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Delaunay Transformations of a Delaunay Polytope*

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Abstract. Let P be a Delaunay polytope in $\mathbb{R}^n$. Let $\mathcal{T}(P)$ denote the set of affine bijections f of $\mathbb{R}^n$ for which $f(P)$ is again a Delaunay polytope. The relation: $f \sim g$ if f, g differ by an orthogonal transformation and/or a translation is an equivalence relation on $\mathcal{T}(P)$. We show that the dimension (in the topological sense) of the quotient set $\mathcal{T}(P)/\sim$ coincides with another parameter of P, namely, with its rank.

Let V denote the set of vertices of P and let d_P denote the distance on V defined by $d_P(u, v) = \|u - v\|^2$ for $u, v \in V$. Assouad [1] shows that d_P belongs to the cone $\mathcal{H}_{|V|} := \{d \mid \sum_{u,v \in V} b_u b_v d(u, v) \leq 0$ for $b \in \mathbb{Z}^V$ with $\sum_{u \in V} b_u = 1\}$. Then, the rank of P is defined as the dimension of the smallest face of the cone $\mathcal{H}_{|V|}$ that contains d_P.

Keywords: Delaunay polytope, affine transformation, lattice, dimension, hypermetric

1. Introduction

This paper is motivated by a question of Billera (private communication, 1994), who asked whether the notion of rank of a Delaunay polytope P, which is defined in [3] in terms of a certain cone $\mathcal{H}_{|V|}$, can be expressed in a more intrinsic way as an invariant of a set of transformations of P. We give a positive answer to this question. Namely, we show that the rank of P is equal to the dimension (in the topological sense) of the set consisting of the affine bijections f (up to Euclidian motions) for which $f(P)$ is again a Delaunay polytope.

In this result, we use the notion of dimension of a topological space. This notion was defined at the beginning of the twentieth century, in particular, after the works of Brouwer, Menger, Urysohn; see, e.g., [5].

> Namely, for a topological space X, its dimension $\dim(X)$ is defined in the following way. If, for any open sets G_i $(1 \leq i \leq s)$ such that $X = \bigcup_{1 \leq i \leq s} G_i$, there exist open sets H_j $(1 \leq j \leq t)$ such that $X = \bigcup_{1 \leq j \leq t} H_j$, each H_j is contained in some G_i, and the intersection of any $n + 2$ H_j's is empty, then $\dim(X) \leq n$. If $\dim(X) \leq n$ but not $\dim(X) \leq n - 1$ then $\dim(X) = n$.

This concept generalizes the usual notion of dimension for a Euclidian space or a polyhedron. We do not need, however, to know the precise definition of this notion of dimension. We will only use the fact that the dimension is a topological invariant, i.e., that two homeomorphic topological spaces have the same dimension.

We recall the definitions for a Delaunay polytope and its rank in Sections 1.1 and 1.2.

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1.1. Delaunay transformations

Let P be an n-dimensional polytope in $\mathbb{R}^n$ with set of vertices V. Then, P is said to be a *Delaunay polytope* if the following conditions hold:

- The set $L := \{\sum_{v \in V} b_v v \mid b \in \mathbb{Z}^V$ and $\sum_{v \in V} b_v = 1\}$ is a lattice (i.e., there exists a nonempty ball centered at each lattice point that contains no other lattice point).
- There exists a sphere S with center c and radius r such that

$$\|x - c\| \geq r \quad \text{for all } x \in L, \tag{1.1}$$

with equality in (1.1) if and only if x is a vertex of P.

In other words, no lattice point lies in the interior of the ball whose boundary sphere is S and the lattice points lying on S are precisely the vertices of P. In particular, P is inscribed on the sphere S. (Here, $\|x\| = \sqrt{x^T x}$ denotes the Euclidian norm of $x \in \mathbb{R}^n$.)

