alpha$ is sometimes called a *descent class*. We define in terms of these classes the *ribbon*, *q-ribbon*, and *(q, t)-ribbon numbers* for a composition α of n:

$$\begin{aligned} r_\alpha &:= |\{w \in W : \alpha = \beta(w)\}|, \\ r_\alpha(q) &:= \sum_{\substack{w \in W: \\ \alpha = \beta(w)}} q^{\ell(w)}, \\ r_\alpha(q,t) &:= \sum_{\substack{w \in W: \\ \alpha = \beta(w)}} \mathrm{wt}(w; q, t). \end{aligned} \tag{9.1}$$

Recall that the partial order by refinement on the 2^{n-1} compositions α of n is isomorphic to the partial order by inclusion of their subsets of partial sums

$$\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\}.$$

From (8.1) and Theorem 8.6 it should be clear that there is an inclusion-exclusion relation between these three kinds of the ribbon numbers and three kinds of multinomials (ordinary, q-, and (q, t)-multinomials).

However, it turns out that the inclusion-exclusion formula for the ribbons collates into a determinantal formula involving factorials. This determinant for ribbon numbers goes back to MacMahon, for q-ribbon numbers to Stanley (see [19, Examples 2.2.5]), and for (q, t)-ribbon numbers is new, although all three are proven in the same way; see Stanley [19, Examples 2.2.4, 2.2.5]).

Proposition 9.1 *For any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n, with partial sums $\sigma_i := \sum_{j=1}^{i} \alpha_j$, one has*

$$r_\alpha = \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \binom{n}{\beta} = n! \det\left(\frac{1}{(\sigma_j - \sigma_{i-1})!}\right)_{i,j=1}^{\ell(\alpha)}$$

$$r_\alpha(q) = \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \begin{bmatrix} n \\ \beta \end{bmatrix}_q = [n]!_q \det\left(\frac{1}{[\sigma_j - \sigma_{i-1}]!_q}\right)_{i,j=1}^{\ell(\alpha)}$$

$$r_\alpha(q,t) = \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \begin{bmatrix} n \\ \beta \end{bmatrix}_{q,t} = n!_{q,t} \det\left(\varphi^{\sigma_{i-1}} \frac{1}{(\sigma_j - \sigma_{i-1})!_{q,t}}\right)_{i,j=1}^{\ell(\alpha)}$$

where $[m]!_q := 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{m-1})$.

By the definition (9.1), it is clear that r_α is nonnegative, that $r_\alpha(q)$ is a polynomial in q with nonnegative coefficients, and that for integers $q \geq 2$ one will have $r_\alpha(q,t)$ a polynomial in t with nonnegative coefficients. It should also be clear that

$$\lim_{q\to 1} r_\alpha(q) = r_\alpha$$

$$\lim_{t\to 1} r_\alpha(q,t) = r_\alpha(q)$$

$$\lim_{q\to 1} r_\alpha(q, t^{\frac{1}{q-1}}) = r_\alpha(t).$$

Our goal in the next section will be to interpret these three ribbon numbers homologically.

10 Homological interpretation of ribbon numbers

The ribbon number r_α has a well-known interpretation as the rank of the only non-vanishing homology group in the α-rank-selected subcomplex $\Delta(W,S)_\alpha$ of the *Coxeter complex* $\Delta(W,S)$ for $W = \mathfrak{S}_n$. For prime powers q, a result of Björner [2, Theorem 4.1] analogously shows that $r_\alpha(q)$ is the rank of the homology in the α-rank-selected subcomplex of the *Tits building* $\Delta(G,B)$ for $G = GL_n(\mathbb{F}_q)$.

Here we use Björner's results to give, in parallel, Hilbert series interpretations for $r_\alpha(t), r_\alpha(q,t)$. These interpretations will be related to graded modules of Hom spaces between the homology representations on $\Delta(W,S)_\alpha$ or $\Delta(G,B)_\alpha$ and appropriate polynomial rings. This generalizes work of Kuhn and Mitchell [8], who dealt with the case where $\alpha = (1,1,\ldots,1) =: 1^n$, in order to determine the (graded) composition multiplicities of the *Steinberg character* of G within the polynomial ring $\mathbb{F}_q[\mathbf{x}]$.

Definition 10.1 Let $W := \mathfrak{S}_n$ and $G := GL_n(\mathbb{F}_q)$. Given a composition α of n, define the virtual sum of induced $\mathbb{Z}W$-modules

$$\chi^\alpha := \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} 1_{W_\beta}^W \tag{10.1}$$

and $\mathbb{F}_q G$-modules

$$\chi_q^\alpha := \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} 1_{P_\beta}^G. \tag{10.2}$$

These virtual modules have been considered by Björner, Bromwich, Curtis, Mathas, Smith, Solomon, Surowski, and others; see [2], [11] and [16] for some of the relevant references. In the special case where $\alpha = 1^n$ is a single column with n cells, χ^α is the *sign* representation of W, and χ_q^α is the *Steinberg representation* of G.

