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Subdirect Decomposition of *n*-Chromatic Graphs

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Abstract. It is shown that any n-chromatic graph is a full subdirect product of copies of the complete graphs K_n and K_{n+1}, except for some easily described graphs which are full subdirect products of copies of $K_{n+1} - \{\circ\!-\!\circ\}$ and $K_{n+2} - \{\circ\!-\!\circ\}$; *full* means here that the projections of the decomposition are epimorphic in edges. This improves some results of Sabidussi. Subdirect powers of K_n or $K_{n+1} - \{\circ\!-\!\circ\}$ are also characterized, and the subdirectly irreducibles of the quasivariety of n-colorable graphs with respect to full and ordinary decompositions are determined.

Keywords: graphs, n-colorable graph, subdirectly irreducible, subdirect product, quasivariety

Introduction

Birkhoff's theorem on the subdirect decomposition of algebras of an equational class into subdirectly irreducible factors [1] has been generalized by several authors to other classes of first-order structures (not to mention generalizations to certain abstract categories). Malcev [10] to classes of structures axiomatized by positive universal axioms, Sabidussi [15] to the class of all graphs, Nešetřil and Pultr [12] to finitely generated quasivarieties of graphs, Pickett [13] and Burris [3] to classes of structures satisfying certain conditions on their lattices of equivalence relations, and this author [6] to classes of structures closed under direct epimorphic limits. In particular, any quasivariety (universal Horn class) or any class of finite structures satisfies a generalized Birkhoff's theorem.

Some of these generalizations are equivalent (see the discussion in Fleischer [8]), but they fall into two groups depending whether we ask the projections of a subdirect product just to be onto on elements of the domain as in [6, 10, 12], or we ask them to be *full* also, that is, onto on atomic relations as in [3, 13, 15]. The first type of subdirectly irreducible structures form a subclass of the irreducibles of the second type; usually there is a tradeoff between simpler decompositions in the first case and tighter decompositions in the second.

For the class of all finite graphs (irreflexive, undirected), the subdirectly irreducible structures in the first sense are the complete graphs K_i, $i = 1, 2, \ldots$, and the irreducibles with respect to full subdirect products are the complete graphs, together with the complete graphs minus and edge: $A_i = K_{i+1} - \{\circ\!-\!\circ\}$, $i = 2, 3, \ldots$. In this context, Sabidussi [15] has shown that any finite graph is a full subdirect product of K_i's, possibly for various i's, or a full subdirect product of A_i's.

We improve Sabidussi's result and answer some of his questions by showing that any n-chromatic graph is a full subdirect product of copies of K_n and K_{n+1}, with the exception of some particular graphs which are full subdirect products of copies of A_n and A_{n+1}. We determine also the full subdirect powers of K_n and the full subdirect powers of A_n. To this end, we examine the subdirectly irreducibles of the quasivariety $\mathcal{C}_n$ of n-colorable graphs for finite n. In the first sense they are

$$K_i \ (1 \leq i \leq n), \quad A_n = K_{n+1} - \{\circ\text{–}\circ\} \quad \text{and} \quad B_n = K_{n+2} - \{\circ\text{–}\circ\text{–}\circ\text{–}\circ\}, \tag{1}$$

where a 3-chain is subtracted from K_{n+2} in the last graph. The subdirectly irreducibles for full decompositions also include $A_1, \ldots, A_{n-1}$ and several weak subgraphs of B_n, their number growing quadratically with n.

It follows from (1) that $\mathcal{C}_n$ is generated as a quasivariety by the single graph B_n, a result first obtained by Nešetřil and Pultr [12]. Wheeler [17] proved independently that $\mathcal{C}_n$ is generated by a single finite graph (distinct from B_n), in order to show that $\mathcal{C}_n$ has a ω-categorical model companion with a primitive recursive axiomatization. This last property has been shown by Burris to hold for any quasivariety generated by a single finite structure [4, 5].

1. Preliminaries

By a *graph* we will always mean a nondirected simple graph without loops; that is, a first order structure $G = (V, E)$, where V is the set of *vertices* and E is a binary relation in V (the set of *edges*) satisfying the universal Horn sentences:

A1. (*irreflexibity*) $\forall x(\neg xEx)$
A2. (*symmetry*) $\forall x \forall y(xEy \to yEx)$.

An element $(u, v) \in E$ will be denoted sometimes $\overline{uv}$, and we will write $u \not{E} v$ to express that (u, v) is not an edge of G.