Delaunay polytopes were introduced by Voronoi [6, 7] (they are also called *L-polytopes* in the literature). They are closely related to the well known Voronoi polytopes. Namely, the vertices of the Voronoi polytope at a lattice point u are precisely the centers of the Delaunay polytopes that have u as a vertex.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine bijection. In general, $f(P)$ is not a Delaunay polytope. For instance, an equilateral triangle is a Delaunay polytope while a triangle with a right angle is *not* a Delaunay polytope; in fact, a triangle is a Delaunay polytope if and only if it has no obtuse angle. We call f a *Delaunay transformation* of P if $f(P)$ is a Delaunay polytope and we let $\mathcal{T}(P)$ denote the set of all Delaunay transformations of P. Observe that all translations, orthogonal transformations, and homotheties are Delaunay transformations of P. Given two affine bijections f, g of $\mathbb{R}^n$, write

$$f \sim g \tag{1.2}$$

if f, g differ only by an orthogonal transformation or a translation, i.e., if there exist an orthogonal transformation h of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$ such that $g(x) = h(f(x)) + a$ for all $x \in \mathbb{R}^n$. The relation $\sim$ is an equivalence relation on $\mathcal{T}(P)$. Let $\mathcal{T}(P)/\sim$ denote the quotient space of $\mathcal{T}(P)$ by $\sim$.

Our goal in this paper is to evaluate the dimension (in the topological sense) of the set $\mathcal{T}(P)/\sim$. In fact, the set $\mathcal{T}(P)/\sim$ can be more simply defined in terms of matrices.

Clearly, we can suppose that the origin is a vertex of P (else, replace P by a translate of it). Then, every equivalence class of $\mathcal{T}(P)/\sim$ contains a representative f, which maps the origin on the origin. Hence, f can be identified with the nonsingular matrix A, which represents f in the canonical basis of $\mathbb{R}^n$. Given two $n \times n$ matrices A, B, write

$$A \sim B \quad \text{if } A^T A = B^T B. \tag{1.3}$$

When restricted to the set $GL(n)$ of the $n \times n$ nonsingular matrices, the definition of the relation $\sim$ from (1.3) is coherent with the one given in (1.2). Namely, for $A, B \in GL(n)$, $A \sim B$ if $A = UB$ for some orthogonal matrix U. Set

$$\mathcal{T}_0(P) := \{A \in GL(n) \mid A(P) \text{ is a Delaunay polytope}\}, \tag{1.4}$$

where $A(P) := \{Ax \mid x \in P\}$ denotes the image of P under A. From the discussion above, it follows that

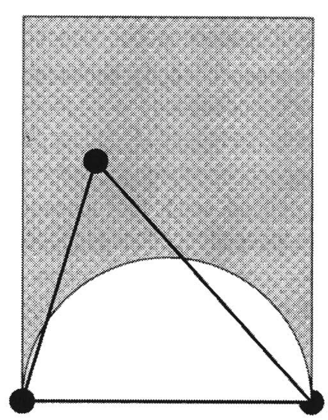
$$\mathcal{T}(P)/\!\sim\, = \mathcal{T}_0(P)/\!\sim . \tag{1.5}$$

As an example, we describe below the Delaunay transformations of the triangle and of the cube.

Example 1.6 Consider the triangle α_2 with vertices $v_0 = (0,0)$, $v_1 = (2,0)$ and $v_3 = (1,2)$; it is a Delaunay polytope. Every class of the set $\mathcal{T}(\alpha_2)/\sim$ admits a representative of the form $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where a, b, c satisfy:

$$\begin{cases} a > 0, \\ 0 < a + 2b < 2a, \\ 4b^2 + 4c^2 > a^2. \end{cases}$$

Indeed, up to rotation, every Delaunay transformation A of α_2 can be supposed to leave the x-axis invariant and, hence, has the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. The conditions on a, b, c express the fact that each angle of the triangle $A(\alpha_2)$ is acute. Geometrically, this means that the point $A(v_2)$ should lie in the shaded region shown in the figure below.



This shows that there are three degrees of freedom for the parameters of a Delaunay transformation (up to orthogonal transformation) of α_2; in other words, the set $\mathcal{T}(\alpha_2)/\sim$ has dimension 3. More generally, an easy induction shows that, for the n-dimensional simplex α_n, $\mathcal{T}(\alpha_n)/\sim$ has dimension $\binom{n+1}{2}$.

Example 1.7 Consider now the square $\gamma_2 = [0,1]^2$. It is easy to see that each class of $\mathcal{T}(\gamma_2)/\sim$ has a representative of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b > 0$. Hence, the set $\mathcal{T}(\gamma_2)/\sim$ has dimension 2. More generally, for the n-dimensional cube γ_n, $\mathcal{T}(\gamma_n)/\sim$ has dimension n.

