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On a Family of Hyperplane Arrangements Related to the Affine Weyl Groups

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Abstract. Let Φ be an irreducible crystallographic root system in a Euclidean space V, with Φ^+ the set of positive roots. For $\alpha \in \Phi$, $k \in \mathbf{Z}$, let $H(\alpha, k)$ be the hyperplane $\{v \in V : \langle \alpha, v \rangle = k\}$. We define a set of hyperplanes $\mathcal{H} = \{H(\delta, 1) : \delta \in \Phi^+\} \cup \{H(\delta, 0) : \delta \in \Phi^+\}$. This hyperplane arrangement is significant in the study of the affine Weyl groups. In this paper it is shown that the Poincaré polynomial of $\mathcal{H}$ is $(1 + ht)^n$, where n is the rank of Φ and h is the Coxeter number of the finite Coxeter group corresponding to Φ.

Keywords: hyperplane arrangement, Weyl group, Poincaré polynomial

1. Introduction

Let Φ be an irreducible crystallographic root system in a Euclidean space V, with Φ^+ the set of positive roots. For $\alpha \in \Phi$, $k \in \mathbf{Z}$, let $H(\alpha, k)$ be the hyperplane $\{v \in V : \langle \alpha, v \rangle = k\}$. We define a set of hyperplanes $\mathcal{H} = \{H(\delta, 1) : \delta \in \Phi^+\} \cup \{H(\delta, 0) : \delta \in \Phi^+\}$. We will refer to $\mathcal{H}$ as the sandwich arrangement of hyperplanes associated to Φ. This set of hyperplanes has appeared in at least two areas of the study of the affine Weyl groups: the Kazhdan-Lusztig representation theory as it applies to these groups [7], and the study of the properties of the language of reduced expressions [3]. In [8] Shi proved the following theorem:

Theorem 1.1 *The number of connected components of $V - \bigcup_{H \in \mathcal{H}} H$ is $(h + 1)^n$, where n is the rank of Φ, and h is the Coxeter number of the associated finite Coxeter group.*

The purpose of this paper is, in some sense, to generalize this result by determining the Poincaré polynomial $P(\mathcal{H}, t)$ of $\mathcal{H}$. The number of connected components of $V - \bigcup_{H \in \mathcal{H}} H$, and the number of these components that are bounded, can both be read off easily from $P(\mathcal{H}, t)$. The Poincaré polynomial has other connections to combinatorial and algebraic properties of $\mathcal{H}$; a good reference is [6].

2. The Poincaré polynomial of $\mathcal{H}$

The intersection poset $L(\mathcal{H})$ of $\mathcal{H}$ is the set of nonempty intersections of elements of $\mathcal{H}$, partially ordered by reverse inclusion. This poset is ranked by codimension, with V the unique element having rank 0. Writing $\mu(x)$ for $\mu(V, x)$, we define the Poincaré polynomial of $\mathcal{H}$ to be

$$P(\mathcal{H}, t) = \sum_{x \in L(\mathcal{H})} \mu(x)(-t)^{\mathrm{rk}(x)}.$$

Theorem 2.1 ([6] (2.3), [9]) *For any set $\mathcal{H}$ of hyperplanes in a real Euclidean space V the number of connected components of $V - \bigcup_{H \in \mathcal{H}} H$ is equal to $P(\mathcal{H}, 1)$. The number of bounded connected components is $|P(\mathcal{H}, -1)|$.*

To proceed to evaluate the Poincaré polynomials for the sandwich arrangement, we need the following simple lemma.

Lemma 2.2 ([6] (2.3)) *If $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a hyperplane arrangement, and $H_1 \perp H_2$ for all $H_1 \in \mathcal{A}_1$, $H_2 \in \mathcal{A}_2$, then*

$$P(\mathcal{A}, t) = P(\mathcal{A}_1, t) P(\mathcal{A}_2, t).$$

Let Φ be a root system, and let $\mathcal{H}$ be the associated sandwich arrangement. Let $\mathcal{H}_0$ be the subarrangement of $\mathcal{H}$ consisting of the hyperplanes that contain the origin of V. For $Y \in L(\mathcal{H}_0)$, let W_Y be the group generated by the reflections through all hyperplanes containing Y. This is a Coxeter group [5].