The notions of *homomorphism*, *isomorphism*, *subgraph*, and *product* are defined for graphs as is costumary for first-order structures in model theory, cf. [9]. An *epimorphism* will be a homomorphism $f : (V, E) \to (V', E')$ onto on vertices. It will be *full* if it is onto also on edges; that is, for every $(u', v') \in E'$ there is $(u, v) \in E$ with $f(u) = u'$ and $f(v) = v'$. It will be *proper* if it is not an isomorphism.

Notice that what is usually called a "subgraph" by graph theorists (cf. [2]) is a homomorphic inclusion or *weak* subgraph but not necessarily a subgraph in the model theoretic sense. The last meaning corresponds to what graph theorists call a *full subgraph*, or *induced subgraph* [2], a meaning that will be utilized in this paper unless stated otherwise.

Given a vertex v in a graph $G = (V, E)$, let $G - v$ denote the result of deleting v and all edges incident with v; that is, the (full) subgraph of G with universe $V - \{v\}$. Similarly, if $\overline{uv}$ is an edge of G then $G - \overline{uv}$ denotes the graph obtained by deleting the edge $\overline{uv}$, but not the vertices u, v. This is not a (full) subgraph of G. These notations have natural generalizations $G - \{v_1, \ldots, v_n\}$ and $G - \{\overline{uv}_1, \ldots, \overline{u_n v_n}\}$ for sets of vertices and edges, respectively.

Definition

(a) A *subdirect product* of a family of graphs $\{G_i\}_{i \in I}$ is a subgraph G of the cartesian product $\prod_i G_i$ such that for each $i \in I, \pi_i \mid G : G \to G_i$ is an epimorphism, where

π_i is the ith projection of the product. A *subdirect decomposition* of a graph G is an isomorphism of G onto a subdirect product.

(b) Given a class $\mathcal{C}$ of graphs, G is a *subdirectly irreducible graph* (*s.i.*) *of* $\mathcal{C}$ if $G \in \mathcal{C}$ and for any subdirect decomposition $f : G \to \prod_i G_i$ of G with G_i in $\mathcal{C}$, there is $i \in I$ such that $\pi_i \circ f$ is an isomorphism.

(c) Subdirect products and decompositions having full projections will be called *full subdirect products* and *full decompositions*, respectively. The irreducibles with respect to full decompositions will be called *full subdirectly irreducibles* (*f.s.i.*).

Let $\mathcal{I}_\mathcal{C}$ (respectively, $\mathcal{FI}_\mathcal{C}$) be the class of s.i. (respectively, f.s.i.) graphs of a class $\mathcal{C}$, then $\mathcal{I}_\mathcal{C} \subseteq \mathcal{FI}_\mathcal{C}$ because every full decomposition is also a subdirect decomposition.

We recall the definition of direct limits for the specific case of graphs, which we need for the next theorem (cf. [9]). A *directed system* of graphs consists of a partially ordered set $(I, \leq)$ where each finite subset has an upper bound, and a family of graph homomorphisms $\mathfrak{D} = \{h_{ij} : G_i \to G_j : i, j \in I, i \leq j\}$ such that $h_{ii} = Id_{G_i}$, and $h_{jk} \circ h_{ij} = h_{ik}$ whenever $i \leq j \leq k$. It is called a *chain* if $(I, \leq)$ is linearly ordered. It is (*fully*) *epimorphic* if each h_{ij} is a (resp., full) epimorphism.

If $G_i = (V_i, E_i)$, the *direct limit of* $\mathfrak{D}$ is the graph $G^* = (V^*, E^*)$ where V^* consists of the equivalence classes $(a, i)_\equiv$ of $\cup_i V_i \times \{i\}$ under the equivalence relation: $(a, i) \equiv (b, j) \Leftrightarrow h_{ik}(a) = h_{jk}(b)$ *for some* k, and the edge relation is: $(a, i)_\equiv E^*(b, j)_\equiv \Leftrightarrow h_{ik}(a) E_i h_{jk}(b)$ *for some* k.

The *induced* homomophisms $h_i : G_i \to G^*$ defined by $h_i(a) = (a, i)_\equiv$ are (full) epimorphisms whenever $\mathfrak{D}$ is (fully) epimorphic.

Subdirect Decomposition Theorem *Any graph of a class $\mathcal{C}$ closed under direct limits of chains of (full) epimorphism is a subdirect product of s.i. of $\mathcal{C}$ (resp. a full subdirect product of f.s.i. of $\mathcal{C}$).*

Proof: Theorem 4 in [6] yields the decomposition into s.i., and it is easily generalized to yield the full decomposition into f.s.i., using the fact that the induced h_i are full epimorphisms when all the h_{ij} are. □

It follows that a class of graphs closed under direct limits of epimorphic directed systems has both kinds of decompositions. The theorem applies in particular to *universal Horn* classes or *quasivarieties*, those axiomatized by *universal Horn* sentences: $\forall x_1 \ldots \forall x_n$ $(\vee_j \neg \theta_j \vee \pm \theta)$ with θ, θ_j atomic, because these sentences are easily seen to be preserved by all direct limits. This is the case of the class of all graphs (axioms A1 and A2) or the class of n-colorable graphs for finite n (next section).