For any composition α of n, these virtual modules χ^α and χ_q^α turn out to be *genuine* $\mathbb{Z}W$ and $\mathbb{Z}G$-modules. They can be defined over the integers because they are the representations on the *top* homology of the (shellable) simplicial complexes $\Delta(W,S)_\alpha$ and $\Delta(G,B)_\alpha$, which are the *rank-selection* (or *type-selection*) of the Tits building $\Delta(G,B)$ to the rank set given by the partial sums $\{\sigma_s\}_{s=1,\dots,\ell-1}$; see [2, §4]. Note that top-dimensional homology groups are always free as $\mathbb{Z}$-modules because they are the group of top-dimensional *cycles*; there are no boundaries to mod out.

In what follows, we will make several arguments about why certain algebraic complexes

$$\cdots \to C_{i+1} \stackrel{d_{i+1}}{\to} C_i \stackrel{d_i}{\to} C_{i-1} \to \cdots$$

are not only acyclic, but actually *chain-contractible*, that is, there exist maps backward $C_{i+1} \stackrel{D_i}{\leftarrow} C_i$ for each i with the property that

$$D_{i-1}d_i + d_{i+1}D_i = 1_{C_i}.$$

We will use repeatedly the following key fact.

Proposition 10.2 *If one applies an additive functor to a chain-contractible complex, the result remains chain-contractible.*

Proof If the complex is called $(\mathcal{C}, d_*)$ and the functor called F, then the maps $F(D_i)$ provide a chain-contraction for $(F(\mathcal{C}), F(d_*))$: additivity and functoriality imply

$$F(D_{i-1})F(d_i) + F(d_{i+1})F(D_i) = 1_{F(C_i)}$$

if F is covariant, and a similar statement if F is contravariant. □

The following key fact was proven by Kuhn and Mitchell for $\alpha = 1^n$; we simply repeat their proof for general α.

Theorem 10.3 *Given a composition α of n, the simplicial chain complex for the type-selection $\Delta(W,S)_\alpha$ or $\Delta(G,B)_\alpha$ gives rise to chain-contractible complexes of $\mathbb{Z}W$ or $\mathbb{F}_q G$-modules*

$$\begin{aligned} 0 &\to \chi^\alpha \to \mathcal{C} \text{ or} \\ 0 &\to \chi_q^\alpha \to \mathcal{C} \end{aligned} \tag{10.3}$$

where the typical term in $\mathcal{C}$ takes the form

$$\bigoplus_{\substack{\beta \text{ refined by } \alpha: \\ \ell(\beta)=k}} 1_{W_\beta}^{W} \quad \textit{or}$$

$$\bigoplus_{\substack{\beta \text{ refined by } \alpha: \\ \ell(\beta)=k}} 1_{P_\beta}^{G}.$$

Proof We give the proof for the case of the Tits building $\Delta(G, B)$; the "$q = 1$ case" for $\Delta(W, S)$ is even easier.

First note that χ_q^α includes in the first (top) chain group as the kernel of the top boundary map, setting up the complex of $\mathbb{Z}G$-modules in (10.3). It remains to prove that it is chain-contractible after tensoring with $\mathbb{F}_q$.

Björner [2], Kuhn and Mitchell [8], and Smith [16] have observed that the shelling order which one uses for the Tits building (or any of its rank-selections) can actually be chosen B-equivariant: one can shell the facets bwP_α in any order that respects the ordering by length of the minimal coset representative $w \in W/W_\alpha$, and the B-action never alters this representative w. This means that the resulting chain-contraction maps can be chosen as $\mathbb{Z}B$-module maps.

Since $[G : B]$ is coprime to the prime p (= the characteristic of $\mathbb{F}_q$), if one tensors the coefficients with the localization $\mathbb{Z}_{(p)}$ at the prime p (i.e. inverting all elements of $\mathbb{Z}$ coprime to p), one can start with these $\mathbb{Z}_{(p)}B$-module maps, and average them over the cosets G/B to obtain $\mathbb{Z}_{(p)}G$-module maps that still give a chain-contraction.

Lastly, one can tensor the coefficients with $\mathbb{F}_q$ and obtain the desired $\mathbb{F}_qG$-module chain-contraction. □

Given an $\mathbb{F}_qG$-module ψ, one can regard the $\mathbb{F}_q$-vector space $\mathrm{Hom}_{\mathbb{F}_qG}(\psi, \mathbb{F}_q[\mathbf{x}])$ as an $\mathbb{F}_q[\mathbf{x}]^G$-module: given f in $\mathbb{F}_q[\mathbf{x}]^G$, and a G-equivariant map $h : \psi \to \mathbb{F}_q[\mathbf{x}]$, the map fh that sends $u \in \psi$ to $f \cdot h(u)$ is also G-equivariant.

