

Journal of Inequalities in Pure and Applied Mathematics

COEFFICIENT INEQUALITY FOR A FUNCTION WHOSE DERIVATIVE HAS A POSITIVE REAL PART

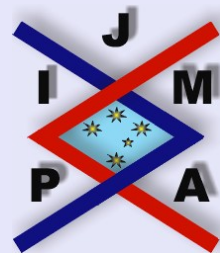
S. ABRAMOVICH, M. KLARIČIĆ BAKULA AND S. BANIĆ

Department of Mathematics
University of Haifa
Haifa, 31905, Israel.
EMail: abramos@math.haifa.ac.il

Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12, 21000 Split
Croatia.
EMail: milica@pmfst.hr

Faculty of Civil Engineering and Architecture
University of Split
Matice hrvatske 15, 21000 Split
Croatia.
EMail: Senka.Banic@gradst.hr

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Abstract

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Abstract

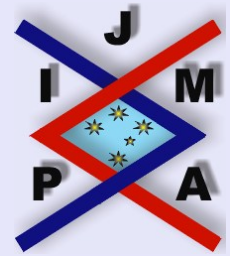
A variant of Jensen-Steffensen's inequality is considered for convex and for superquadratic functions. Consequently, inequalities for power means involving not only positive weights have been established.

2000 Mathematics Subject Classification: 26D15, 26A51.

Key words: Jensen-Steffensen's inequality, Monotonicity, Superquadraticity, Power means.

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1. Introduction.

Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ a convex function on I . If $\xi = (\xi_1, \dots, \xi_m)$ is any m -tuple in I^m and $\mathbf{p} = (p_1, \dots, p_m)$ any nonnegative m -tuple such that $\sum_{i=1}^m p_i > 0$, then the well known Jensen's inequality (see for example [7, p. 43])

$$(1.1) \quad f\left(\frac{1}{P_m} \sum_{i=1}^m p_i \xi_i\right) \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(\xi_i)$$

holds, where $P_m = \sum_{i=1}^m p_i$.

If f is strictly convex, then (1.1) is strict unless $\xi_i = c$ for all $i \in \{j : p_j > 0\}$.

It is well known that the assumption “ \mathbf{p} is a nonnegative m -tuple” can be relaxed at the expense of more restrictions on the m -tuple ξ .

If \mathbf{p} is a real m -tuple such that

$$(1.2) \quad 0 \leq P_j \leq P_m, \quad j = 1, \dots, m, \quad P_m > 0,$$

where $P_j = \sum_{i=1}^j p_i$, then for any monotonic m -tuple ξ (increasing or decreasing) in I^m we get

$$\bar{\xi} = \frac{1}{P_m} \sum_{i=1}^m p_i \xi_i \in I,$$

and for any function f convex on I (1.1) still holds. Inequality (1.1) considered under conditions (1.2) is known as the Jensen-Steffensen's inequality [7, p. 57] for convex functions.

In his paper [5] A. McD. Mercer considered some monotonicity properties of power means. He proved the following theorem:



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Theorem A. Suppose that $0 < a < b$ and $a \leq x_1 \leq x_2 \leq \dots \leq x_n \leq b$ hold with at least one of the x_k satisfying $a < x_k < b$. If $\mathbf{w} = (w_1, \dots, w_n)$ is a positive n -tuple with $\sum_{i=1}^n w_i = 1$ and $-\infty < r < s < +\infty$, then

$$a < Q_r(a, b; \mathbf{x}) < Q_s(a, b; \mathbf{x}) < b,$$

where

$$Q_t(a, b; \mathbf{x}) \equiv \left(a^t + b^t - \sum_{i=1}^n w_i x_i^t \right)^{\frac{1}{t}}$$

for all real $t \neq 0$, and

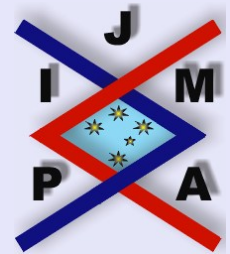
$$Q_0(a, b; \mathbf{x}) \equiv \frac{ab}{G}, \quad G = \prod_{i=1}^n x_i^{w_i}.$$

In his next paper [6], Mercer gave a variant of Jensen's inequality for which Witkowski presented in [8] a shorter proof. This is stated in the following theorem:

