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# Direct and indirect influences of Jakob Bernoulli's *Ars conjectandi* in 18th century Great Britain

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**Abstract**

The *Ars Conjectandi* influenced British mathematicians in three different ways. By the second book, Bernoulli showed himself as a pioneer of combinatorial methods whose work was eventually translated into English in 1795. Jakob Bernoulli's tract on infinite series, which was appended to the Ar*s Conjectandi* was quoted by mathematicians concerned with the infinitesimal calculus, the most important being Colin Maclaurin. The *Ars Conjectandi* had the greatest impact through its first and fourth book and here especially on one author, Abraham de Moivre. This is visible in the transition from the basic concept of odds used in de Moivre's *Mensura Sortis* to the concept of probability in *The Doctrine of Chances*, for which de Moivre offers a first, albeit modest, theory. De Moivre's greatest achievement in the new theory of probability created by Jakob Bernoulli and himself was a form of the central limit theorem. It was inspired by Bernoulli's law of large numbers. Using an asymptotic series for n!, which he had developed together with James Stirling, and Jakob Bernoulli's formula for sums of powers of integers contained in the second book of the *Ars Conjectandi,* de Moivre was able to approximate the binomial distribution, as developed in the first book of the *Ars Conjectandi,* by the normal distribution. A similar deduction by Thomas Simpson, who followed closely the model of de Moivre, can be attributed to the indirect influence of the *Ars Conjectandi*. The problems arising from de Moivre's finding and his interpretation of it eventually led to Thomas Bayes' *Essay* and the discussion of its theological implications.

**Résumé**

L'*Ars Conjectandi* a influencé les mathématiciens Britanniques selon trois directions. Dans la deuxième partie de son livre Bernoulli apparaît comme un pionnier des méthodes combinatoires dont le travail fut finalement traduit en anglais en 1795. Le petit traité de Jakob Bernoulli sur les séries infinies, qui fut ajouté à l'*Ars Conjectandi*, fut cité par les mathématiciens qui s'intéressaient au calcul infinitésimal, le plus important d'entre eux étant Colin Maclaurin. C'est avec ses première et quatrième parties que l'*Ars Conjectandi* eut le plus grand impact, en particulier sur un auteur : Abraham de Moivre. On peut constater cela dans le passage du concept fondamental de cas utilisé par de Moivre dans son *De Mensura Sortis* au concept de probabilité qu'il utilise dans son *Doctrine of Chances* pour lequel il présente une première, quoique modeste, théorie. Le plus important perfectionnement apporté par de Moivre, de la nouvelle théorie des probabilités créée par Jakob Bernoulli et lui-même, fut une

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version du théorème de la limite centrale. Il lui fut inspiré par la loi des grands nombres de Bernoulli. Utilisant une série asymptotique pour n!, qu'il avait mise au point avec James Stirling, et la formule de Jakob Bernoulli pour les sommes de puissances des entiers contenue dans la deuxième partie de l'*Ars Conjectandi*, de Moivre fut en mesure d'approcher la distribution binomiale, telle qu'elle était présentée dans la première partie de l'*Ars Conjectandi*, par la distribution normale. Une déduction similaire de Thomas Simpson, qui suivit de très près le modèle de de Moivre, peut être attribuée à l'influence indirecte de l'*Ars Conjectandi*. Les problèmes issus de l'invention de de Moivre et l'interprétation qu'il en fit, conduisirent par la suite à l'*Essay* de Thomas Bayes et à la discussion de ses conséquences sur le plan théologique.

# 1. Introduction

Eight years after Jakob Bernoulli's death appeared his unfinished manuscript *Ars Conjectandi* unaltered in print in August 1713 together with a tract about infinite series and a letter in French on the *Jeu de Paume*, a predecessor of tennis[2]. The *Tractatus de seriebus infinitis* constitutes a reprint of the five “dissertations” about infinite series published between 1686 and 1704, which were conceived by Jakob Bernoulli from the very beginning as a coherent whole. Unlike the French letter on the *Jeu de Paume,* the tract on infinite series did not remain unnoticed by British mathematicians.

A short preface to the *Ars Conjectandi* was contributed by Nikolaus Bernoulli, Jakob's nephew, in which he asked Pierre Rémond de Montmort, the anonymous author of the *Essay sur les jeux de Hazard*, and Abraham de Moivre to complete his uncle's work.