We come to the main result of this section, whose assertion for $\mathfrak{S}_n$-representations is known in characteristic zero; see the extended Remark 10.5 below.

Theorem 10.4 *Given a composition α of n, the $\mathbb{Z}[\mathbf{x}]^W$-module*

$$M := \mathrm{Hom}_{\mathbb{Z}W}(\chi^\alpha, \mathbb{Z}[\mathbf{x}])$$

is free over $\mathbb{Z}[\mathbf{x}]^W$, with

$$\mathrm{Hilb}(M/\mathbb{Z}[\mathbf{x}]_+^W M, t) = r_\alpha(t).$$

Analogously, for q a prime power, the $\mathbb{F}_q[\mathbf{x}]^G$-module

$$M := \mathrm{Hom}_{\mathbb{F}_qG}(\chi_q^\alpha, \mathbb{F}_q[\mathbf{x}])$$

is free over $\mathbb{F}_q[\mathbf{x}]^G$, with

$$\mathrm{Hilb}(M/\mathbb{F}_q[\mathbf{x}]_+^G M, t) = r_\alpha(q, t).$$

Proof As with the previous theorem, we give the proof only for the assertions about G; the proof for the assertions about W are analogous and easier.

Start with the chain-contractible $\mathbb{F}_q GL_n$-complex from Theorem 10.3. Applying the functor $\mathrm{Hom}_{\mathbb{F}_q GL_n}(-, \mathbb{F}_q[\mathbf{x}])$ to this, one obtains (via Proposition 10.2) a chain-contractible complex of $\mathbb{F}_q[\mathbf{x}]^G$-modules that looks like

$$\mathcal{C}' \to M \to 0$$

and where the typical term in $\mathcal{C}'$ is a direct sum of terms of the form

$$\mathrm{Hom}_{\mathbb{F}_q G}(1_{P_\beta}^G, \mathbb{F}_q[\mathbf{x}]) \cong \mathbb{F}_q[\mathbf{x}]^{P_\beta}.$$

Since every ring $\mathbb{F}_q[\mathbf{x}]^{P_\beta}$ is a free $\mathbb{F}_q[\mathbf{x}]^G$-module by Corollary 7.2, this is actually a free $\mathbb{F}_q[\mathbf{x}]^G$-resolution of M. Thus it can be used to compute $\mathrm{Tor}^{\mathbb{F}_q[\mathbf{x}]^G}(M, \mathbb{F}_q)$: tensoring $\mathcal{C}'$ over $\mathbb{F}_q[\mathbf{x}]^G$ with $\mathbb{F}_q$ gives (via Proposition 10.2) a chain-contractible complex $\mathcal{C}''$ of $\mathbb{F}_q$-vector spaces, whose homology computes this Tor. But since the complex $\mathcal{C}''$ is chain-contractible, $\mathrm{Tor}_i^{\mathbb{F}_q[\mathbf{x}]^G}(M, \mathbb{F}_q)$ vanishes for $i > 0$, that is, M is a free $\mathbb{F}_q[\mathbf{x}]^G$-module, giving the first assertion of the theorem.

For the second assertion, note that the resolution $\mathcal{C}'$ of the $\mathbb{F}_q[\mathbf{x}]^G$-module M shows

$$\mathrm{Hilb}(M, t) = \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^{P_\beta}, t).$$

Since M and every one of the $\mathbb{F}_q[\mathbf{x}]^{P_\beta}$'s are all free as $\mathbb{F}_q[\mathbf{x}]^G$-modules by Corollary 7.2, one can divide both sides by $\mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^G, t)$ to obtain

$$\begin{aligned} \mathrm{Hilb}(M/\mathbb{F}_q[\mathbf{x}]^G_+ M, t) &= \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \frac{\mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^{P_\beta}, t)}{\mathrm{Hilb}(\mathbb{F}_q[\mathbf{x}]^G, t)} \\ &= \sum_{\beta \text{ refined by } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \begin{bmatrix} n \\ \beta \end{bmatrix}_{q,t} \\ &= r_\alpha(q, t). \ \square \end{aligned}$$

□

Remark 10.5 We sketch here how the assertion in Theorem 10.4 for $W = \mathfrak{S}_n$ follows from known results in the literature, when one considers $\mathbb{C}W$-modules rather than $\mathbb{Z}W$-modules; see Roichman [15] for generalizations and more recent viewpoints on some of these results.