1.2. Rank of a Delaunay polytope

Let P be a Delaunay polytope with set of vertices V. We consider the cone $\mathcal{H}_{|V|}$ in the space $\mathbb{R}^{\binom{V}{2}}$ (indexed by the pairs of elements of V) defined by the inequalities:

$$\sum_{\substack{u,v\in V\\ u<v}} b_u b_v x_{uv} \leq 0 \tag{1.8}$$

for all $b \in \mathbb{Z}^V$ such that $\sum_{u\in V} b_u = 1$. (We suppose that the elements of V are ordered and $u < v$ refers to that order.) We use the notation $\mathcal{H}_{|V|}$ as the cone $\mathcal{H}_{|V|}$ depends only on $|V|$. The cone $\mathcal{H}_{|V|}$ is known as the *hypermetric cone*. Note that $\mathcal{H}_{|V|}$ is defined by infinitely many inequalities. However, it is shown in [4] that $\mathcal{H}_{|V|}$ is a polyhedral cone, i.e., that a finite subset of the inequalities (1.8) suffices to describe $\mathcal{H}_{|V|}$.

One can observe that one point d_P belonging to the hypermetric cone $\mathcal{H}_{|V|}$ can be constructed from P. Namely, set

$$d_P(u, v) := \|u - v\|^2 \tag{1.9}$$

for $u, v \in V$. Then, the vector $d_P := (d_P(u, v))_{u,v\in V, u<v}$ belongs to $\mathcal{H}_{|V|}$. To see it, take $b \in \mathbb{Z}^V$ such that $\sum_{u\in V} b_u = 1$. Let c and r denote the center and radius of the sphere circumscribing P. Then,

$$\begin{aligned}\sum_{u,v\in V} b_u b_v d_P(u, v) &= \sum_{u,v\in V} b_u b_v \|u - v\|^2\\ &= \sum_{u,v\in V} b_u b_v \|(u - c) - (v - c)\|^2\\ &= \sum_{u,v\in V} b_u b_v (2r^2 - 2(u - c)^T (v - c))\\ &= 2r^2 - 2\left\| \sum_{u\in V} b_u u - c\right\|^2 \leq 0,\end{aligned}$$

where the last inequality follows from (1.1). This property was observed by Assouad [1]. Assouad proved, moreover, that, conversely, every point of the cone $\mathcal{H}_{|V|}$ arises in some sense from a Delaunay polytope. More precisely, let d be an arbitrary point of the cone $\mathcal{H}_{|V|}$. Then, there exists a Delaunay polytope Q with set of vertices W and a mapping $\varphi : V \to W$ such that

$$d(u, v) = \|\varphi(u) - \varphi(v)\|^2$$

for all $u, v \in V$. We refer to [2] for a detailed survey on the connections between Delaunay polytopes and the hypermetric cone.

This leads to the following notion of rank for a Delaunay polytope, introduced in [3].

Definition 1.10 Let P be a Delaunay polytope with set of vertices V and let d_P denote the point of the cone $\mathcal{H}_{|V|}$ defined by (1.9). Then, the *rank* of P is defined as the dimension of the smallest face of $\mathcal{H}_{|V|}$ that contains d_P.

For instance, the n-simplex α_n has rank $\binom{n+1}{2}$ and the n-cube has dimension n (see [3]).

1.3. The main result

The following is the main result of the paper. The proof is given in Section 2.

Theorem 1.11 *Let P be a Delaunay polytope. Then, its rank is equal to the dimension of the quotient set $\mathcal{T}(P)/\sim$ of Delaunay transformations of P.*

Remark 1.12 Note that the dimension of $\mathcal{T}(P)/\sim$ is always greater or equal to 1, as the homotheties are Delaunay transformations of any Delaunay polytope. It is shown in [3] that P has rank 1 if and only if the homotheties are the only Delaunay transformations of P (up to translations and orthogonal transformations), i.e., if the dimension of $\mathcal{T}(P)/\sim$ is equal to 1. Hence, Theorem 1.11 holds for rank one Delaunay polytopes. Several examples of rank one Delaunay polytopes are described in [3].