In 1714 an English edition of Huygens' tract *De ratiociniis in ludo aleae* was published in London. The translator of this treatise is not mentioned on the titlepage of this booklet[3]; his name W. Browne is found at the end of the dedication for the famous Newtonian physician Richard Mead. In the advertisement to the reader Browne justifies his translation by hinting to the quality and "great scarcity" of Huygens' treatise, notwithstanding an earlier English edition of Huygens' tract by "the learned Dr. Arbuthnot"[4]. Brown refers also to "our excellent analyst M. De Moivre" who "likewise had wonderfully improv'd the Subject …", but he thinks that Huygens treatise was not made superfluous by de Moivre and his "more comprehensive and general" presentation of the subject, because Huygens' work can be used as a kind of introduction to de Moivre's. Browne also mentions that he has received information "within these few days" "that M. Montmort's French Piece is just newly reprinted at Paris, with very considerable Additions"[5]. Hard to believe, but of course pos sible, that Browne, who seems so well informed about contemporary works concerning games of chance and their mathematical treatment, did not know about the posthumous edition of the *Ars*

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[2] [Bernoulli, 1713]
[3] [Brown, 1714]
[4] [Arbuthnot, 1692] This monography was extended by the combinatorial methods as used by Pascal and Fermat. It was more than just an English translation of Huygens' treatise from 1657. Later editions up to the fourth revised by John Ham, which appeared in 1738, followed.
[5] [Montmort, 1713]

*conjectandi*. If he knew of it, one possible reason for not mentioning it was certainly that it could be considered as a competitor to his own English edition of Huygens' tract since this tract constitutes, together with Jakob Bernoulli's annotations, the first part of the *Ars conjectandi*. Concerning the Latin, every educated potential buyer in Great Britain could read it as easily as an English text, whereas the number of those who could read French in the United Kingdom of the 18th century was considerably smaller. De Moivre confirms this in a chapter of his *Miscellanea Analytica* from 1730 where he states that there are not so many who can compare Montmort's French text with the English text of the *Doctrine of Chances*[6]. In order to facilitate such a comparison de Moivre uses Latin in the *Miscellanea Analytica*.

Considering the demand for Huygens' treatise as claimed by Browne, and at the same time for more advanced treatments of the subject, the *Ars conjectandi* should have had better chances in the British book-market than Montmort's *Essay.*

If the number of British book and auction catalogues of the 18th century containing the *Ars conjectandi* or the *Essay* is indicative of the relative distribution of the two books in Great Britain then Bernoulli's posthumously published work was much more popular than Montmort's French book. Based on the online catalogue of Eighteenth Century Collections of British Books (ECCO) one finds that Montmort's *Essay* was only once mentioned in an auction catalogue of the library of Topham Beauclerk from 1781, which contained the second edition of de Moivre's *Doctrine of chances* from 1738 and also the first edition of his *Annuities upon lives* from 1725[7]. In contrast to Montmort's *Essay,* Jakob Bernoulli's *Ars conjectandi* appears in many catalogues for books on sale or for auctions in the United Kingdom of the 18th century. From such catalogues, it can be seen that book collectors like Martin Folkes, president of the Royal Society, whose library was sold in February 1756, William Burnet, the former governor of New England, Archibald Campbell (1682-1761), the third duke of Argyll, or the physician Thomas Pellet (*c*.1671-1744) owned the *Ars conjectandi.* Not so surprising, because of its intended juridical applications, it can also be found in the library of the faculty of advocates in Edinburgh.

Of course it can be doubted that all those who owned or had owned the *Ars conjectandi* had familiarized themselves with its content.

This is certainly different in the case of the authors who mentioned Jakob Bernoulli's *Ars conjectandi* in their publications and through them acted as disseminators of the results contained in this book. Of course, there are certainly a number of English authors who took

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[6] [Moivre, 1730] p. 152.

[7] [Paterson, 1781] Montmort's *Essay* from 1708 appears as number 4932. The following two numbers are de Moivre's two works.

advantage of the *Ars conjectandi* without quoting it. Reasons for doing so can include in the worst-case blunt plagiarism and can range from the desire to quote only compatriots to a kind of Newtonian chauvinism of those who considered Jakob Bernoulli an arch-Leibnizian.