It is known from work of Hochster and Eagon (see [18, Theorem 3.10]) that for *any* $\mathbb{C}W$-module χ, the Hom-space $M^\chi := \mathrm{Hom}_{\mathbb{C}W}(\chi, \mathbb{C}[\mathbf{x}])$ is always free as a $\mathbb{C}[\mathbf{x}]^W$-module. One can compute its Hilbert series via a *Molien series* calculation

[18, Theorem 2.1] as

$$\begin{aligned}\mathrm{Hilb}(M^\chi,t) &= \tfrac{1}{n!}\textstyle\sum_{w\in W=\mathfrak{S}_n}\frac{\chi(w)}{\det(1-tw)}\\ &= \tfrac{1}{n!}\textstyle\sum_{w\in\mathfrak{S}_n}\chi(w)\cdot p_{\lambda(w)}(1,t,t^2,\ldots)\\ &= s_\chi(1,t,t^2,\ldots).\end{aligned} \tag{10.4}$$

Here $\lambda(w)$ denotes the partition of n that gives w's cycle type, $p_\lambda(x_1,x_2,\ldots)$ is the *power sum* symmetric function corresponding to λ, and $s_\chi(x_1,x_2,\ldots)$ is the symmetric function which is the image of the character χ under the *Frobenius characteristic map* from $\mathfrak{S}_n$-characters to symmetric functions. Thus one has

$$\begin{aligned}\mathrm{Hilb}(M^\chi/\mathbb{C}[\mathbf{x}]^W_+ M^\chi,t) &= \frac{\mathrm{Hilb}(M^\chi,t)}{\mathrm{Hilb}(\mathbb{C}[\mathbf{x}]^W,t)}\\ &= (t;t)_n\, s_\chi(1,t,t^2,\ldots).\end{aligned}$$

When χ is a *skew-character* $\chi^{\lambda/\mu}$ of $\mathfrak{S}_n$, then $s_\chi = s_{\lambda/\mu}$ is a skew Schur function, and one has the explicit formula [20, Proposition 7.19.11]

$$(t;t)_n\, s_{\lambda/\mu}(1,t,t^2,\ldots) = f^{\lambda/\mu}(t) := \sum_Q t^{\mathrm{maj}(Q)} \tag{10.5}$$

where Q runs over standard Young tableaux of shape λ/μ, and $\mathrm{maj}(Q)$ is the sum of the entries i in the *descent set* defined by

$$\mathrm{Des}(Q) := \{i\in\{1,2,\ldots,n-1\} : i+1 \text{ appears in a lower row of } Q \text{ than } i\}.$$

Given a composition $\alpha=(\alpha_1,\ldots,\alpha_\ell)$, the $\mathbb{C}W$-module χ^α on the top homology of the subcomplex $\Delta(W,S)_\alpha$ turns out to be the skew-character $\chi^{\lambda/\mu}$ for the ribbon skew shape λ/μ having α_i cells in its i^{th} lowest row: Solomon [17] used the Hopf trace formula to express the homology representation χ^α as the virtual sum in (10.1), and this can then be re-intepreted as the *Jacobi-Trudi* formula for the skew-character of this ribbon skew shape. Consequently, if $M := M^{\chi^\alpha}$ then

$$\mathrm{Hilb}(M/\mathbb{C}[\mathbf{x}]^W_+ M,t) = \sum_{Q:\lambda(Q)=\alpha} t^{\mathrm{maj}(Q)} = \sum_{\substack{w\in\mathfrak{S}_n:\\ \beta(w)=\alpha}} t^{\mathrm{maj}(w)} = \sum_{\substack{w\in\mathfrak{S}_n:\\ \beta(w)=\alpha}} t^{\ell(w)} = r_\alpha(t).$$

Here the second equality uses a well-known bijection that reads a standard tableaux Q filling the ribbon shape and associates to it a permutation w in W having descent composition $\beta(w)=\alpha$, while the third equality is a well-known result of MacMahon (see e.g., [3]).

11 The coincidence in the case of hooks

As mentioned in the Introduction, there is an important coincidence that occurs in the special case of the principal specialization of $\mathbf{S}_\lambda$ when λ is a *hook shape* $(m,1^k)$, leading to a relation with the (q,t)-ribbon number for the reverse hook composition $\alpha = (1^k, m)$.

We begin with a simplification in the product formula for the principal specialization when λ is a hook.

Proposition 11.1 *For* $n \geq k$,

$$\mathbf{S}_{(m,1^k)}(1,t,\cdots,t^n) = \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \prod_{i=1}^{k} \frac{t^{q^{m+k}} - t^{q^i}}{t^{q^i} - t} = \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \mathbf{S}_{(m,1^k)}(1,t,\cdots,t^k).$$

Proof The second equation is a consequence of the first. The first equation is a straightforward consequence of (6.1), deduced similarly to the proof of Theorem 6.2 and equation (6.7). □

This implies the following relation, that we will use below for an induction.