Theorem B. If f is a convex function on an interval containing an n -tuple $\mathbf{x} = (x_1, \dots, x_n)$ such that $0 < x_1 \leq x_2 \leq \dots \leq x_n$ and $\mathbf{w} = (w_1, \dots, w_n)$ is a positive n -tuple with $\sum_{i=1}^n w_i = 1$, then

$$f \left(x_1 + x_n - \sum_{i=1}^n w_i x_i \right) \leq f(x_1) + f(x_n) - \sum_{i=1}^n w_i f(x_i).$$

This theorem is a special case of the following theorem proved in [4] by Abramovich, Klaričić Bakula, Matić and Pečarić:



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Theorem C ([4, Th. 2]). Let $f : I \rightarrow \mathbb{R}$, where I is an interval in \mathbb{R} and let $[a, b] \subseteq I$, $a < b$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a monotonic n -tuple in $[a, b]^n$ and $\mathbf{v} = (v_1, \dots, v_n)$ a real n -tuple such that $v_i \neq 0$, $i = 1, \dots, n$, and $0 \leq V_j \leq V_n$, $j = 1, \dots, n$, $V_n > 0$, where $V_j = \sum_{i=1}^j v_i$. If f is convex on I , then

$$(1.3) \quad f\left(a + b - \frac{1}{V_n} \sum_{i=1}^n v_i x_i\right) \leq f(a) + f(b) - \frac{1}{V_n} \sum_{i=1}^n v_i f(x_i).$$

In case f is strictly convex, the equality holds in (1.3) iff one of the following two cases occurs:

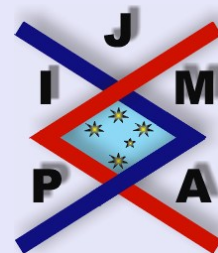
1. either $\bar{x} = a$ or $\bar{x} = b$,
2. there exists $l \in \{2, \dots, n-1\}$ such that $\bar{x} = x_1 + x_n - x_l$ and

$$(1.4) \quad \begin{cases} x_1 = a, x_n = b \text{ or } x_1 = b, x_n = a, \\ \bar{V}_j (x_{j-1} - x_j) = 0, j = 2, \dots, l, \\ V_j (x_j - x_{j+1}) = 0, j = l, \dots, n-1, \end{cases}$$

where $\bar{V}_j = \sum_{i=j}^n v_i$, $j = 1, \dots, n$ and $\bar{x} = (1/V_n) \sum_{i=1}^n v_i x_i$.

In the special case where $\mathbf{v} > 0$ and f is strictly convex, the equality holds in (1.4) iff $x_i = a$, $i = 1, \dots, n$, or $x_i = b$, $i = 1, \dots, n$.

Here, as in the rest of the paper, when we say that an n -tuple ξ is increasing (decreasing) we mean that $\xi_1 \leq \xi_2 \leq \dots \leq \xi_n$ ($\xi_1 \geq \xi_2 \geq \dots \geq \xi_n$). Similarly,



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when we say that a function $f : I \rightarrow \mathbb{R}$ is increasing (decreasing) on I we mean that for all $u, v \in I$ we have $u < v \Rightarrow f(u) \leq f(v)$ ($u < v \Rightarrow f(u) \geq f(v)$).

In Section 2 we refine Theorems A, B, and C. These refinements are achieved by superquadratic functions which were introduced in [1] and [2].

As Jensen's inequality for convex functions is a generalization of Hölder's inequality for $f(x) = x^p$, $p \geq 1$, so the inequalities satisfied by superquadratic functions are generalizations of the inequalities satisfied by the superquadratic functions $f(x) = x^p$, $p \geq 2$ (see [1], [2]).

First we quote some definitions and state a list of basic properties of superquadratic functions.

Definition 1.1. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$(1.5) \quad f(y) - f(x) - f(|y - x|) \geq C(x)(y - x)$$

for all $y \geq 0$.

Definition 1.2. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be strictly superquadratic if (1.5) is strict for all $x \neq y$ where $xy \neq 0$.