Lemma 2.3 *For $Y \in L(\mathcal{H}_0)$, let $W_{Y,1} \times \cdots \times W_{Y,m}$ be the decomposition of W_Y into irreducible Coxeter groups. Let $\mathcal{H}(W_{Y,i})$ be the sandwich arrangement associated to the Coxeter group $W_{Y,i}$. Then*

$$[t^l] P(\mathcal{H}, t) = [t^l] \sum_{Y \in L(\mathcal{H}_0) \colon \mathrm{rk}(Y) = l} P(\mathcal{H}(W_{Y,1}), t) \cdots P(\mathcal{H}(W_{Y,m}, t)).$$

Proof: For any $X \in L(\mathcal{H})$, let X_0 be the unique translate of X that passes through the origin. Since the hyperplanes that intersect to form X all have translates in $\mathcal{H}_0$, $X_0 \in L(\mathcal{H}_0)$. For $Y \in L(\mathcal{H}_0)$ with $\mathrm{rk}(Y) = l$, let $\mathcal{H}_Y = \{H \in \mathcal{H} : H_0 \supseteq Y\}$. By the decomposition of the Coxeter group W_Y and by the previous lemma, $P(\mathcal{H}_Y, t) = P(\mathcal{H}(W_{Y,1}), t) \cdots P(\mathcal{H}(W_{Y,m}), t)$. We have

$$\begin{aligned}[t^l] P(\mathcal{H}(W_{Y,1}), t) \cdots P(\mathcal{H}(W_{Y,m}), t) &= \sum_{X \in L(\mathcal{H}_Y) \colon \mathrm{rk}(X) = l} (-1)^l \mu(X) \\ &= \sum_{X \in L(\mathcal{H}) \colon X_0 = Y} (-1)^l \mu(X).\end{aligned}$$

Thus

$$\begin{aligned}&[t^l] \sum_{Y \in L(\mathcal{H}_0) \colon \mathrm{rk}(Y) = l} P(\mathcal{H}(W_{Y,1}), t) \cdots P(\mathcal{H}(W_{Y,m}), t) \\ &\quad = \sum_{Y \in L(\mathcal{H}_0) \colon \mathrm{rk}(Y) = l} \sum_{X \in L(\mathcal{H}) \colon X_0 = Y} (-1)^l \mu(X) \\ &\quad = \sum_{X \in L(\mathcal{H}) \colon \mathrm{rk}(X) = l} (-1)^l \mu(X).\end{aligned}$$

□

Theorem 2.4 *Let Φ be an irreducible crystallographic root system, W the associated finite group, and $\mathcal{H}$ the associated sandwich arrangement. We have*

$$P(\mathcal{H}, t) = (1 + ht)^n,$$

where h is the Coxeter number and n is the rank of the associated finite Coxeter group W.

We prove the theorem by induction on the number of generators, using the previous lemma. We will determine every coefficient of $P(\mathcal{H}, t)$ except that of t^n. Since we know $P(\mathcal{H}, 1)$ from Theorem 1.1, this will determine the polynomial. The analysis will be done case-by-case.

A_n: There is a bijection between $L(\mathcal{H}_0)$ and the partitions of $[n + 1]$. It is given by matching the partition $B = (B_1, \dots, B_m)$ with

$$Y = \cap\{x_i - x_j = 0 : \ i, j \text{ are in the same block of } B\}.$$

The Coxeter group W_Y is isomorphic to $A_{|B_1|-1} \times \cdots \times A_{|B_m|-1}$, and $\mathrm{rk}(Y) = n + 1 - m$. By Lemma 2.3, for $l < n$ we have

$$[t^l]P(\mathcal{H}, t) = \sum |B_1|^{|B_1|-1} \cdots |B_{n+1-l}|^{|B_{n+1-l}|-1},$$

where the sum is taken over all partitions of $[n+1]$ into $n+1-l$ blocks. This is recognized to be the number of labeled forests on $n + 1$ vertices of $n + 1 - l$ rooted trees. From [4] we have

$$[t^l]P(\mathcal{H}, t) = (n + 1)^l \binom{n}{n - l}.$$

We have shown that the coefficients of t^l in $P(\mathcal{H}, t)$ and $(1 + (n + 1)t)^n$ are the same for $1 \le l \le n-1$. Since $P(\mathcal{H}, t)$ is an nth degree polynomial and $P(\mathcal{H}, 1) = (n+2)^n$, $P(\mathcal{H}, t)$ is in fact equal to $(1 + (n + 1)t)^n$.