## 2. The impact of the mathematical methods contained in the *Ars Conjectandi*

Indicative for the specific interest in a book in a foreign country are translations into the vernacular. The *Ars Conjectandi* was not translated in total into English in the 18th century, but the second book of it was translated into English and published late in the 18th century[8]. The most interesting result in the second part of the *Ars Conjectandi* was certainly the general formula for sums of powers of integers, which, in contrast to most of the other results, was entirely due to Jakob Bernoulli, and which had considerable influence especially in Great Britain. That other parts of Jakob Bernoulli's combinatorics were noticed in England testify e.g. the quarrels between Edward Waring and William Samuel Powell, a Cambridge celebrity, who had questioned Waring's mathematical ability in three published pamphlets. Waring had distributed a portion of his *Miscellanea analytica* in support of his candidature, which Powell anonymously attacked in order to serve the interests of William Ludlam, one of the competitors of Waring, and like Powell a fellow from St John's College, in *Observations on the First Chapter of a Book called 'Miscellanea analytica'* (1760). Waring replied to the piece, to which Powell answered, again anonymously, in his *Defence of the Observations*. Two more publications appeared, one by Waring and one by Powell. Waring had credited Jakob Bernoulli on the basis of the second part of the *Ars Conjectandi,* with the invention of what is now called mathematical induction. Powell tried to ridicule the new Professor and holder of the Lucasian chair by referring to Euclid who according to Powell had already used the same kind of proof[9]. Without judging the soundness of the ar guments exchanged between the two opponents, it is clear that both Waring and Powell were familiar at least with parts of the *Ars Conjectandi.* It should be added that Waring, whose *Miscellanea Analytica* were published in 1762, and in a second edition in 1785, mentioned Jakob Bernoulli several times

[8] [Bernoulli 1795 a and 1795 b]

[9] [Powell, 1760] p. 17-20.

in the preface[10]. Waring was especially interested in Bernoulli's tract on infinite series [11], which was recommended to his readers also by John Bonnycastle[12].

Perhaps less effective for the impact of the *Ars conjectandi* were hints in encyclopaedias like Ephraim Chambers', where Jakob Bernoulli is mentioned in the article on gaming[13]. Other examples of encyclopaedic works, which refer to Jakob Bernoulli's contributions to a theory of probability, are those of Croker and Hall. Both mention, in the article "algebra", the application of algebraic methods to the doctrine of chances by de Moivre and Jakob Bernoulli[14]. Croker and Hall belong to the second half of the 18th century in which Bernoulli's *Ars Conjectandi* could hardly stimulate the development of new mathematical results. Rather these references reflect a historical mood, typical for the time. This becomes more explicit with Charles Hutton and his *Tracts, Mathematical and Philosophical* published in 1786 in which Jakob Bernoulli is mentioned as one amongst "the more modern mathematicians" who gave demonstrations of the binomial theorem using mathematical induction[15].

Surprisingly I could not find any direct hint to the *Ars conjectandi* in the English philosophical and psychological literature of the 18th century, like in David Hume or in David Hartley. Especially Hartley was interested in questions, which had been asked by Jakob Bernoulli and de Moivre, but he mentioned only de Moivre when dealing with the problem of motivating Bernoulli's law of large numbers and refers to an unnamed "ingenious friend" for the solution of the inverse problem of estimating the unknown probability of an event from the number of times an event has occured and failed in $n$ independent trials [16]. One possible reason why Bernoulli was ignored by British philosophers is that he wrote as a mathematician a mathematical tract, which contained only in the fourth part a few passages capable of inducing further philosophical or psychological discussion.

From a mathematical point of view, the visible impact of the *Ars conjectandi* concentrates next to the annotated re-edition of Huygens' tract in book one, in which Jakob introduced the binomial or Bernoulli distribution, on the combinatorics of book two and on

[10] [Waring, 1762 and 1785] See there pp. XX, XXII and XXIV.
[11] See [Waring, 1784] especially p. 395 and 403.
[12] [Bonnycastle, 1788] where on p. 177 in a footnote different authors are mentioned who published on the "summation of series" amongst them "James Bernoulli"
[13] [Chambers, 1738]
[14] [Croker, 1764-1766] in vol. 1 and [Hall, 1791] also in vol. 1.
[15] [Hutton, 1786] p. 71 f.
[16] [Hartley, 1749] proposition 87.

book four, which introduces as the main concept for a theory of games of chance, probability, together with a measure of probability and a proof for the law of large numbers. In addition some English authors like Edward Waring refer to the tract on infinite series, which was added to the *Ars Conjectandi* proper.

## 3. The general influence of the *Ars conjectandi* on the *Doctrine of Chances* of de Moivre and his British successors

The mathematician in Great Britain who was most deeply influenced by Bernoulli's *Ars Conjectandi* and who seems to be the main source of later references to the *Ars Conjectandi* and Jakob Bernoulli was Abraham de Moivre, a Huguenot, who became a naturalized Englishman in 1705. De Moivre who made his living as a private teacher of mathematics had but little success in his attempts to gain reputation by concentrating on the new infinitesimal calculus[17]. He turned to the new field of the calculus of games of chance for which he published the tract *De mensura sortis* in the *Phil. Trans*. for 1711[18]. When de Moivre published his *Doctrine of chances*, the first edition of which appeared in 1718[19], he was familiar with the *Ars Conjectandi* and the second edition of Montmort's *Essay*, which was extended by the correspondence between Montmort and Jakob Bernoulli's nephew Nikolaus Bernoulli up to the end of 1713. Both books had been sent to de Moivre early in 1714[20]. De Moivre refers explicitly to the *Ars Conjectandi* at the end of his preface to the *Doctrine*, where he thanks Nikolaus Bernoulli for the invitation to continue the research program of his uncle together with Montmort.