Lemma A ([2, Lemma 2.3]). Suppose that f is superquadratic. Let $\xi_i \geq 0$, $i = 1, \dots, m$, and let $\bar{\xi} = \sum_{i=1}^m p_i \xi_i$, where $p_i \geq 0$, $i = 1, \dots, m$, and $\sum_{i=1}^m p_i = 1$. Then

$$\sum_{i=1}^m p_i f(\xi_i) - f(\bar{\xi}) \geq \sum_{i=1}^m p_i f(|\xi_i - \bar{\xi}|).$$

Lemma B ([1, Lemma 2.2]). Let f be superquadratic function with $C(x)$ as in Definition 1.1. Then:



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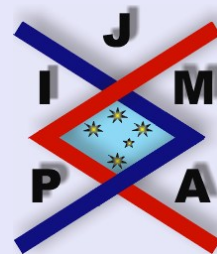


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(i) $f(0) \leq 0$,

(ii) if $f(0) = f'(0) = 0$ then $C(x) = f'(x)$ whenever f is differentiable at $x > 0$,

(iii) if $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

In [3] the following refinement of Jensen's Steffensen's type inequality for nonnegative superquadratic functions was proved:

Theorem D ([3, Th. 1]). Let $f : [0, \infty) \rightarrow [0, \infty)$ be a differentiable and superquadratic function, let ξ be a nonnegative monotonic m -tuple in \mathbb{R}^m and p a real m -tuple, $m \geq 3$, satisfying

$$0 \leq P_j \leq P_m, \quad j = 1, \dots, m, \quad P_m > 0.$$

Let $\bar{\xi}$ be defined as

$$\bar{\xi} = \frac{1}{P_m} \sum_{i=1}^m p_i \xi_i.$$

Then

$$(1.6) \quad \sum_{i=1}^m p_i f(\xi_i) - P_m f(\bar{\xi}) \\ \geq \sum_{i=1}^{k-1} P_i f(\xi_{i+1} - \xi_i) + P_k f(\bar{\xi} - \xi_k) \\ + \bar{P}_{k+1} f(\xi_{k+1} - \bar{\xi}) + \sum_{i=k+2}^m \bar{P}_i f(\xi_i - \xi_{i-1})$$

$$\begin{aligned} &\geq \left[\sum_{i=1}^k P_i + \sum_{i=k+1}^m \bar{P}_i \right] f \left(\frac{\sum_{i=1}^m p_i |\bar{\xi} - \xi_i|}{\sum_{i=1}^k P_i + \sum_{i=k+1}^m \bar{P}_i} \right) \\ &\geq (m-1) P_m f \left(\frac{\sum_{i=1}^m p_i |\bar{\xi} - \xi_i|}{(m-1) P_m} \right), \end{aligned}$$

where $\bar{P}_i = \sum_{j=i}^m p_j$ and $k \in \{1, \dots, m-1\}$ satisfies

$$\xi_k \leq \bar{\xi} \leq \xi_{k+1}.$$

In case f is also strictly superquadratic, inequality

$$\sum_{i=1}^m p_i f(\xi_i) - P_m f(\bar{\xi}) > (m-1) P_m f \left(\frac{\sum_{i=1}^m p_i (|\xi_i - \bar{\xi}|)}{(m-1) P_m} \right)$$

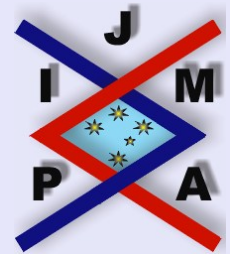
holds for $\xi > \mathbf{0}$ unless one of the following two cases occurs:

1. either $\bar{\xi} = \xi_1$ or $\bar{\xi} = \xi_m$,
2. there exists $k \in \{3, \dots, m-2\}$ such that $\bar{\xi} = \xi_k$ and

$$(1.7) \quad \begin{cases} P_j (\xi_j - \xi_{j+1}) = 0, & j = 1, \dots, k-1 \\ \bar{P}_j (\xi_j - \xi_{j-1}) = 0, & j = k+1, \dots, m. \end{cases}$$

In these two cases

$$\sum_{i=1}^m p_i f(\xi_i) - P_m f(\bar{\xi}) = 0.$$



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In Section 2 we refine Theorem B and Theorem C for functions which are superquadratic and positive. One of the refinements is derived easily from Theorem D.

We use in Section 3 the following theorem [7, p. 323] to give an alternative proof of Theorem B.