Corollary 11.2

$$\mathbf{S}_{(m,1^k)}(1,t,t^2,\dots,t^n) + \mathbf{S}_{(m+1,1^{k-1})}(1,t,t^2,\dots,t^{n-1}) = \mathbf{H}_m(1,t,t^2,\dots,t^n) \cdot \mathbf{E}_k(1,t,t^2,\dots,t^{n-1})$$

Proof Apply Proposition 11.1 to the left side:

$$\begin{aligned}
&\mathbf{S}_{(m,1^k)}(1,t,t^2,\dots,t^n) + \mathbf{S}_{(m+1,1^{k-1})}(1,t,t^2,\dots,t^{n-1}) \\
&= \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \prod_{i=1}^{k} \frac{t^{q^{m+k}} - t^{q^i}}{t^{q^i} - t} + \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \prod_{i=1}^{k-1} \frac{t^{q^{m+k}} - t^{q^i}}{t^{q^i} - t} \\
&= \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \left(\left(\frac{t^{q^{m+k}} - t^{q^k}}{t^{q^k} - t} + 1 \right) \prod_{i=1}^{k-1} \frac{t^{q^{m+k}} - t^{q^i}}{t^{q^i} - t} \right) \\
&= \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \prod_{i=1}^{k} \frac{t^{q^{m+k}} - t^{q^{i-1}}}{t^{q^i} - t} \\
&= \begin{bmatrix} n+m \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \left(\begin{bmatrix} m+k \\ k \end{bmatrix}_{q,t} \prod_{i=1}^{k} \frac{t^{q^k} - t^{q^{i-1}}}{t^{q^i} - t} \right) \\
&= \begin{bmatrix} n+m \\ n \end{bmatrix}_{q,t} \cdot \begin{bmatrix} n \\ n-k \end{bmatrix}_{q,t} \varphi^{n-k} \prod_{i=1}^{k} \frac{t^{q^k} - t^{q^{i-1}}}{t^{q^i} - t} \\
&= \mathbf{H}_m(1,t,t^2,\dots,t^n) \cdot \mathbf{E}_k(1,t,t^2,\dots,t^{n-1}).
\end{aligned}$$

The last equality used (6.7). □

Theorem 11.3

$$\mathbf{S}_{(m,1^k)}(1,t,t^2,\ldots,t^n) = \begin{bmatrix} m+n \\ n-k \end{bmatrix}_{q,t} r_{(1^k,m)}(q,t^{q^{n-k}}).$$

Proof By the second equation in Proposition 11.1, it suffices to prove this in the case $n=k$, that is,

$$\mathbf{S}_{(m,1^k)}(1,t,t^2,\ldots,t^k) = r_{(1^k,m)}(q,t). \tag{11.1}$$

Let $LHS(m,k)$, $RHS(m,k)$ denote the left, right sides in (11.1). We will show they are equal by induction on k; in the base case $k=0$ both are easily seen to equal 1.

For the inductive step, note that Corollary 11.2 gives the following recurrence on k for $LHS(m,k)$:

$$LHS(m,k) = -LHS(m+1,k-1) + \begin{bmatrix} m+k \\ k \end{bmatrix}_{q,t} LHS(1,k-1).$$

To show $RHS(m,k)$ satisfies the same recurrence, start with the summation expression for $RHS(m,k)$ given in Theorem 10.4:

$$RHS(m,k) = \sum_{\beta \text{ refined by } (1^k,m)} (-1)^{k+1-\ell(\beta)} \begin{bmatrix} m+k \\ \beta \end{bmatrix}_{q,t}.$$

Classify the terms indexed by β in this sum according to whether the composition β ends in a last part strictly larger than m, or equal to m. The former terms correspond to compositions β refined by $(1^{k-1}, m+1)$, and their sum gives rise to the desired first term $-RHS(m+1,k-1)$ in the recurrence. The latter terms correspond, by removing the last part of β of size m, to compositions $\hat{\beta}$ refined by 1^k, and their sum is

$$\begin{aligned} &\sum_{\substack{\beta \text{ refined by } (1^k,m) \\ \text{ending in } m}} (-1)^{k+1-\ell(\beta)} \begin{bmatrix} m+k \\ \beta \end{bmatrix}_{q,t} \\ &= \begin{bmatrix} m+k \\ k \end{bmatrix}_{q,t} \cdot \sum_{\hat{\beta} \text{ refined by } 1^k} (-1)^{k-\ell(\hat{\beta})} \begin{bmatrix} k \\ \hat{\beta} \end{bmatrix}_{q,t} \\ &= \begin{bmatrix} m+k \\ k \end{bmatrix}_{q,t} \cdot RHS(1,k-1), \end{aligned}$$

that is, the desired second term in the recurrence. □

Remark 11.4 Note that equation (11.1), together with Theorem 10.4, gives the principal specialization $\mathbf{S}_{(m,1^k)}(1,t,t^2,\ldots,t^n)$ an algebraic interpretation in the special case $n = k$. However, this generalizes in a straightforward fashion when $n \geq k$, as the same methods that prove Theorem 10.4 can be used to prove the following.