What distinguishes de Moivre's *Doctrine* from the works of his predecessors is that he tried to familiarize his reader in an "introduction", which constitutes a kind of prototheory of the calculus of probabilities, with the main concepts and methods used in the book. Apparently de Moivre's introduction depends on results contained in the first part of the *Ars Conjectandi*, albeit without explicit reference to its author.

Bernoulli had started with Huygens' generalized concept of expectation *E*, according to which, if there are $p_i$ Chances to attain the amount $a_i$, $i = 1, 2, ..., n$

[17] [Schneider, 2004]
[18] [Moivre, 1711]
[19] [Moivre, 1718]
[20] So de Moivre confirms the arrival of the *Ars Conjectandi* in a letter to Nikolaus Bernoulli from March 3, 1714.

$$E = \frac{\sum_{i=1}^{n} p_i a_i}{\sum_{i=1}^{n} p_i}.$$

Unlike Huygens, Jakob Bernoulli understood the $a_i$ not only as amounts of money but also as "some kind of prize, laurels, victory, social status of a person or thing, public office, some kind of work, life or death". In addition in the solution of Huygens' problems Bernoulli normalized Huygens' $E$ by assuming $\sum_{i=1}^{n} a_i = 1$. If, as in all cases with two players contending for the stake $a$, $a = a_1 = 1$ and $a_2 = 0$, signifying loss of the game, the value of the expectation coincides with the probability of winning the game. In this case Huygens' expectation equals the classical measure of probability as introduced by Jakob Bernoulli in the first chapter of the fourth part of the *Ars Conjectandi*.

Huygens' propositions X to XIV deal with dicing problems of the type: What are the odds of throwing a given number of points with two or three dice? or: With how many throws of a die can one undertake it to throw a six or a double six? In proposition XII Huygens solved the problem "To find how many dice should one take to throw two sixes at the first throw."

Bernoulli's generalization of Huygens' propositions leads to the problem of finding the expectation, or, since the stakes are normalized to the amount 1, the probability, of a player who contends to achieve in a series of $n$ independent trials at least $m$ successes, if the chances of success and failure are as $b : c$ and $b + c = a$, or the probability of success is $\frac{b}{a}$ and that of failure $\frac{c}{a}$. Bernoulli finds inductively that this expectation or probability is

$$\sum_{\nu=0}^{n-m} \binom{n}{m+\nu} \left(\frac{b}{a}\right)^{m+\nu} \left(\frac{c}{a}\right)^{n-m-\nu}.$$

For this, Bernoulli considers the expectation $B(m,n)$ of the opponent who contends that there will be no more than $m-1$ successes in $n$ independent trials. Bernoulli uses the reduction formula $B(m,n) = \frac{b \cdot B(m-1,n-1) + c \cdot B(m,n-1)}{a}$ where $B(0,n) = 0$, $B(1,n) = \left(\frac{c}{a}\right)^n$, and $B(n,n) = 1 - \left(\frac{b}{a}\right)^n$ for all $n \geq 1$. He calculates $B(\nu,\mu)$, which he tabulates for $\nu \leq 4$ and $\mu \leq 6$ and extrapolates by incomplete induction

$$B(m,n) = \sum_{\mu=0}^{m-1} \binom{n}{\mu} \left(\frac{b}{a}\right)^{\mu} \left(\frac{c}{a}\right)^{n-\mu} .$$

He indicates how the same result can be achieved with combinatorial methods which were not used by Huygens and which Bernoulli is going to develop in the second part.

Bernoulli's procedure presupposes the equivalent of the multiplication rule for independent events, which he formulates most explicitly on p. 44. Following the multiplication rule, he determines the probability of exactly $m$ successes in $n$ independent trials if the probabilty of success is $\frac{b}{a}$ and that of failure $\frac{c}{a}$ and if the order of successes and failures does not matter, as

$$\binom{n}{m} \frac{b^m c^{n-m}}{a^n} = \binom{n}{n-m} \frac{b^m c^{n-m}}{a^n} .$$

This is the binomial, or as it was later called too, Bernoulli distribution.

De Moivre includes the binomial distribution only in the introduction to the *Doctrine* of 1738 and 1756, whereas the introduction to the first edition of the *Doctrine* ends with Bernoulli's solution of two special cases of Huygens' generalized poblems XI and XII. De Moivre uses, in part in his introduction, the same problems as Huygens and Jakob Bernoulli, in order to derive what is today called the negative binomial distribution and the binomial distribution.