Theorem E. *Let I be an interval in \mathbb{R} , and ξ, η two decreasing m -tuples such that $\xi, \eta \in I^m$. Let p be a real m -tuple such that*

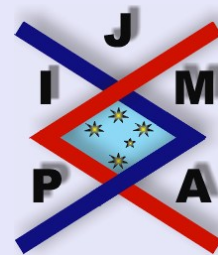
$$(1.8) \quad \sum_{i=1}^k p_i \xi_i \leq \sum_{i=1}^k p_i \eta_i$$

for $k = 1, 2, \dots, m - 1$, and

$$(1.9) \quad \sum_{i=1}^m p_i \xi_i = \sum_{i=1}^m p_i \eta_i .$$

Then for every continuous convex function $f : I \rightarrow \mathbb{R}$ we have

$$(1.10) \quad \sum_{i=1}^m p_i f(\xi_i) \leq \sum_{i=1}^m p_i f(\eta_i) .$$



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2. Variants of Jensen-Steffensen's Inequality for Positive Superquadratic Functions

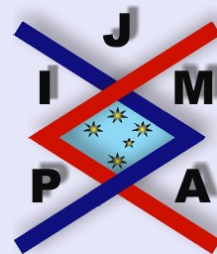
In this section we refine in two ways Theorem C for functions which are superquadratic and positive. The refinement in Theorem 2.1 follows by showing that it is a special case of Theorem D for specific p . The refinement in Theorem 2.2 follows the steps in the proof of Theorem B given by Witkowski in [8]. Therefore the second refinement is confined only to the specific p given in Theorem B, which means that what we get is a variant of Jensen's inequality and not of the more general Jensen-Steffensen's inequality.

Theorem 2.1. *Let $f : [0, \infty) \rightarrow [0, \infty)$ and let $[a, b] \subseteq [0, \infty)$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a monotonic n -tuple in $[a, b]^n$ and $\mathbf{v} = (v_1, \dots, v_n)$ a real n -tuple such that $v_i \neq 0, i = 1, \dots, n, 0 \leq V_j \leq V_n, j = 1, \dots, n$, and $V_n > 0$, where $V_j = \sum_{i=1}^j v_i$. If f is differentiable and superquadratic, then*

$$(2.1) \quad f(a) + f(b) - \frac{1}{V_n} \sum_{i=1}^n v_i f(x_i) - f\left(a + b - \frac{1}{V_n} \sum_{i=1}^n v_i x_i\right) \\ \geq (n+1) f\left(\frac{b - a - \frac{1}{V_n} \sum_{i=1}^n v_i \left|a + b - x_i - \frac{1}{V_n} \sum_{j=1}^n v_j x_j\right|}{n+1}\right).$$

In case f is also strictly superquadratic and $a > 0$, inequality (2.1) is strict unless one of the following two cases occurs:

1. either $\bar{x} = a$ or $\bar{x} = b$,



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2. there exists $l \in \{2, \dots, n-1\}$ such that $\bar{x} = x_1 + x_n - x_l$ and

$$(2.2) \quad \begin{cases} x_1 = a, x_n = b \text{ or } x_1 = b, x_n = a \\ \bar{V}_j(x_{j-1} - x_j) = 0, j = 2, \dots, l, \\ V_j(x_j - x_{j+1}) = 0, j = l, \dots, n-1, \end{cases}$$

where $\bar{V}_j = \sum_{i=j}^n v_i$, $j = 1, \dots, n$, and $\bar{x} = \frac{1}{V_n} \sum_{i=1}^n v_i x_i$.

In these two cases we have

$$f(a) + f(b) - \frac{1}{V_n} \sum_{i=1}^n v_i f(x_i) - f\left(a + b - \frac{1}{V_n} \sum_{i=1}^n v_i x_i\right) = 0.$$

In the special case where $v > \mathbf{0}$ and f is also strictly superquadratic, the equality holds in (2.1) iff $x_i = a$, $i = 1, \dots, n$, or $x_i = b$, $i = 1, \dots, n$.

Proof. Suppose that \mathbf{x} is an increasing n -tuple in $[a, b]^n$. The proof of the theorem is an immediate result of Theorem D, by defining the $(n+2)$ -tuples ξ and p as

$$\begin{aligned} \xi_1 &= a, & \xi_{i+1} &= x_i, \quad i = 1, \dots, n, & \xi_{n+2} &= b \\ p_1 &= 1, & p_{i+1} &= -v_i/V_n, \quad i = 1, \dots, n, & p_{n+2} &= 1. \end{aligned}$$

Then we get (2.1) from the last inequality in (1.6) and from the fact that in our special case we have

$$\sum_{i=1}^k P_i + \sum_{i=k+1}^{n+2} \bar{P}_i \leq n + 1,$$



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for $P_i = \sum_{j=1}^i p_j$ and $\bar{P}_i = \sum_{j=i}^{n+2} p_j$, and

$$\bar{\xi} = \frac{1}{P_m} \sum_{i=1}^m p_i \xi_i = a + b - \frac{1}{V_n} \sum_{i=1}^n v_i x_i = a + b - \bar{x}.$$

The proof of the equality case and the special case where $\mathbf{v} > \mathbf{0}$ follows also from Theorem **D**.