2. Proof of Theorem 1.11

In what follows, P denotes an n-dimensional Delaunay polytope in $\mathbb{R}^n$ with set of vertices V and admitting the origin as a vertex. As the hypermetric cone $\mathcal{H}_{|V|}$ is a polyhedral cone, there exists a finite set $\mathcal{B}_P \subset \{b \in \mathbb{Z}^V \mid \sum_{u\in V} b_u = 1\}$ such that

$$\mathcal{H}_{|V|} = \left\{x \in \mathbb{R}^{\binom{V}{2}} \mid \sum_{\substack{u,v\in V\\ u<v}} b_u b_v x_{uv} \leq 0 \quad \text{for all } b \in \mathcal{B}_P\right\}. \tag{2.1}$$

Let d_P denote the point of $\mathcal{H}_{|V|}$ defined by (1.9). Let F_P denote the smallest face of the cone $\mathcal{H}_{|V|}$ that contains d_P. Then, F_P is defined by

$$F_P = \left\{x \in \mathcal{H}_{|V|} \mid \sum_{\substack{u,v\in V\\ u<v}} b_u b_v x_{uv} = 0 \quad \text{for all } b \in \mathcal{A}_P\right\} \tag{2.2}$$

for some subset $\mathcal{A}_P \subseteq \mathcal{B}_P$.

2.1. A characterization of the Delaunay transformations of P

We start with an easy result of linear algebra. We use the following notation: For two $n \times n$ matrices A, B,

$$\langle A, B\rangle := \sum_{1\leq i,j\leq n} a_{ij} b_{ij}$$

denotes the usual scalar product, with an $n \times n$ matrix being viewed as an n^2-vector. Recall the identity: $x^T A x = \langle A, xx^T\rangle$ for an $n \times n$ matrix A and $x \in \mathbb{R}^n$.

Lemma 2.3 *Let $x_1, \ldots, x_n$ be n linearly independent vectors in $\mathbb{R}^n$. Then, the system $\mathcal{S} = \{x_i x_i^T (1 \leq i \leq n),\ (x_i - x_j)(x_i - x_j)^T (1 \leq i < j \leq n)\}$ is linearly independent.*

Proof: As $\mathcal{S}$ consists of $n + \binom{n}{2} = \binom{n+1}{2}$ elements, it suffices to show that, if X is a symmetric $n \times n$ matrix orthogonal to all members of $\mathcal{S}$, then X is the zero matrix. By assumption, $\langle X, x_i x_i^T \rangle = x_i^T X x_i = 0$ for $i = 1, \ldots, n$, and $\langle X, (x_i - x_j)(x_i - x_j)^T \rangle = (x_i - x_j)^T X (x_i - x_j) = 0$, implying that $x_i^T X x_j + x_j^T X x_i = 0$ for $1 \le i < j \le n$. We check that $x^T X x = 0$ for all $x \in \mathbb{R}^n$. Indeed, let $x = \sum_{1 \le i \le n} \alpha_i x_i$ for some scalars α_i. Then, $x^T X x = \sum_{1 \le i \le n} \alpha_i^2 x_i^T X x_i + \sum_{1 \le i < j \le n} \alpha_i \alpha_j (x_i^T X x_j + x_j^T X x_i) = 0$. This implies that $X = 0$; indeed, if x is an eigenvector of X for the eigenvalue λ, then $0 = x^T X x = \lambda \|x\|^2$, yielding $\lambda = 0$. □

The following result of [3] plays a crucial role in the proof, as it will enable us to characterize the Delaunay transformations of P in Theorem 2.6.

Theorem 2.4 [3] *Let P be an n-dimensional Delaunay polytope in $\mathbb{R}^n$ with set of vertices V and such that $0 \in V$. Let F_P denote the smallest face of the cone $\mathcal{H}_{|V|}$ containing d_P.*

(i) *Let $A \in GL(n)$. If $A(P)$ is a Delaunay polytope, then the vector $d_{A(P)} \in \mathbb{R}^{\binom{V}{2}}$ defined by*

$$d_{A(P)}(u, v) = \|Au - Av\|^2 \tag{2.5}$$

for all $u, v \in V$, belongs to the relative interior of F_P.

(ii) *Let d be a vector that lies in the relative interior of F_P. Then, there exists $A \in GL(n)$ such that $A(P)$ is a Delaunay polytope and d coincides with the point $d_{A(P)}$ defined by* (2.5).