B_n: The elements of $L(\mathcal{H}_0)$ of dimension l (rank $n - l$) are somewhat harder to describe than in the A_n case. We can start by taking a subset $J \subseteq [n]$ and partitioning it into l non-empty blocks $X = (X_1, \dots, X_l)$. Define a sign function sgn: $J \to \{1, -1\}$ so that $\mathrm{sgn}(j) = 1$ whenever j is the smallest element in its block. For a given partition of J, there are $2^{|J|-l}$ ways to do this. The partition and the function sgn together determine the intersection

$$\begin{aligned} Y = {} & \cap\{\mathrm{sgn}(i)x_i - \mathrm{sgn}(j)x_j = 0 : i, j \text{ are in the same block of } X\} \\ & \cap\{x_k = 0 : k \in [n] - J\}. \end{aligned}$$

We have $W_Y \cong A_{|X_1|-1} \times \cdots \times A_{|X_l|-l} \times B_{n-|J|}$, and the contribution of Y to the coefficient of t^{n-l} in $P(\mathcal{H}, t)$ is $|X_1|^{|X_1|-1} \cdots |X_l|^{|X_l|-1}(2(n-|J|))^{n-|J|}$. If we sum $\Pi|X_i|^{|X_i|-1}$ over all partitions of J into l blocks, we get $|J|^{|J|-l}\binom{|J|-1}{l-1}$, the coefficient of $t^{|J|-l}$ in $P(\mathcal{H}(A_{|J|-1}), t)$. Putting this all together, the coefficient of t^{n-l} in $P(\mathcal{H}(B_n), t)$ is

$$\sum_{k=l}^{n} \binom{n}{k} (2k)^{k-l} \binom{k-1}{l-1} (2(n-k))^{n-k}.$$

We would like to show that this is equal to the coefficient of t^{n-l} in $(1+2nt)^n$, which is $\binom{n}{l}(2n)^{n-l}$. We can remove a factor of 2^{n-l} so that we have

$$\sum_{k=l}^{n} \binom{n}{k} k^{k-l} \binom{k-1}{l-1} (n-k)^{n-k} = \binom{n}{l} n^{n-l},$$

which is a consequence of Abel's Identity [2].

C_n: The calculations are the same as for B_n.

D_n: This is very similar to the B_n case. If $|J| \neq n-1$, the intersection Y determined by X, J and sgn is

$$\begin{aligned} Y = &\cap \{\operatorname{sgn}(i)x_i - \operatorname{sgn}(j)x_j = 0 : i, j \text{ are in the same block of } X\} \\ &\cap \{x_k - x_l = 0 : k, l \in [n] - J\} \\ &\cap \{x_k + x_l = 0 : k, l \in [n] - J\}. \end{aligned}$$

If $|J| = n-1$, there is no corresponding Y. We have $W_Y \cong A_{|X_1|-1} \times \cdots \times A_{|X_l|-1} \times D_{n-|J|}$, and the identity to be proved is

$$\sum_{k=l}^{n} \binom{n}{k} k^{k-l} \binom{k-1}{l-1} ((n-k)-1)^{n-k} = \binom{n}{l} (n-1)^{n-l},$$

which is again a consequence of Abel's Identity.

For the exceptional groups we use the data from [5]. The integers $n(R, T)$ listed there give the number of $Y \in L(\mathcal{H}_0(T))$ such that $W_Y \cong R$. As before, we need only show that the coefficients of $t^0, \ldots, t^{n-1}$ match the coefficients of $(1+ht)^n$. The calculations are shown in the tables that follow. In these tables, $c(R)$ is the leading coefficient of $P(\mathcal{H}(R_1), t) \cdots P(\mathcal{H}(R_m), t)$, where $R_1 \times \cdots \times R_m$ is the decomposition of R into irreducible factors.

As a corollary of Theorem 2.1, we have the following.

Corollary 2.5 *Let $\mathcal{H}$, h, and n be as in Theorem* 1.1. *The number of bounded components of $V - \bigcup_{H \in \mathcal{H}} H$ is $(h-1)^n$.*

3. Tables

Table 1. E_6.

	R	$n(R, E_6)$	$n(R, E_6) \cdot c(R)$
t^5	$A_1 \times A_2^2$	360	58320
	$A_1 \times A_4$	216	270000
	A_5	36	279936
	D_5	27	884736
			1492992
t^4	$A_1^2 \times A_2$	1080	38880
	A_2^2	120	9720
	$A_1 \times A_3$	540	69120
	A_4	216	135000
	D_4	45	58320
			311040
t^3	A_1^3	540	4320
	$A_1 \times A_2$	720	12960
	A_3	270	17280
			34560
t^2	A_1^2	270	1080
	A_2	120	1080
			2160
t^1	A_1	36	72
t^0	A_0	1	1

Table 2. E_7.