Despite the methods he had developed, de Moivre depended to a certain degree on Jakob Bernoulli concerning the combinatorial and analytical methods applied by him to problems concerning games of chance. Even if de Moivre practically never refers to the second part of the *Ars Conjectandi* when he is dealing with combinatorial methods, but rather to van Schooten, Wallis, and later to Pascal, he must have read even this part of the *Ars Conjectandi,* as his later use of Bernoulli's formula for sums of powers of integers testifies. It was de Moivre who baptized the constants used by Bernoulli for the formation of the coefficients in his formula Bernoulli-numbers.

Concerning the problems dealing with contemporary popular games of chance, de Moivre seems to have been inspired more by Montmort and his own clients who informed him about the most fashionable games in French and British society, than by Bernoulli's *Ars Conjectandi*.

The increased number of references to Jakob Bernoulli and his *Ars conjectandi* in the *Miscellanea Analytica* and in the second edition of the *Doctrine* suggest that de Moivre read and used the *Ars conjectandi* repeatedly, when preparing the *Miscellanea Analytica* in the early 1720ies and again especially the second and fourth part in the early 1730ies when he derived his approximation of the binomial distribution by the normal distribution.

De Moivre's *Doctrine* is in part the result of a competition between de Moivre on one hand and Montmort together with Nikolaus Bernoulli on the other. De Moivre claimed, very much resented by Montmort, that his representation of the solutions of the then current problems tended to be more general than those of Montmort. This led to some arguments between the two men, which seemed to have been resolved during Montmort's visit in London in 1715, and after the publication of the first edition of the *Doctrine,* by Montmort's premature death in 1719. Some critical statements concerning de Moivre's conduct against Monmort, in the eloge for Montmort written by the secretary of the French Academy of Science, de Fontenelle, induced de Moivre to answer these criticisms as late as 1730 in his *Miscellanea Analytica*. There are eight instances where de Moivre refers to Jakob Bernoulli in the *Miscellanea Analytica*[21] including a long quotation from the *Ars Conjectandi* concerning Bernoulli's version of the law of large numbers[22]. Other references concern the determi nation of the maximum term in the binomial $(a+b)^n$, the formula for sums of powers of integers and the two main methods used by Jakob Bernoulli for the summation of series. One gets the impression that at least part of these references served also as arguments against the former claims of Montmort. This holds especially for Bernoulli's tract on series, of which de Moivre took some examples, in order to show how easy it was for Montmort to deduce from it results he claimed as his own[23].

Bernoulli's first method for the summation of series consists in subtracting from a determinate infinite series the same series without the first $n$ terms, so that the sum of the infinite series, the terms of which are the differences between term number $i$ and term number $(i+n+1)$ of the original series is equal to the sum of the first $n$ terms of the original series.

With the second method Bernoulli had according to de Moivre derived the following result[24]:

$$\sum_{i=0}^{\infty}\binom{i+d}{i}\cdot r^i=\sum_{i=0}^{\infty}\binom{i+d}{d}\cdot r^i=(1-r)^{-d-1}.$$

[21] [Moivre, 1730] p. 96-98, 107 f., 113, 125 f., 129-131, 138 f., 161 f. and 167.
[22] [Moivre, 1730 p.96-98.
[23] [Moivre, 1730] p. 160-163
[24] [Bernoulli, 1713] p. 245-248.

Starting with the convergent geometrical series $\sum_{i=0}^{\infty} r^i = (1-r)^{-1}$ Bernoulli considered the equations

$$\begin{aligned} 1 + r + r^2 + r^3 + r^4 + r^5 + \cdots &= (1-r)^{-1} \\ r + r^2 + r^3 + r^4 + r^5 + \cdots &= r \cdot (1-r)^{-1} \\ r^2 + r^3 + r^4 + r^5 + \cdots &= r^2 \cdot (1-r)^{-1} \\ r^3 + r^4 + r^5 + \cdots &= r^3 \cdot (1-r)^{-1} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

which he summed up

$$\sum_{j=0}^{\infty} r^j \sum_{i=0}^{\infty} r^i = \sum_{k=0}^{\infty} (k+1) \cdot r^k = \sum_{k=0}^{\infty} \binom{k+1}{1} \cdot r^k = \sum_{j=0}^{\infty} r^j (1-r)^{-1} = (1-r)^{-2};$$

in the same way he calculated

$$\sum_{j=0}^{\infty} r^j \sum_{i=0}^{\infty} \binom{i+1}{1} \cdot r^i = \sum_{k=0}^{\infty} r^k \sum_{i=0}^{k} \binom{i+1}{1} = \sum_{k=0}^{\infty} \binom{k+2}{2} \cdot r^k = \sum_{j=0}^{\infty} r^j \cdot (1-r)^{-2} = (1-r)^{-3}$$

and in general

$$\sum_{j=0}^{\infty} r^j \sum_{i=0}^{\infty} \binom{i+d}{d} \cdot r^i = \sum_{k=0}^{\infty} r^k \sum_{i=0}^{k} \binom{i+d}{d} = \sum_{k=0}^{\infty} \binom{k+d+1}{d+1} \cdot r^k = \sum_{j=0}^{\infty} r^j \cdot (1-r)^{-d-1} = (1-r)^{-d-2}$$

De Moivre made explicit the few steps necessary for Montmort in order to achieve his result.