Theorem 2.6 *Let P be an n-dimensional Delaunay polytope in $\mathbb{R}^n$ having the origin as a vertex and let F_P denote the smallest face of $\mathcal{H}_{|V|}$ containing d_P. Let $A \in GL(n)$. Then, $A(P)$ is a Delaunay polytope if and only if the vector $d_{A(P)}$ defined by* (2.5) *lies in the relative interior of F_P.*

Proof: Necessity follows from Theorem 2.4 (i). Conversely, suppose that $d_{A(P)}$ lies in the relative interior of F_P. By Theorem 2.4 (ii), there exists $B \in GL(n)$ such that $B(P)$ is a Delaunay polytope and $d_{A(P)} = d_{B(P)}$. Then, $(u - v)^T A^T A(u - v) = (u - v)^T B^T B(u - v)$ for all $u, v \in V$. As V has full dimension n, we deduce from Lemma 2.3 that $A^T A = B^T B$. Hence, $(BA^{-1})^T (BA^{-1}) = I$, i.e., BA^{-1} is an orthogonal matrix. This shows that the polytope $A(P)$ can be obtained from $B(P)$ by applying an orthogonal transformation. Therefore, $A(P)$ is a Delaunay polytope. □

Corollary 2.7 *Let P be an n-dimensional Delaunay polytope in $\mathbb{R}^n$ with set of vertices V and such that $0 \in V$. Let $A \in GL(n)$. Then, $A(P)$ is a Delaunay polytope if and only if the following holds:*

$$\sum_{u,v \in V} b_u b_v \|Au - Av\|^2 = 0 \quad \textit{for all } b \in \mathcal{A}_P,$$

$$\sum_{u,v \in V} b_u b_v \|Au - Av\|^2 < 0 \quad \textit{for all } b \in \mathcal{B}_P \setminus \mathcal{A}_P$$

(where the sets $\mathcal{A}_P$ and $\mathcal{B}_P$ define F_P as in relation (2.2)).

We conclude with an auxiliary result that will be needed in the next section.

Lemma 2.8 *Let P be an n-dimensional Delaunay polytope in $\mathbb{R}^n$ with set of vertices V and such that $0 \in V$. Let $A_1, \dots, A_k \in GL(n)$ and, for $h = 1, \dots, k$, let d_h be defined by $d_h(u, v) = \|A_h u - A_h v\|^2$ for $u, v \in V$. The following assertions are equivalent.*

(i) *$d_1, \dots, d_k$ are linearly independent.*
(ii) *$A_1^T A_1, \dots, A_k^T A_k$ are linearly independent.*

Proof: (i) $\Rightarrow$ (ii) Suppose that $\sum_{1 \le h \le k} \alpha_h A_h^T A_h = 0$ for some scalars α_h's. Then, $(u - v)^T (\sum_{1 \le h \le k} \alpha_h A_h^T A_h)(u - v) = 0$, i.e., $\sum_{1 \le h \le k} \alpha_h d_h(u, v) = 0$ for all $u, v \in V$. Hence, $\sum_{1 \le h \le k} \alpha_h d_h = 0$, implying that $\alpha_h = 0$ for all h.
(ii) $\Rightarrow$ (i) Suppose that $\sum_{1 \le h \le k} \alpha_h d_h = 0$. Then, $(u - v)^T (\sum_{1 \le h \le k} \alpha_h A_h^T A_h)(u - v) = 0$ for all $u, v \in V$. As V is full dimensional and contains the origin, we deduce from Lemma 2.3 that $\sum_{1 \le h \le k} \alpha_h A_h^T A_h = 0$. Therefore, $\alpha_h = 0$ for all h. □

2.2. *The cone $\mathcal{C}_P$*

By the considerations in Section 2.1, we are led to define the set $\mathcal{C}_P$, which consists of the $n \times n$ symmetric positive semidefinite matrices M that satisfy:

(a) $\sum_{u,v \in V} b_u b_v (u - v)^T M (u - v) = 0$ for all $b \in \mathcal{A}_P$,

(b) $\sum_{u,v \in V} b_u b_v (u - v)^T M (u - v) \le 0$ for all $b \in \mathcal{B}_P \backslash \mathcal{A}_P$.