Let $A := \mathbb{F}_q[x_1, \dots, x_{m+n}]$ with its usual action of $G_{m+n} := GL_{m+n}(\mathbb{F}_q)$. Given α a composition of n, consider the induced $\mathbb{F}_q G_{m+n}$-module

$$\chi_q^{\alpha,m+n} := \mathrm{Ind}_{P_{(m,n)}}^{G_{m+n}} \chi_q^{\alpha}$$

where one considers the homology representation χ_q^{α} for $GL_n(\mathbb{F}_q)$ as also a representation for the parabolic subgroup $P_{(m,n)}$, via the homomorphism $P_{(m,n)} \to GL_n(\mathbb{F}_q)$ that ignores all but the lower right $n \times n$ submatrix.

Theorem 11.5 *In the above situation, the $A^{G_{m+n}}$-module*

$$M := \mathrm{Hom}_{\mathbb{F}_q G_{m+n}}(\chi_q^{\alpha,m+n}, A)$$

is free over $A^{G_{m+n}}$, with

$$\mathrm{Hilb}(M/A_+^{G_{m+n}} M, t) = \begin{bmatrix} m+n \\ n-k \end{bmatrix}_{q,t} r_\alpha(q, t^{q^{n-k}}).$$

When $\alpha = (1^k, m)$, the right side above is $\mathbf{S}_{(m,1^k)}(1, t, t^2, \dots, t^n)$, by Theorem 11.3.

12 Questions

12.1 Bases for the quotient rings and Schubert calculus

Is there a simple explicit basis one can write down for $\mathbb{F}_q[\mathbf{x}]^{P_\alpha}/(\mathbb{F}_q[\mathbf{x}]_+^G)$? By analogy to Schubert polynomial theory, it would be desirable to have a basis when $\alpha = 1^n$, containing the basis for any other α as a subset.

12.2 The meaning of the principal specializations

What is the algebraic (representation-theoretic, Hilbert series?) meaning of $\mathbf{S}_{\lambda/\mu}(1, t, \dots, t^n)$, or perhaps just the non-skew special case where $\mu = \varnothing$? Is there an algebraic meaning to the elements $\mathbf{S}_{\lambda/\mu}(x_1, x_2, \dots, x_n)$ lying in $\hat{\mathbb{Q}}[\mathbf{x}]$?

12.3 A (q, t) version of the fake degrees?

The sum appearing in (10.5) is a skew generalization of the usual *fake-degree* polynomial

$$f^\lambda(t) = \sum_Q t^{\mathrm{maj}(Q)} = q^{b(\lambda)} \frac{[n]!_q}{\prod_x [h(x)]_q} \tag{12.1}$$

where $b(\lambda) = \sum_{i \geq 1} \binom{\lambda'_i}{2}$, and $h(x)$ is the *hook length* of the cell x of λ; see [20, Corollary 7.12.5]. The fake degree polynomial $f^\lambda(q)$ has a different meaning when

q is a prime power, giving the dimension of the complex *unipotent representations* of $GL_n(\mathbb{F}_q)$ considered originally by Steinberg [21]; see [5, 13].

Is there a (q,t)-fake degree polynomial $f^{\lambda/\mu}(q,t) \in \hat{\mathbb{Q}}(\mathbf{t})$ having any or all of the following properties (a)-(f)?:

(a) $\lim_{t\to 1} f^{\lambda/\mu}(q,t) = f^{\lambda/\mu}(q)$.
(b) $\lim_{q\to 1} f^{\lambda/\mu}(q, t^{\frac{1}{q-1}}) = f^{\lambda/\mu}(t)$.
(c) Better yet, a summation formula generalizing (10.5) of the form

$$f^{\lambda/\mu}(q,t) = \sum_Q \mathrm{wt}(Q; q, t)$$

where Q runs over all standard Young tableau of shape λ/μ. Here $\mathrm{wt}(Q; q, t)$ should be an element of $\hat{\mathbb{Q}}(\mathbf{t})$ with a product formula that shows it is a polynomial in t with nonnegative coefficients for integers $q \geq 2$, and that

$$\lim_{t\to 1} \mathrm{wt}(Q; q, t) = q^{\mathrm{maj}(Q)} \text{ and } \lim_{q\to 1} \mathrm{wt}(Q; q, t^{\frac{1}{q-1}}) = t^{\mathrm{maj}(Q)}.$$