Colin Maclaurin was one of the subscribers to the *Miscellanea Analytica* who had ordered six copies of the book, five for Scottish friends and colleagues who do not figure on the list of subscribers. He had studied his own copy of de Moivre's book carefully as one can see from his quotations in the two volumed *Treatise on Fluxions* from 1742. It could be that the references to Jakob Bernoulli and the *Ars Conjectandi* are originally motivated by reading the *Miscellanea Analytica*, because they concern nearly the same items, but the fact that Maclaurin gives exact page numbers where they are lacking in the *Miscellanea Analytica* constitutes an argument for Maclaurins personal acquaintance with the *Ars Conjectandi*.

Unlike de Moivre, Maclaurin seems to be less interested in a description of the results and methods in his references to Jakob Bernoulli and others, but rather in the fact that Bernoulli had found the same results, albeit with the help of another method. In detail Maclaurin refers to Jakob Bernoulli concerning the two methods for finding the sums of infinite series, which had been mentioned by de Moivre in the *Miscellanea Analytica*[25], Bernoulli's formula for the sums of powers of integers[26] and the properties of the binomial

[25] [Maclaurin, 1742] p. 294 (§ 354) and p. 301 (§360).
[26] [Maclaurin, 1742] p. 677 (§ 833).

coefficients of a binomial “when raised to high power”[27]. James Stirling, who entertained closed connections to Maclaurin and de Moivre, mentioned two estimates for the elastic curve contained in Bernoulli's tract on infinite series in his *Methodus Differentialis* from 1730[28].

Induced by the impact of de Moivre’s teaching and publications, or independently of them, some authors of books on games like Richard Seymour, who confesses in the dedication of the second part of his *Court gamester*[29] “I was very agreeable entertained by the writings of Ughen[30], Bernoulli and Montmort” or the immensely popular Edmond Hoyle in his numerous books on games[31] give direct or indirect hints to the *Ars Conjectandi.*

Certainly more serious from a mathematical point of view, are the remarks of James Dodson, a former student of de Moivre, who in the preface to the second volume of his *Mathematical repository* mentions the names of Huygens, Montmort, Bernoulli, de Moivre and Simpson as relevant for the “Doctrine of Chances”[32]. In 1753 when the second volume of Dodson’s *Mathematical repository* appeared the third posthumous edition of de Moivre’s *Doctrine* had to wait another three years. So Dodson’ reference was to the second edition of the *Doctrine* and, according to de Moivre, Simpson’s short version of it[33].

Compared with the first edition, de Moivre had added to the second (and third) edition of the *Doctrine* not only a much improved introduction but also many new problems and methods for their solution, which eventually made Montmort’s *Essay* obsolete.

## 4. Bernoulli's *Ars Conjectandi* and de Moivre's central limit theorem with its implications

De Moivre's greatest mathematical achievement in the later editions of the *Doctrine* is considered a form of the central limit theorem, which he found in 1733 at the age of 66 and which he communicated to a small group of his friends and students in printed form without considering this communication as a first publication. He understood his central limit theorem as a generalization and a sharpening of Bernoulli’s main theorem of the *Ars Conjectandi*,

27 [Maclaurin, 1742] p. 685 f. (§ 844)
28 [Stirling, 1730] p. 56 f. where he refers to a result contained in the *Ars Conjetandi*, p. 300.
29 [Seymour, 1732]
30 Not easy to recognize as a misspelling of Huygens. This and the difficulty to be „agreeable entertained“ by the writings of Huygens, Jakob Bernoulli and Montmort leave some doubts concerning Seymour’s reading of these works.
31 See [Hoyle, 1761] or [Hoyle, 1754].
32 [Dodson, 1748/1753] vol. 2, London 1753, p. VII.
33 [Simpson, 1740]

which was later named the law of large numbers by Poisson. In it, Bernoulli had shown that the relative frequency $h_{nt}$ of an event with probability $p = \frac{r}{t}$, $t = r + s$, in $nt$ independent trials converges in probability to $p$. More precisely, he had shown that, for any given small positive number $\varepsilon = \frac{1}{t}$ and any given large natural number $c$, for sufficiently large $n$ the inequality

$$\frac{\Pr\{|h_{nt} - p| \le \frac{1}{t}\}}{\Pr\{|h_{nt} - p| > \frac{1}{t}\}} > c$$

holds, which is equivalent to

$$\Pr\{|h_{nt} - p| \le \tfrac{1}{t}\} > \frac{c}{c+1} \text{ or } 1 > \Pr\{|h_{nt} - p| \le \tfrac{1}{t}\} > 1 - \frac{1}{c+1}$$

or to what is now called Bernoulli's weak law of large numbers.