(Recall the definition of the sets $\mathcal{B}_P$, $\mathcal{A}_P$ from (2.1) and (2.2).) Hence, $\mathcal{C}_P$ is a closed cone, whose relative interior $\mathring{\mathcal{C}}_P$ consists of the symmetric positive definite matrices M that satisfy (a), and satisfy (b) with strict inequalities. As an immediate consequence of Corollary 2.7, the set $\mathcal{T}_0(P)$ defined in (1.4) can be rewritten as

$$\mathcal{T}_0(P) = \{A \in GL(n) \mid A^T A \in \mathring{\mathcal{C}}_P\}. \tag{2.9}$$

We can express the dimension of the cone $\mathcal{C}_P$ in terms of the rank of P. Namely,

Proposition 2.10 *The dimension of the cone $\mathcal{C}_P$ is equal to the rank of P.*

Proof: Let k denote the rank of P and let p denote the dimension of the cone $\mathcal{C}_P$. As the face F_P has dimension k, we can find k linearly independent points $d_1, \dots, d_k$ lying in the relative interior of F_P. By Theorem 2.4 (ii) and Corollary 2.7, there exist $A_1, \dots, A_k \in GL(n)$ such that $A_h^T A_h \in \mathring{\mathcal{C}}_P$ and $d_h(u, v) = \|A_h u - A_h v\|^2$ for $u, v \in V$, $h = 1, \dots, k$. By Lemma 2.8, $A_1^T A_1, \dots, A_k^T A_k$ are linearly independent. This shows that $k \le p$. Now, as the cone $\mathcal{C}_P$ has dimension p, we can find p linearly independent points $M_1, \dots, M_p$ in the relative interior of $\mathcal{C}_P$. Since M_h is positive definite, it has the form $M_h = A_h^T A_h$ for some $A_h \in GL(n)$, for $h = 1, \dots, p$. Then, the point d_h defined from (2.5) using A_h lies in the relative interior of F_P, for $h = 1, \dots, p$. Moreover, $d_1, \dots, d_p$ are linearly independent by Lemma 2.8. This shows that $p \le k$. Hence, we have the equality: $p = k$. □

2.3. The homeomorphism θ

We show here that Theorem 1.11 holds, i.e., that the dimension of the set $\mathcal{T}(P)/\sim$ is equal to the rank of P. Recall from (1.5) that the set $\mathcal{T}(P)/\sim$ coincides with the set $\mathcal{T}_0(P)/\sim$, where $\mathcal{T}_0(P)$ is defined by relations (1.4) or (2.9). Consider the mapping

$$\begin{aligned} \mathcal{M}_n &\rightarrow PSD_n \\ A &\mapsto A^T A, \end{aligned}$$

where $\mathcal{M}_n$ denotes the set of $n \times n$ matrices and PSD_n the set of $n \times n$ positive semidefinite matrices. By definition of the equivalence relation $\sim$ from (1.3), we have a bijection

$$\begin{aligned} \theta : \mathcal{M}_n/\sim &\rightarrow PSD_n \\ \bar{A} \quad &\mapsto A^T A, \end{aligned}$$

where $\bar{A}$ denotes the class of $A \in \mathcal{M}_n$ in the quotient set $\mathcal{M}_n/\sim$.

Lemma 2.11 *θ is a homeomorphism between the sets $\mathcal{M}_n/\sim$ and PSD_n.*

Proof: The mapping θ is clearly continuous. We show that its inverse θ^{-1} is also continuous. For this, we show that the image $\theta(C)$ of any closed set C in $\mathcal{M}_n/\sim$ is a closed set. Indeed, consider a sequence $(A^i)_{i \in \mathbb{N}}$ of matrices of $\mathcal{M}_n$ for which the class $\bar{A}^i$ of A^i in $\mathcal{M}_n/\sim$ belongs to C for all $i \in \mathbb{N}$, and the sequence $((A^i)^T A^i)_{i \in \mathbb{N}}$ is convergent, with limit $M \in PSD_n$. We show that $M \in \theta(C)$. As the sequence $((A^i)^T A^i)_{i \in \mathbb{N}}$ is convergent, this implies easily that all the entries of the matrices A^i ($i \in \mathbb{N}$) are bounded. Hence, we can find a convergent subsequence $(A^{i_j})_{j \in \mathbb{N}}$ of $(A^i)_{i \in \mathbb{N}}$. Denote by $A \in \mathcal{M}_n$ the limit of $(A^{i_j})_{j \in \mathbb{N}}$. Therefore, $\bar{A}$ belongs to C, since $\bar{A}^{i_j} \in C$ for all $j \in \mathbb{N}$ and C is closed. Moreover, the sequence $((A^{i_j})^T A^{i_j})_{j \in \mathbb{N}}$ converges to $A^T A$, from which we deduce that

$$M = A^T A.$$

This shows that $M = \theta(\bar{A})$ belongs to $\theta(C)$. □

Corollary 2.12 *The spaces $\mathcal{T}_0(P)/\sim$ and $\mathring{\mathcal{C}}_P$ are homeomorphic, via the mapping θ.*

Proof: This follows from Lemma 2.11, as the mapping θ is one-to-one between the sets $\mathcal{T}_0(P)/\sim$ and $\mathring{\mathcal{C}}_P$. □

Therefore, both sets $\mathcal{T}_0(P)/\sim$ and $\mathring{\mathcal{C}}_P$ have the same dimension, which is equal to the rank of P, by Proposition 2.10. This concludes the Proof of Theorem 1.11.