(Note that property (c) would imply properties (a), (b).)
(d) A Hilbert series interpretation q-analogous to (10.5) of the form

$$\begin{aligned}\mathrm{Hilb}(M,t) &= \frac{f^{\lambda/\mu}(q,t)}{(1-t^{q^n-1})(1-t^{q^n-q})\cdots(1-t^{q^n-q^{n-1}})} \\ &= f^{\lambda/\mu}(q,t)\cdot \mathrm{Hilb}(K[\mathbf{x}]^G, t)\end{aligned} \tag{12.2}$$

Here M should be a graded $K[\mathbf{x}]^G$-module (not necessarily free), where K is some extension field of $\mathbb{F}_q$ and $G = GL_n(\mathbb{F}_q)$ acts in the usual way on $K[\mathbf{x}] := K[x_1, \ldots, x_n]$. Equivalently this would mean that

$$f^{\lambda/\mu}(q,t) = \sum_{i\geq 0} (-1)^i \, \mathrm{Hilb}(\mathrm{Tor}_i^{K[\mathbf{x}]^G}(M, K), t).$$

(e) A reinterpretation of the power series in (12.2) as a principal specialization of some symmetric function in an infinite variable set, generalizing (10.5).
(f) When $\mu = \varnothing$, a product (q,t)-*hook length formula* for $f^{\lambda}(q,t)$ generalizing (12.1).

When $\lambda = (m, 1^k)$ is a hook shape, one can define $f^{\lambda}(q,t) := r_\alpha(q,t)$ for $\alpha = (1^k, m)$. Then our previous results on $r_\alpha(q,t)$ can be loosely re-interpreted as verifying properties (a),(b),(c),(d),(f) (but not, as far as we know, (e)). Here the desired module M is $\mathrm{Hom}_{\mathbb{F}_q G}(\chi_q^\alpha, \mathbb{F}_q[\mathbf{x}])$, where χ_q^α was the homology representation on the α-type-selected subcomplex of the Tits building $\Delta(G, B)$ considered in Section 10. It is known that this homology representation χ_q^α is an integral lift of the complex unipotent character considered by Steinberg in this case.

12.4 An equicharacteristic q-Specht module for $\mathbb{F}_q GL_n(\mathbb{F}_q)$?

The discussion in Section 12.3 and Remark 10.5 perhaps suggests the existence of a generalization of the $\mathbb{F}_q G$-module χ_q^α for $G = GL_n(\mathbb{F}_q)$ from ribbon shapes α to *all* skew shapes λ/μ.

Question 12.1 Can one find a field extension K of $\mathbb{F}_q$ and a KG-module $\chi_q^{\lambda/\mu}$ which is a q-analogue of the skew Specht module $\chi^{\lambda/\mu}$ for $\mathfrak{S}_n$, generalizing the homology representation $\chi_q^\alpha = \chi_q^{\lambda/\mu}$ when α is a ribbon skew shape as in Remark 10.5, and playing the following three roles?

Role 1. Let λ/μ be an arbitrary skew shape. It is known from work of James and Peel [6] that any skew Specht modules $\chi^{\lambda/\mu}$ for $W = \mathfrak{S}_n$ has a characteristic-free *Specht series*, that is, a filtration in which each factor is isomorphic to a *non-skew* Specht module χ^ν, and where the number of factors isomorphic to χ^ν is equal to the Littlewood-Richardson number $c^\nu_{\lambda/\mu} = c^\lambda_{\mu,\nu}$.

Question 12.2 Does the hypothesized q-skew Specht KG-module $\chi_q^{\lambda/\mu}$ from Question 12.1 have a KG-module filtration in which each factor is isomorphic to one of the non-skew q-Specht modules χ_q^ν, and where the number of factors isomorphic to χ_q^ν is equal to the Littlewood-Richardson number $c^\nu_{\lambda/\mu}$?

In particular, this would force $\dim_K\left(\chi_q^{\lambda/\mu}\right) = f^{\lambda/\mu}(q)$, and would answer a question asked by Björner [2, §6, p. 207]. It suggests that perhaps there is a construction of such a χ_q^λ in the spirit of the *cross-characteristic* q-analogue of Specht modules defined by James [5], which also has dimension given by $f^\lambda(q)$.

Role 2. Let λ/μ be an arbitrary skew shape.

Question 12.3 Does the hypothesized module $\chi_q^{\lambda/\mu}$ from Question 12.1 allow one to define the module

$$M := \mathrm{Hom}_{\mathbb{F}_q G}(\chi_q^{\lambda/\mu}, \mathbb{F}_q[\mathbf{x}]) \tag{12.3}$$

giving a definition for the (q,t)-fake degree $f^{\lambda/\mu}(q,t)$ as in part (d) of Section 12.3?