Let ISP$(\mathcal{A})$ denote the class of structures isomorphic to substructures of products of elements of a class $\mathcal{A}$. Quasivarieties are closed under products and subgraphs; hence, the subdirect decomposition theorem implies for them: $\mathcal{C} = \text{ISP}(\mathcal{I}_\mathcal{C}) = \text{ISP}(\mathcal{FI}_\mathcal{C})$.

The theorem applies also to nonaxiomatizable classes, as the class of n-chromatic graphs which is also closed under all direct limits (but not under subgraphs or products): if each graphs G_i of a directed system is n-chromatic, then the direct limit G^* is n-colorable, being a direct limit of n-colorable graphs. Moreover, the existence of induced homomorphisms $h_i : G_i \to G^*$ implies $n = \chi(G_i) \leq \chi(G^*)$; hence, $\chi(G^*) = n$.

It applies as well to any class of finite graphs, say the class of finite planar graphs, since an epimorphic system of finite graphs has for limit any graph in the system with a maximum number of edges among those having the minimum number of vertices.

Lemma 1 *$G = (V, E)$ is a s.i. (resp., f.s.i.) graph of a class $\mathcal{C}$ of graphs if and only if there are vertices $a, b \in V$ satisfying one of the following conditions*:

1. *$a \neq b$ and $h(a) = h(b)$ for any (resp., full) proper epimorphism $h : G \to G'$ with $G' \in \mathcal{C}$.*
2. *$a \not\!E b$ and $h(a) E h(b)$ for any (resp., full) proper epimorphism $h : G \to G'$ with $G' \in \mathcal{C}$.*

Proof: By Lemma 1 in [6], which also holds for the full epimorphism version. □

The pair of vertices $\{a, b\}$ given by the lemma will be called a *critical pair of G of type* 1 (*modulo* $\mathcal{C}$) when it satisfies condition 1, and *of type* 2 if it satisfies condition 2.

2. The quasivariety of *n*-colorable graphs

K_n will denote the *complete graph in n vertices* $(\{1, \dots, n\}, \{(i, j) \mid i \neq j\})$. An *$n$-coloring* of a graph G is a homomorphism $f : G \to K_n$. A graph is *n-colorable* if it has an n-coloring. G is *n-chromatic* if it is n-colorable but not $(n-1)$-colorable.

The class $\mathcal{C}_n$ of n-colorable graphs is an universal Horn class, although it is not finitely axiomatizable for $n \geq 2$, cf. [7, 16]. A simple axiomatization is obtained from axioms A1, A2 in Section 1, adding the sentences:

$$\theta_k^n : \forall x_1 \dots \forall x_k \left[\bigvee_{R \in \mathcal{R}_k^n} \left(\bigwedge_{(i,j) \in R} \neg x_i E x_j \right) \right], \quad k = 1, 2, \dots,$$

where $\mathcal{R}_k^n$ is the class of all equivalence relations in $\{1, \dots, k\}$ having at most n equivalence classes. Clearly, $G \models \theta_k^n$ if and only if all subgraphs of G of power at most k may be partitioned in at most n blocks of independent vertices, that is, they are n-colorable. Therefore, using a compactness argument in case G is infinite, $G \models \{\theta_k^n : k \in \omega\}$ if and only if G is n-colorable. Now, using distributivity of $\vee$ and $\forall$ over $\wedge$, each θ_k^n becomes equivalent to the set of universal Horn sentences of the form

$$\theta_f : \forall x_1 \dots \forall x_k \left[\bigvee_{R \in \mathcal{R}_k^n} \neg x_{f_1(R)} E x_{f_2(R)} \right],$$

where $f = (f_1, f_2)$ runs over all choice functions of $\mathcal{R}_k^n$, that is, $f(R) = (f_1(R), f_2(R)) \in R$.