	R	$n(R, E_7)$	$n(R, E_7) \cdot c(R)$
t^6	$A_1 \times A_2 \times A_3$	5040	5806080
	$A_2 \times A_4$	2016	11340000
	$A_1 \times A_5$	1008	15676416
	A_6	288	33882912
	$A_1 \times D_5$	378	24772608
	D_6	63	63000000
	E_6	28	83607552
			238085568

(*Continued on next page.*)

Table 2. (*Continued.*)

	R	$n(R, E_7)$	$n(R, E_7) \cdot c(R)$
t^5	$A_1^3 \times A_2$	5040	362880
	$A_1 \times A_2^2$	10080	1632960
	$A_1^2 \times A_3$	7560	1935360
	$A_2 \times A_3$	5040	2903040
	$A_1 \times A_4$	6048	7560000
	A_5	1344	10450944
	$A_1 \times D_4$	945	2449440
	D_5	378	12386304
			39680928
t^4	A_1^4	3780	60480
	$A_1^2 \times A_2$	15120	544320
	A_2^2	3360	272160
	$A_1 \times A_3$	8820	1128960
	A_4	2016	1260000
	D_4	315	408240
			3674160
t^3	A_1^3	4095	32760
	$A_1 \times A_2$	5040	90720
	A_3	1260	80640
			204120
t^2	A_1^2	945	3780
	A_2	336	3024
			6804
t^1	A_1	63	126
t^0	A_0	1	1

Table 3. E_8.

	R	$n(R, E_8)$	$n(R, E_8) \cdot c(R)$
t^7	$A_1 \times A_2 \times A_4$	241920	2721600000
	$A_3 \times A_4$	120960	4838400000
	$A_1 \times A_6$	34560	8131898880
	A_7	8640	18119393280
	$A_2 \times D_5$	30240	8918138880
	D_7	1080	38698352640
	$A_1 \times E_6$	3360	20065812480
	E_7	120	73466403840
			174960000000

(*Continued on next page.*)

Table 3. (*Continued.*)

	R	$n(R, E_8)$	$n(R, E_8) \cdot c(R)$
t^6	$A_1^2 \times A_2^2$	604800	195955200
	$A_1 \times A_2 \times A_3$	604800	696729600
	$A_1^2 \times A_4$	362880	907200000
	A_3^2	151200	619315200
	$A_2 \times A_4$	241920	1360800000
	$A_1 \times A_5$	120960	1881169920
	A_6	34560	4065949440
	$A_2 \times D_4$	50400	587865600
	$A_1 \times D_5$	45360	2972712960
	D_6	3780	3780000000
	E_6	1120	3344302080
			20412000000
t^5	$A_1^3 \times A_2$	604800	43545600
	$A_1 \times A_2^2$	403200	65318400
	$A_1^2 \times A_3$	453600	116121600
	$A_2 \times A_3$	302400	174182400
	$A_1 \times A_4$	241920	302400000
	A_5	40320	313528320
	$A_1 \times D_4$	37800	97977600
	D_5	7560	247726080
			1360800000
t^4	A_1^4	113400	1814400
	$A_1^2 \times A_2$	302400	10886400
	A_2^2	67200	5443200
	$A_1 \times A_3$	151200	19353600
	A_4	24192	15120000
	D_4	3150	4082400
			56700000
t^3	A_1^3	37800	302400
	$A_1 \times A_2$	40320	725760
	A_3	7560	483840
			1512000
t^2	A_1^2	3780	15120
	A_2	1120	10080
			25200
t^1	A_1	120	240
t^0	A_0	1	1

Table 4. F_4.

	R	$n(R, F_4)$	$n(R, F_4) \cdot c(R)$
t^3	$A_1 \times A_2$	96	1728
	B_3	12	2592
	C_3	12	2592
			6912
t^2	A_2	32	288
	$A_1 \times A_1$	72	288
	B_2	18	288
			864
t^1	A_1	24	48
t^0	A_0	1	1

Table 5. G_2.

	R	$n(R, G_2)$	$n(R, G_2) \cdot c(R)$
t^1	A_1	6	12
t^0	A_0	1	1

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