We have

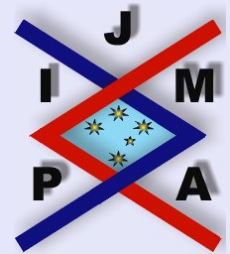
$$P_j = \frac{\bar{V}_j}{V_n}, \quad j = 1, \dots, n, \quad P_{n+1} = 0, \quad P_{n+2} = 1,$$

$$\bar{P}_1 = 1, \quad \bar{P}_2 = 0, \quad \bar{P}_j = \frac{V_{j-2}}{V_n}, \quad j = 3, \dots, n+2.$$

Obviously, $\bar{\xi} = \xi_1$ is equivalent to $\bar{x} = b$ and $\bar{\xi} = \xi_{n+2}$ is equivalent to $\bar{x} = a$. Also, the existence of some $k \in \{3, \dots, m-2\}$ such that $\bar{\xi} = \xi_k$ and that (1.7) holds is equivalent to the existence of some $l \in \{2, \dots, n-1\}$ such that $\bar{x} = x_1 + x_n - x_l = a + b - x_l$ and that (2.2) holds. Therefore, applying Theorem **D** we get the desired conclusions. In the case when \mathbf{x} is decreasing we simply replace \mathbf{x} and \mathbf{v} with $\tilde{\mathbf{x}} = (x_n, \dots, x_1)$ and $\tilde{\mathbf{v}} = (v_n, \dots, v_1)$, respectively, and then argue in the same manner.

In the special case that $\mathbf{v} > \mathbf{0}$ also, $V_i > 0$ and $\bar{V}_j > 0$, $i = 1, \dots, n$, and therefore according to (2.2) equality holds in (2.1) only when either $x_1 = \dots = x_n = a$ or $x_1 = \dots = x_n = b$. \square

In the following theorem we will prove a refinement of Theorem **B**. Without loss of generality we assume that $\sum_{i=1}^n v_i = 1$.



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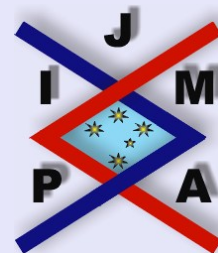
Theorem 2.2. Let $f : [0, \infty) \rightarrow [0, \infty)$ and let $[a, b] \subseteq [0, \infty)$, $a < b$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be an n -tuple in $[a, b]^n$ and $\mathbf{v} = (v_1, \dots, v_n)$ a real n -tuple such that $\mathbf{v} \geq \mathbf{0}$ and $\sum_{i=1}^n v_i = 1$. If f is superquadratic we have

$$\begin{aligned}
 & f(a) + f(b) - \sum_{i=1}^n v_i f(x_i) - f\left(a + b - \sum_{i=1}^n v_i x_i\right) \\
 & \geq \sum_{i=1}^n v_i f\left(\left|\sum_{j=1}^n v_j x_j - x_i\right|\right) \\
 & \quad + 2 \sum_{i=1}^n v_i \left[\frac{x_i - a}{b - a} f(b - x_i) + \frac{b - x_i}{b - a} f(x_i - a) \right] \\
 (2.3) \quad & \geq \sum_{i=1}^n v_i f\left(\left|\sum_{j=1}^n v_j x_j - x_i\right|\right) + 2 \sum_{i=1}^n v_i f\left(\frac{2(x_i - a)(b - x_i)}{b - a}\right).
 \end{aligned}$$

If f is strictly superquadratic and $\mathbf{v} > \mathbf{0}$ equality holds in (2.3) iff $x_i = a$, $i = 1, \dots, n$, or $x_i = b$, $i = 1, \dots, n$.