The complete graphs K_i, $i \leq n$, are s.i. in $\mathcal{C}_n$ because there is no proper epimorphism defined in any of them (irreflexivity). The graphs:

$$A_n = K_{n+1} - \{\circ\!-\!\circ\} = K_n + \{\overline{i\ n+1} : 2 \leq i \leq n\}$$
$$B_n = K_{n+2} - \{\circ\!-\!\circ\!-\!\circ\!-\!\circ\} = K_n + \{\overline{i\ n+1}, \overline{j\ n+2} : 1 \leq i, j \leq n, i \neq 1, j \neq 2\}$$

are also s.i. in $\mathcal{C}_n$ by Lemma 1. To see this first notice that $f(i) = i$ (mod n) colors both graphs. Moreover, any proper epimorphism $h : A_n \to H \in \mathcal{C}_n$ must have $h(1)Eh(n+1)$ or $h(1) = h(n+1)$; otherwise, $H \approx A_n$. Since the first possibility yields $H \approx K_{n+1} \notin \mathcal{C}_n$, we must have the second, showing that $\{1, n+1\}$ is a critical pair of type 1 for A_n. Similarly, for any proper epimorphism $h : B_n \to H \in \mathcal{C}_n$ at least one of the pairs $\{h(1), h(n+1)\}$, $\{h(n+1), h(n+2)\}$ or $\{h(n+2), h(2)\}$ must be a singleton or an edge. The alternatives $h(1)Eh(n+1)$, $h(n+1) = h(n+2)$, or $h(n+2)Eh(2)$, imply that H contains a copy of K_{n+1}. That leaves the cases $h(1) = h(n+1)$, $h(n+1)Eh(n+2)$, or $h(n+2) = h(2)$; all of which imply $h(n+1)Eh(n+2)$, because $h(1)Eh(n+2)$ and $h(2)Eh(n+1)$, showing that $\{n+1, n+2\}$ is a critical pair of type 2 for B_n.

Theorem 1 *Modulo isomorphism*, $K_1, \ldots, K_n$, A_n *and* B_n *are all the subdirectly irreducible graphs of* $\mathcal{C}_n$, $n \geq 2$.

Proof: By the previous remarks these graphs are s.i. They are the only ones by Corollary 2(b) in Section 3 and the Definition of s.i. □

Corollary 1 (Nešetřil and Pultr [12]) $\mathcal{C}_n = \mathrm{ISP}(B_n)$, *that is*, *any n-colorable graph is isomorphic to a subgraph of a power of* B_n.

Proof: Since $\mathcal{C}_n$ is a quasivariety, $\mathcal{C}_n = \mathrm{ISP}(\mathcal{I}_{\mathcal{C}_n})$ by the Subdirect Decomposition Theorem, but all subdirectly irreducible graphs of $\mathcal{C}_n$ are easily seen to be subgraphs of B_n. □

Figure 1 gives the list of subdirectly irreducible n-colorable graphs for $n = 2, 3, 4$:

Remarks

1. The only critical pair of A_n is $\{1, n+1\}$, of type 1, and the only one of B_n is $\{n+1, n+2\}$, of type 2. Moreover, these graphs are uniquely n-colorable.
2. Since B_4 is planar, Corollary 1 and the **4-Color Theorem** imply that $\mathcal{C}_4$ is the smallest universal Horn class containing the class $\mathcal{P}$ of all planar graphs. Hence, $\mathcal{C}_4$ and $\mathcal{P}$ share all properties describable by classes of universal Horn sentences, and have the same subdirectly irreducibles. It may be shown, without using the 4-Color Theorem, that any subdirectly irreducible graph of $\mathcal{P}$ must be maximal planar, and therefore any planar graph is a subdirect product of maximal planar graphs.
3. In [12], B_n is described as a disjoint sum of K_{n-2} and a 3-chain. Wheeler [17] gives a distinct generator for $\mathcal{C}_n$. His generator for $\mathcal{C}_4$ is not planar since it contains a subgraph homeomorphic to K_5.

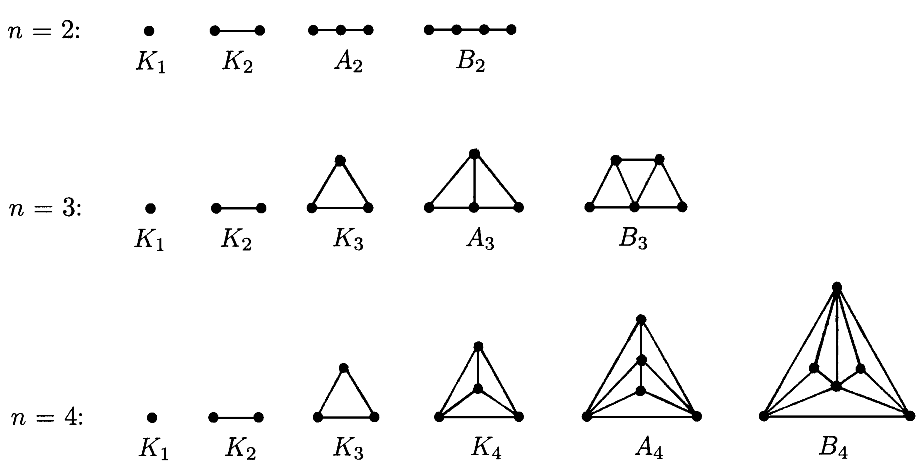

Figure 1.