We conclude with two remarks. The first one illustrates the difficulties encountered when trying to compute the usual linear dimension of the quotient set $\mathcal{T}(P)/\sim$. The second one shows that, although θ extends to a homeomorphism between the closure of the set $\mathcal{T}(P)/\sim$ and the cone $\mathcal{C}_P$, this yields no further result in terms of Delaunay transformations of P.

Remark 2.13 Quite naturally, one may wonder why we did not try to compute the usual linear dimension of the set $\mathcal{T}(P)/\sim$ (i.e., its maximum number of linearly independent points). It turns out however that this is not a well defined notion as it depends on the choice of the representatives in the equivalence classes of $\mathcal{T}(P)/\sim$.

To see it, consider again the case of the square $\gamma_2 = [0, 1]^2$ from Example 1.7. The matrices $A_1 := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $A_2 := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, and $A_3 := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ are Delaunay transformations of γ_2, which belong to distinct classes in the quotient set $\mathcal{T}(\gamma_2)/\sim$. Clearly, A_3 is an orthogonal matrix, i.e., belongs to the same class as the identity matrix I. Observe now that A_1, A_2, A_3 are linearly independent, while the set $\{A_1, A_2, I\}$ has rank 2. This shows that the rank depends on the representatives we use in each class.

Remark 2.14 The closure $cl(\mathcal{T}_0(P))$ of the set $\mathcal{T}_0(P)$ is defined by

$$cl(\mathcal{T}_0(P)) = \{A \in \mathcal{M}_n \mid A^T A \in \mathcal{C}_P\}.$$

As the mapping θ is one-to-one between the sets $cl(\mathcal{T}_0(P))/\sim$ and $\mathcal{C}_P$, we deduce that these two sets are homeomorphic. Therefore, the dimension of the set $cl(\mathcal{T}_0(P))/\sim$ is also equal to the rank of P. Note, however, that the set $cl(\mathcal{T}_0(P))/\sim$ has no immediate interpretation in terms of Delaunay transformations of P. In particular, the analogue of (1.4) does not hold, i.e., it is not true that, for any $A \in \mathcal{M}_n$,

$$A^T A \in \mathcal{C}_P \;\Leftrightarrow\; A(P) \text{ is a Delaunay polytope.}$$

For instance, take for P the unit square $[0, 1]^2$, with vertices $v_0 = (0, 0)$, $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 1)$, and let $A := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then, $Av_0 = (0, 0)$, $Av_1 = Av_2 = (1, 1)$, $Av_3 = (2, 2)$, and $A(P)$ is the segment $[(0, 0), (2, 2)]$. Hence, $A(P)$ is a Delaunay polytope, but the matrix $A^T A$ does not belong to the cone $\mathcal{C}_P$. Indeed, points of $\mathcal{C}_P$ should satisfy the triangle equality: $x_{v_1v_2} = x_{v_0v_1} + x_{v_0v_2}$ (because d_P satisfies it, as $\|v_1 - v_2\|^2 = \|v_0 - v_1\|^2 + \|v_0 - v_2\|^2$), but $0 = \|Av_1 - Av_2\|^2 \neq \|Av_0 - Av_1\|^2 + \|Av_0 - Av_2\|^2 = 2$. Conversely, if we choose for P an equilateral triangle and for A a transformation of $\mathbb{R}^2$ mapping P on a triangle with a right angle, then $A(P)$ is not a Delaunay polytope, while the matrix $A^T A$ clearly belongs to the cone $\mathcal{C}_P$. (In these examples, we use the fact that, for $|V| \leq 4$, the hypermetric cone $\mathcal{H}_{|V|}$ is defined by the triangle inequalities: $x_{uv} \leq x_{uw} + x_{vw}$ for distinct $u, v, w \in V$.)

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