Role 3. Let λ/μ be an arbitary skew shape. It follows from Schur-Weyl duality that one can re-interpret the usual Schur function principal specialization as a Hilbert series in the following way:

$$s_{\lambda/\mu}(1, t, \dots, t^{N-1}) := \mathrm{Hilb}\left(\mathrm{Hom}_{\mathbb{C}W}(\chi^{\lambda/\mu}, V^{\otimes n}), t\right). \tag{12.4}$$

Here $V = \mathbb{C}^N$ is viewed as a graded vector space having basis elements in degrees $(0, 1, 2, \dots, N-1)$, inducing a grading on the n-fold tensor space $V^{\otimes n}$, and $W = \mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting tensor positions.

Question 12.4 Does the hypothesized KG-module $\chi_q^{\lambda/\mu}$ from Question 12.1, playing the role of $\chi^{\lambda/\mu}$ in (12.4), have a hypothesized accompanying graded KG-module $V(N,n,q)$, playing the role of $V^{\otimes n}$ in (12.4), so that

$$\mathbf{S}_{\lambda/\mu}(1,t,\dots,t^{N-1}) := \mathrm{Hilb}\left(\mathrm{Hom}_{KG}(\chi_q^{\lambda/\mu}, V(N,n,q)), t\right)?$$

Acknowledgements The authors thank Peter Webb for helpful conversations, and thank two anonymous referees for their suggestions.

References

1. Benson, D.J.: Polynomial Invariants of Finite Groups. London Mathematical Society Lecture Note Series, vol. 190. Cambridge University Press, Cambridge (1993)
2. Björner, A.: Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings. Adv. Math. **52**(3), 173–212 (1984)
3. Foata, D., Schützenberger, M.-P.: Major index and inversion number of permutations. Math. Nachr. **83**, 143–159 (1978)
4. Hewett, T.J.: Modular invariant theory of parabolic subgroups of $\mathrm{GL}_n(F_q)$ and the associated Steenrod modules. Duke Math. J. **82**(1), 91–102 (1996)
5. James, G.D.: Representations of General Linear Groups. London Mathematical Society Lecture Note Series, vol. 94. Cambridge University Press, Cambridge (1984)
6. James, G.D., Peel, M.H.: Specht series for skew representations of symmetric groups. J. Algebra **56**, 343–364 (1979)
7. Kac, V., Cheung, P.: Quantum Calculus. Universitext. Springer, New York (2002)
8. Kuhn, N., Mitchell, S.: The multiplicity of the Steinberg representation of $\mathrm{GL}_n F_q$ in the symmetric algebra. Proc. Amer. Math. Soc. **96**(1), 1–6 (1986)
9. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. With Contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1995)
10. Macdonald, I.G.: Schur functions: theme and variations. In: Séminaire Lotharingien de Combinatoire, Saint-Nabor, 1992. Publ. Inst. Rech. Math. Av., vol. 498, pp. 5–39. Univ. Louis Pasteur, Strasbourg (1992)
11. Mathas, A.: A q-analogue of the Coxeter complex. J. Algebra **164**(3), 831–848 (1994)
12. Mui, H.: Modular invariant theory and cohomology algebras of symmetric groups. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22**(3), 319–369 (1975)
13. Olsson, J.B.: On the blocks of $GL(n,q)$. I. Trans. Amer. Math. Soc. **222**, 143–156 (1976)
14. Reiner, V., Stanton, D., White, D.: The cyclic sieving phenomenon. J. Combin. Theory Ser. A **108**, 17–50 (2004)
15. Roichman, Y.: On permutation statistics and Hecke algebra characters. In: Combinatorial Methods in Representation Theory, Kyoto, 1998. Adv. Stud. Pure Math., vol. 28, pp. 287–304. Kinokuniya, Tokyo (2000)
16. Smith, S.D.: On decomposition of modular representations from Cohen-Macaulay geometries. J. Algebra **131**(2), 598–625 (1990)
17. Solomon, L.: A decomposition of the group algebra of a finite Coxeter group. J. Algebra **9**, 220–239 (1968)
18. Stanley, R.P.: Invariants of finite groups and their applications to combinatorics. Bull. Amer. Math. Soc. (N.S.) **1**, 475–511 (1979)
19. Stanley, R.P.: Enumerative Combinatorics, Vol. 1. Cambridge Studies in Advanced Mathematics, vol. 49. Cambridge University Press, Cambridge (1997)
20. Stanley, R.P.: Enumerative Combinatorics, Vol. 2. Cambridge Studies in Advanced Mathematics, vol. 62. Cambridge University Press, Cambridge (1999)
21. Steinberg, R.: A geometric approach to the representations of the full linear group over a Galois field. Trans. Amer. Math. Soc. **71**, 274–282 (1951)