3. Full subdirect decompositions

In addition to the complete graphs, the following graphs are f.s.i. in $\mathcal{C}_n$ due to Lemma 1:

$$A_i = K_{i+1} - \{\circ\!\!-\!\!\circ\}, \quad i = 1, \dots, n,$$

because any proper full epimorphism defined in A_i must identify its unique unconnected pair, making it critical. Moreover, the following weak subgraphs of B_n:

$$B_{n,j,k} = B_n(S, T) = K_n + \{\overline{i\ n+1},\ \overline{m\ n+2} : i \in S, m \in T\},$$

where $S \cup T = V(K_n)$, $j = |S - T|$, $k = |T - S|$, and $1 \leq j \leq k$, because any proper full epimorphism defined in one of them must identify two elements at least (otherwise, it would not be full). Now, the only possible identifications are: $n + 1$ with some $i \notin S$, or $n + 2$ with some $m \notin T$, since identifying $n + 1$ with $n + 2$ would produce K_{n+1}. In any case, $n + 1$ becomes connected to $n + 2$, showing that $\{n + 1, n + 2\}$ is a critical pair for full epimorphisms.

A straightforward counting shows that there are $[n/2] + (n(n-1) - [n/2])/2 = [n^2/4]$; therefore, the number of such graphs grows quadratically. Evidently, $B_n = B_{n,1,1}$. For bipartite graphs, they reduce to B_2. For $n = 3, 4$ they are shown in figure 2 (except B_n).

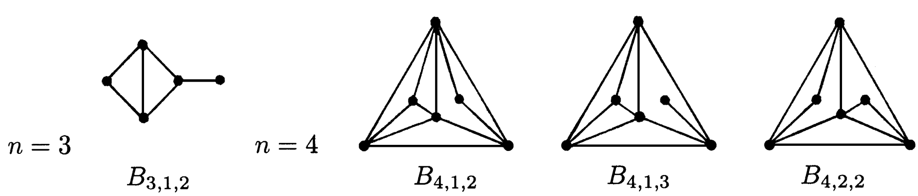

Figure 2.

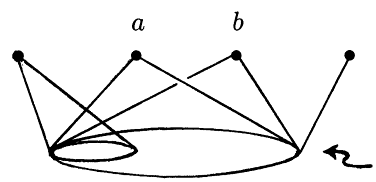

Figure 3.

Theorem 2 K_i, A_i $(1 \le i \le n)$ *and* $B_{n,j,k}$ $(2 \le j + k \le n)$ *are all the f.s.i. of* $\mathcal{C}_n$.

Proof: By the previous remarks and Corollary 2(a) below. □

Definition A graph G will be a *crown* if it has at least two vertices (a and b in figure 3) connected to all vertices of a covering of G.

Recall that a (*vertex*) *covering* of G is a set of vertices B such that any edge of G has at least one end vertex in B (cf. [2]). The covering of a crown will be called its *base*. The vertices not in the base form an independent set (a and b at least). They will be called the *top* vertices of the crown.

Theorem 3 *Any n-chromatic graph distinct form K_n is a full subdirect product of copies of A_n and the $B_{n,j,k}$. If it is not a crown then it is a full subdirect product of copies of the $B_{n,j,k}$.*

Proof: We check that for any pair of distinct vertices a, b in G with $a \not E b$ there is a full epimorphism $h_{ab} = h'$ onto one of the required graphs such that $h'(a) \neq h'(b)$ and $h'(a) \not E h'(b)$. If G is not K_n, at least one such pair of vertices will exist, and the nonempty family $\{h_{ab}\}_{ab}$ will induce the required embedding.

Case 1 $h(a) \neq h(b)$ for any n-coloring h of G. Fix h, then $|h^{-1}(h(a))| \ge 2$ and $|h^{-1}(h(b))| \ge 2$; otherwise, for example $h^{-1}(h(b)) = \{b\}$, we could change the color of a to that of b because $a \not E b$, obtaining a coloring that identifies them. Moreover, $G - \{a, b\}$ is n-chromatic, otherwise we could $n-1$ color this subgraph and assign to a, b the nth color, which also contradicts the hypothesis. The function $h' : G \to B_n$. defined by

$$h'(x) = \begin{cases} h(x) & \text{if } x \neq a, b \\ n+1 & \text{if } x = a \\ n+2 & \text{if } x = b, \end{cases}$$

will be an epimorphism in vertices with $h'(a) \not E h'(b)$, $h'(a) \neq h'(b)$. Moreover, its full image $h'(G)$ is some $B_{n,j,k}$. By n-chromaticity, $h'(G - \{a, b\}) = K_n$. If the edges from $n+1$ and $n+2$ to K_n in $h'(G)$ did not cover together K_n then we could assign the noncovered color to $n+1$ and $n+2$, obtaining again an n-coloring which identifies a and b.