In modern terms, de Moivre was, as opposed to Jakob Bernoulli, interested in the determination of $\varepsilon$ as a function of $n$ and $d$ in the equation:

$$\Pr\left(|h_n - p| \le \varepsilon\right) = d,$$

where $\Pr$ signifies the probability that the relative frequency $h_n$ of the appearances of an event in $n$ independent trials does not deviate from its expected value $p$ by more than $\varepsilon$.

De Moivre based his work in part on results achieved by Jakob and Nikolaus Bernoulli concerning binomial coefficients and their sums.

He started with the simplest case, the symmetric binomial ($p = \frac{1}{2}$) and $\varepsilon = \frac{l}{n}$. First he estimated the maximum term $b(m)$ in the sum

$$\sum_{k=m-l}^{m+l} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = 2^{-n} \cdot \sum_{k=m-l}^{m+l} \binom{n}{k} =: \sum_{i=-l}^{l} b(m+i)$$

for large $n$ $(n = 2m)$ as

$$b(m) \cong \frac{2}{\sqrt{2\pi n}}.$$

In a second step he considered the ratio of a term $b(m \pm l)$ with a distance $l$ ($l = O(\sqrt{n})$) from the maximum to the maximum term for which he found for large $n$:

$$\ln \frac{b(m \pm l)}{b(m)} \cong -\frac{2l^2}{n}.$$

De Moivre had worked on these estimations between 1721 and 1733; important for the final form of these estimations were asymptotic series for $n!$, $n$ large, to which both de Moivre and Stirling had contributed and which were published in 1730. For the development of this asymptotic series de Moivre used Jakob Bernoulli's formulas for sums of powers of integers extensively. With his estimations de Moivre could give the following approximation for large $n$:

$$P\left(\left|h_n - \frac{1}{2}\right| \le \frac{l}{n}\right) = 2^{-n} \sum_{k=m-l}^{m+l} \binom{n}{k} \cong \frac{4}{\sqrt{2\pi n}} \sum_{k=0}^{l} e^{-2k^2/n} \cong \frac{4}{\sqrt{2\pi n}} \int_{x=0}^{l} e^{-2x^2/n} dx.$$

De Moivre did not use this form of representation; especially the last integral was represented, typically for the Newtonian school, in form of the series

$$\frac{4}{\sqrt{2\pi n}} \sum_{i=0}^{\infty} \frac{(-1)^i 2^i l^{2i+1}}{i!(2i+1)n^i}.$$

De Moivre saw that this series converges numerically very fast for $l = \frac{1}{2}\sqrt{n}$. He calculated its value for $l = \frac{s}{2}\sqrt{n}$, $s$ = 1, 2, and 3.

In addition he gave the corresponding estimations for the general binomial (for any $p \neq \frac{1}{2}$). It can be shown that these estimations lead to the series[34]

$$\frac{2}{\sqrt{2\pi pqn}} \sum_{i=0}^{\infty} \frac{(-1)^i l^{2i+1}}{i!(2i+1)(2pqn)^i}$$

for

$$\Pr\left(\left|h_n - p\right| \le \tfrac{l}{n}\right)$$

which for $l = s\sqrt{npq}$ is the same as the one found by de Moivre for the case $p = \frac{1}{2}$. All this shows that de Moivre understood intuitively the importance of what was later called the standard deviation.

In this way de Moivre could show with his approximation of the binomial through the normal distribution, which he used in order to avoid the tedious calculations of the binomial distribution, that for large $n$ and an $\varepsilon = s\frac{\sqrt{npq}}{n}$ the probability $\Pr\left(\left|h_n - p\right| \le \varepsilon\right)$ is approximately 0,684... for $s = 1$, 0,954... for $s = 2$, and 0, 998... for $s = 3$.

The approximation of the binomial through the normal distribution with its consequences was the culmination of the *Doctrine* from the second edition on. He used this purely mathematical result as a means in order to combine the theory of probability with natural religion. In this sense de Moivre considered his form of the central limit theorem as a stochastic proof for the existence of order and design in nature or in his words[35]:

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[34] [Schneider, 1996]
[35] [Moivre, 1738] p. 234.

altho' Chance produces Irregularities, still the Odds will be infinitely great, that in process of Time, those Irregularities will bear no proportion to the recurrency of that Order which naturally results from Original Design.