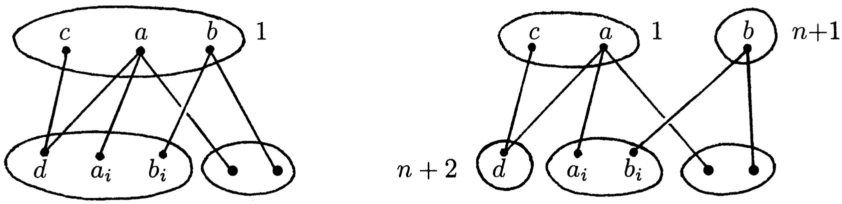

Figure 4.

Case 2 $h(a) = h(b)$ for some n-coloring h. Fix one such h with $h(a) = h(b) = 1$ and $h^{-1}(1)$ extended to a maximal independent set of vertices (Zorn Lemma). Then we have:

(i) $B = \bigcup_{i \neq 1} h^{-1}(i)$ is a vertex covering such that any $x \in B$ is connected to some vertex in $h^{-1}(1)$, otherwise x could be moved to $h^{-1}(1)$ contradicting its maximality.
(ii) a is connected to some vertex of $a_i \in h^{-1}(i)$ for each $i \neq 1$; otherwise we could change the color of a to i, contradicting (i). Similarly, $b\ E\ b_i \in h^{-1}(i)$ for each $i \neq 1$.

Changing the color $h(b)$ to $n+1$, yields a full epimorphism $h' : G \to A_n$ which separates and disconnects a and b.

Subcase 2.1 Assume in addition that G is not a crown, then some $d \in B$ is not connected to a or b, since B satisfies all other properties to make G a crown. However, d is connected to some $c \in h^{-1}(1)$ as noticed in (i) above. Suppose that there is no edge from d to b, for example, as shown in figure 4. Then $d \neq b_i$ for $i \neq 1$ (but d may be a_i) and we may define a full epimorphism $h' : G \to B_n$ which separates and disconnects a and b by

$$h'(x) = \begin{cases} h(x) & \text{if } x \neq b, d \\ n+1 & \text{if } x = b \\ n+2 & \text{if } x = d. \end{cases}$$

h' is full at least onto $B_{n,1,n-1}$ because the edges leaving b cover all original colors distinct from 1 by condition (ii) and the fact that $d \neq b_i$, and the edge (d, c) covers the color 1. Moreover, $h'(G - \{b, d\}) = K_n$ fully; otherwise, we could $n-1$-color $G - \{b, d\}$ and give the color n to b and d, contradicting (i). □

Lemma 3 *Each $B_{n,j,k}$ is a full subdirect product of copies of K_n and K_{n+1}.*

Proof: The following family of full epimorphisms, where $B = B_n(\{1, \ldots, p\}, \{m, \ldots, n\})$, induces a full subdirect decomposition $f : B \to K_{n+1} \times K_n^{m+n-p-1}$.

$$f_0 : B \to K_{n+1}, \quad \text{where } i \mapsto i \text{ for } 1 \leq i \leq n+1, \text{ and } n+2 \mapsto n+1,$$

which separates any pair of distinct vertices in B, except $\{n+1, n+2\}$, and maintains the disconnection between $n+1$ and $n+2$ because $f_0(n+1) = f_0(n+2)$;

$$f_r : B \to K_n\ (r < m), \quad \text{where } i \mapsto i \text{ for } 1 \leq i \leq n, n+1 \mapsto n, \text{ and } n+2 \mapsto r,$$

which maintains the disconnection between $n+1$ and r, and separates $\{n+1, n+2\}$;

$$f_s : B \to K_n (s > p), \quad \text{where } i \mapsto i \text{ for } 1 \leq i \leq n, n+1 \mapsto s, \text{ and } n+2 \mapsto 1,$$

which maintains the disconnection between $n+2$ and s. □

Corollary 2
(a) *Any n-colorable graph is a full subdirect product of the graphs K_i, A_i $(1 \leq i \leq n)$, and $B_{n,j,k}$ $(2 \leq j+k \leq n)$.*
(b) *Any n-colorable graph is a subdirect product* (*not necessarily full*) *of the graphs K_i $(1 \leq i \leq n)$, A_n, and B_n.*

Proof:

(a) For n-chromatic graphs apply Theorem 3. For i -chromatic graphs with $i < n$, Theorem 3 yields a full decomposition onto A_i, $B_{i,j,k}$ which Lemma 3 allows to transform in a full decomposition onto A_i, K_i, K_{i+1}.
(b) A full epimorphism $f : G \to A_i$ $(i \leq n-1)$ induces an ordinary epimorphism $f : G \to K_{i+1}$ $(i+1 \leq n)$, and a full epimorphism $f : G \to B_{n,j,k}$ induces an epimorphism $f : G \to B_n$. Hence, the full subdirect decomposition provided by (a) becomes an ordinary subdirect decomposition onto K_i $(i \leq n)$, A_n, and B_n. □

4. Pure subdirect decompositions

Sabidussi [15, Theorem 3.6] has shown that any finite graph is a "pure" full subdirect product containing only K_i's (*pure complete representation* of [15]) or containing only A_i's (*pure almost complete representantion*). The next Corollary 3 improves Sabidussi's theorem by specifying optimally the values of i, and Corollary 4 gives a simple answer to his question [15, p. 1208] about characterizing graphs with "pure complete" representations. Moreover, the subdirect powers of K_n alone or A_n alone, $n \geq 2$, are characterized in Corollary 5 of the next section.

Lemma 4 *An n-chromatic crown is a full subdirect product of copies of A_n and A_{n+1}.*

Proof: Given an n-chromatic crown G, the subgraph B induced by its base is necessarily $(n-1)$-chromatic. Fix an $(n-1)$-coloring $h : B \to K_{n-1}$ (it must be full), and let a and b be the two vertices connected to all vertices in B. We must show that for any pair of distinct nonincident vertices x, y, there are full epimorphisms onto A_n or A_{n+1} maintaining them distinct and disconnected.

Case 1 x, y are top vertices, then one of them (say x) is distinct from b. Extend h to $h_{xy} : G \to A_n$ by sending a together with x to n, and the other top vertices to $n+1$. Then h_{xy} is a full epimorphism which separates and disconnects x and y. There is at least the epimorphism h_{ab} in this family.

Case 2 x in the top, y in the base, then h_{ab} described in Case 1 separates them. To get an epimorphism disconnecting them, identify x and y. This yields a crown G', having an $(n-1)$-chromatic or n-chromatic base. Proceed as in Case 1 to obtain a full epimorphism onto A_n or A_{n+1} which identifies and therefore disconnects x and y.

Case 3 x, y in the base. Identifying them and proceeding as in Case 2 we obtain a full epimorphism onto A_n or A_{n+1} which maintains them disconnected. If h may be chosen so that $h(x) \neq h(y)$, the extension h_{ab} maintains them distinct. Otherwise, $h(x) = h(y) = i$ and each of x, y must be connected to some vertex in $h^{-1}(j)$, for all $j \neq i, 1 \leq j \leq n-2$ (if not, colors could be changed to have $h(x) \neq h(y)$). Modify and extend h by sending x to n, all vertices in $h^{-1}(i) - \{x\}$ to $n+1$, and all the top vertices to i. This yields a full epimorphism onto A_n which separates x and y. □

Corollary 3 *Any n-chromatic graph G is a full subdirect product of copies of K_n and K_{n+1} or a full subdirect product of copies of A_n and A_{n+1}.*

Proof: If G is not a crown, it is a subdirect product of K_n and K_{n+1} by Theorem 3 and Lemma 3. If G is a crown, apply Lemma 4. □

Corollary 4 *The following are equivalent for a finitely colorable graph*:

(i) *G is a full subdirect product of complete graphs.*

(ii) *G is a full subdirect product of copies of K_n or K_{n+1} where $n = \chi(G)$.*

(iii) *G is not a crown.*

Proof:
(iii) ⇒ (ii) ⇒ (i) If G is not a crown, apply Theorem 3 and Lemma 3. (i) ⇒ (iii) A crown G cannot be a full subdirect product of K_i's because no full epimorphism $h : G \to K_i$ separates the top vertices a and b. Any edge (u, v) in G has a vertex in the base which must be incident with both a and b , assume it is u, so that $h(u) \neq h(a), h(b)$. Then $(h(u), h(v))$ cannot connect $h(a)$ with $h(b)$ in K_i. As any edge of K_i is of this form if h is full, then $h(a) = h(b)$. □

5. Freely jointly and disjointly n-colorable graphs

A graph G is *jointly* (respectively, *disjointly*) n -colorable if for any pair of vertices a, b in G with $a \neq b$, $a \not{E} b$ there is an n-coloring c of G with $c(a) = c(b)$ (respectively, $c(a) \neq c(b)$). G is *freely* n-colorable if both conditions are satisfied.