In a second remark, which appeared only in the third edition of the *Doctrine* he added:

As upon the Supposition of a certain determinate Law according to which any Event is to happen, we demonstrate that the Ratio of Happenings will continually approach to that Law, as the Experiments or Observations are multiplied: so, *conversely*, if from numberless Observations we find the Ratio of the Events to converge to a determinate quantity, as to the Ratio of P to Q; then we conclude that this Ratio expresses the determinate Law according to which the Event is to happen. [...]
those laws serve to wise, useful and beneficent purposes; to preserve the stedfast Order of the Universe, to propagate the several Species of Beings, and furnish to the sentient Kind such degrees of happiness as are suited to their State. [...]
And hence [...] we shall be led, by a short and obvious way, to the acknowledgment of the great MAKER and GOVERNOUR of all; *Himself all-wise, all-powerful* and *good*.

De Moivre offers a solution of the inverse problem, namely to determine the unknown probability $p$ of an event by taking the known relative frequency $r_n$ of its occurence in a great number $n$ of independent trials as an estimate of it, and interprets this solution as a proof for the existence of God, whose attributes and function conform to the creator as imagined by Newton and Bayes.

For the general significance of his limit theorem and the seriousness of his investigations, de Moivre cites at the end of this second remark Jakob Bernoulli's own announcement of the law of large numbers in the *Ars Conjectandi* in English translation[36]:

*This is the problem which I am now to impart to the Publick, after having kept it by me for twenty years: new it is, and difficult; but of such excellent use, that it gives a high value and dignity to every other Branch of this Doctrine.*

De Moivre continues turned against his critics:

Yet ther are Writers, of a Class indeed very different from that of *James Bernoulli*, who insinuate as if the *Doctrine of Probabilities* could have no place in any serious Enquiry; and that Studies of this kind , trivial and easy as they be, rather disqualify a man for reasoning on every other subject.

De Moivre had addressed with his proof and the interpretation of the central limit theorem, as well as with the results achieved earlier in his *Miscellanea analytica,* all the topics dealt with by Bayes in his published papers. These are: Bayes' first publication on God's benevolence[37], to which de Moivre's two remarks following his central limit theorem in 1738 and 1756 could be considered as a reaction. In addition there are two papers published posthumously by Richard Price in the *Phil. Trans.* for 1764[38]. The first deals with the (asymptotic) series for $\log n!$ in the form developed by de Moivre in 1730 in his *Miscellaneis*

[36] [Moivre, 1756] p. 254.
[37] [Bayes, 1731]
[38] [Bayes, 1764] and [Bayes, 1763].

*Analyticis Supplementum* (p. 11). Bayes' second paper offers an attempt to solve a different form of the inverse problem, where the relative frequency of the occurence of an event with unknown probability is known only for a small numbers of trials; this reads

> Given the number of times in which an unknown event has happened and failed: Required the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named.

At least according to Price's introduction to Bayes' *Essay*, in which Price refers to de Moivre as "the great improver of this part of mathematics" and to Jakob Bernoulli as his predecessor, it is clear that the solution of the inverse problem serves even better the purpose to prove in the last instance the existence of God.

Recent research has revealed that Bayes' notebook contains excerpts from Colin Maclaurin's *Treatise of Fluxions* amongst them paragraph 844. In it Maclaurin deals with the ratio between the maximum term of $(a+b)^n$ and $(a+b)^n$ and remarks after references to de *Moivre's Miscellanea Analytica* and Stirling's *Methodus differentialis* that if the exponent of the binomial is great, the works to look at are the *Ars conjectandi* of Jakob Bernoulli and the *Doctrine of Chances* of de Moivre. So next to the work of de Moivre's, which is based in part on the *Ars Conjectandi*, we have another hint for at least a motive for Bayes to read Bernoulli's work.

But even if Bayes had read the *Ars Conjectandi* he had less reason to refer to it or to de Moivre's *Doctrine* than Simpson since Bayes tackled problems, which had not been solved before or which had been solved in an unsatisfactory way. Simpson, however, who according to his own preface depended to a great degree on the *Doctrine*, never mentioned Jakob Bernoulli or the *Ars Conjectandi*. Therefore the work of both Bayes and Simpson is at least in part due to the indirect influence of Jakob Bernoulli mediated by de Moivre.

That references to Jakob Bernoulli and his *Ars Conjectandi* in Britain after the publication of Bayes' *Essay* were restricted to bio-bibliographical or historical remarks has to do with the fact that the further rapid development of probability theory was carried out outside Britain. The exponent of this new phase of probability theory was Laplace who benefited from the immense progress of analysis on the continent, which had started with Leibniz, Jakob and Johann Bernoulli.

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