The classes $\mathcal{J}_n$, $\mathcal{D}_n$, and $\mathcal{F}_n$, of jointly, disjointly, and freely n-colorable graphs, respectively, are universal Horn classes. An axiomatization of $\mathcal{J}_n$ is given by the sentences

$$\forall x_1 \ldots \forall x_k \left[x_1 E x_2 \vee \bigvee_{(1,2) \in R \in \mathcal{R}_k^n} \left(\bigwedge_{(i,j) \in R} \neg x_i E x_j \right) \right], \quad k = 1, 2, \ldots$$

which state the existence of the desired colorings for all subgraphs of size $k = 1, 2 \dots$ Similarly, $\mathcal{D}_n$, is axiomatized by the sentences

$$\forall x_1 \dots \forall x_k \left[x_1 = x_2 \vee \bigvee_{(1,2) \notin R \in \mathcal{R}_k^n} \left(\bigwedge_{(i,j) \in R} \neg x_i E x_j \right) \right], \quad k = 1, 2, \dots$$

and $\mathcal{F}_n$ is axiomatized by the union of both sets of sentences. As in Section 2, these sentences may be reduced to universal Horn sentences. None of these quasivarieties may be finitely axiomatizable, cf. [7].

Theorem 4 $\mathcal{F}_n = ISP(K_n)$, $\mathcal{J}_n = ISP(A_n)$.

Proof: If G is freely n-colorable and not complete, no pair $\{a, b\}$ of G with $a \neq b$ or $a \not{E} b$ can be critical since there are proper epimorphic colorings $h : G \to K_m \in \mathcal{F}_n$ with $h(a) \neq h(b)$ or $h(a) = h(b)$ (which implies $h(a) \not{E} h(b)$). Hence, only complete graphs may be s.i. in $\mathcal{F}_n$. Therefore, K_n generates $\mathcal{F}_n$ by the subdirect decomposition theorem.

Since $A_n \in \mathcal{J}_n$ and $\mathcal{J}_n$ is a quasivariety then ISP$(A_n) \subseteq \mathcal{J}_n$. Reciprocally, if $G \in \mathcal{J}_n$ is n-chromatic distinct from K_n then for pairs of G only Case 2 of the proof of Theorem 3 arises, and so G becomes a (full) subdirect power of A_n. If G is K_n or i-chromatic for $i < n$ then it is freely n -colorable, hence, embeddable in a power of K_n, and so embeddable in one of A_n. □

Corollary 5

(a) *An n-chromatic graph is a full subdirect power of K_n if and only if it is freely n-colorable.*

(b) *An n-chromatic graph is a full subdirect power of A_n if and only if it is jointly n-colorable, and distinct from K_n.*

Proof:

(a) By Theorem 4, G is freely n-colorable if and only if it has an embedding into K_n^I. Each projection of this representation must be full if G is n-chromatic.

(b) Shown in the second part of the proof of Theorem 4. □

We have proper inclusions: $\mathcal{F}_n \subset \mathcal{J}_n$ and $\mathcal{C}_n$, $\mathcal{D}_n \subset \mathcal{C}_n$ since $A_n \in \mathcal{J}_n - \mathcal{D}_n$ and $B_n \in \mathcal{C}_n - (\mathcal{J}_n \cup \mathcal{D}_n)$. The inclusion $\mathcal{F}_n \subseteq \mathcal{D}_n$ is more problematic. It is proper for $n \geq 4$ because $B_{n,2,2} \in \mathcal{D}_n - \mathcal{J}_n$, where

$$B_{n,2,2} = K_n + \{\overline{i\ n+1}, \overline{j\ n+2} : 1 \leq i, j \leq n, i \neq 1, 2, j \neq 3, 4\},$$

since colors 1 and 2 are available for the vertex $n + 1$ and colors 3, 4 for the vertex $n + 2$, and only those. But we do not know if the inclusion $\mathcal{F}_3 \subseteq \mathcal{D}_3$ is proper. On the other hand, it may be seen that $\mathcal{F}_2 = \mathcal{D}_2$, utilizing Theorem 4, because ISP$(K_2) = \{$disjoint unions of isolated vertices and edges$\} = \mathcal{D}_2$.

$B_{n,2,2}$ is subdirectly irreducible in $\mathcal{D}_n$ $(n \geq 4)$ with critical pair $\{b+1, n+2\}$; however, we have not been able to determine generators for the class $\mathcal{D}_n$, $n \geq 3$.

Questions Does $\mathcal{F}_3 = \mathcal{D}_3$? Does $B_{n,2,2}$ generate $\mathcal{D}_n$ for $n \geq 4$?

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