Proof. The proof follows the technique in [8] and refines the result to positive superquadratic functions. From Lemma A we know that for any $\lambda \in [0, 1]$ the following holds:

$$\begin{aligned}
 & \lambda f(a) + (1 - \lambda) f(b) - f(\lambda a + (1 - \lambda) b) \\
 & \geq \lambda f(|a - \lambda a - (1 - \lambda) b|) + (1 - \lambda) f(|b - \lambda a - (1 - \lambda) b|) \\
 & = \lambda f(|(1 - \lambda)(a - b)|) + (1 - \lambda) f(|\lambda(b - a)|) \\
 (2.4) \quad & = \lambda f((1 - \lambda)(b - a)) + (1 - \lambda) f(\lambda(b - a)).
 \end{aligned}$$



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Also, for any $x_i \in [a, b]$ there exists a unique $\lambda_i \in [0, 1]$ such that $x_i = \lambda_i a + (1 - \lambda_i) b$. We have

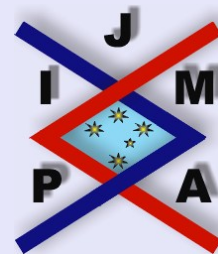
$$(2.5) \quad f(a) + f(b) - \sum_{i=1}^n v_i f(x_i) = f(a) + f(b) - \sum_{i=1}^n v_i f(\lambda_i a + (1 - \lambda_i) b).$$

Applying (2.4) on every $x_i = \lambda_i a + (1 - \lambda_i) b$ in (2.5) we obtain

$$\begin{aligned} & f(a) + f(b) - \sum_{i=1}^n v_i f(x_i) \\ & \geq f(a) + f(b) + \sum_{i=1}^n v_i \left[-\lambda_i f(a) - (1 - \lambda_i) f(b) \right. \\ & \quad \left. + \lambda_i f((1 - \lambda_i)(b - a)) + (1 - \lambda_i) f(\lambda_i(b - a)) \right] \\ (2.6) \quad & = \sum_{i=1}^n v_i [(1 - \lambda_i) f(a) + \lambda_i f(b)] \\ & \quad + \sum_{i=1}^n v_i [\lambda_i f((1 - \lambda_i)(b - a)) + (1 - \lambda_i) f(\lambda_i(b - a))]. \end{aligned}$$

Applying again (2.4) on (2.6) we get

$$\begin{aligned} (2.7) \quad & f(a) + f(b) - \sum_{i=1}^n v_i f(x_i) \geq \sum_{i=1}^n v_i f((1 - \lambda_i) a + \lambda_i b) \\ & \quad + 2 \sum_{i=1}^n v_i [\lambda_i f((1 - \lambda_i)(b - a)) + (1 - \lambda_i) f(\lambda_i(b - a))]. \end{aligned}$$



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Applying again Lemma A on (2.7) we obtain

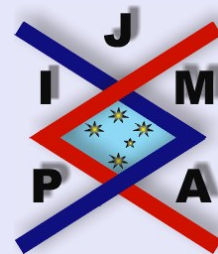
$$\begin{aligned}
 & f(a) + f(b) - \sum_{i=1}^n v_i f(x_i) \\
 & \geq f\left(\sum_{i=1}^n v_i [(1 - \lambda_i)a + \lambda_i b]\right) \\
 & \quad + \sum_{i=1}^n v_i f\left(\left|(1 - \lambda_i)a + \lambda_i b - \sum_{j=1}^n v_j [(1 - \lambda_j)a + \lambda_j b]\right|\right) \\
 & \quad + 2 \sum_{i=1}^n v_i [(1 - \lambda_i)f(\lambda_i(b - a)) + \lambda_i f((1 - \lambda_i)(b - a))] \\
 & = f\left(a + b - \sum_{i=1}^n v_i x_i\right) + \sum_{i=1}^n v_i f\left(\left|\sum_{j=1}^n v_j x_j - x_i\right|\right) \\
 (2.8) \quad & + 2 \sum_{i=1}^n v_i \left[\frac{x_i - a}{b - a} f(b - x_i) + \frac{b - x_i}{b - a} f(x_i - a)\right],
 \end{aligned}$$

and this is the first inequality in (2.3).

Since f is a nonnegative superquadratic function, from Lemma B we know that it is also convex, so from (2.8) we have

$$\sum_{i=1}^n v_i \left[\frac{x_i - a}{b - a} f(b - x_i) + \frac{b - x_i}{b - a} f(x_i - a)\right] \geq \sum_{i=1}^n v_i f\left(\frac{2(b - x_i)(x_i - a)}{b - a}\right),$$

hence, the second inequality in (2.3) is proved.



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For the case when f is strictly superquadratic and $\mathbf{v} > \mathbf{0}$ we may deduce that inequalities (2.6) and (2.7) become equalities iff each of the λ_i , $i = 1, \dots, n$, is either equal to 1 or equal to 0, which means that $x_i \in \{a, b\}$, $i = 1, \dots, n$. However, since we also have

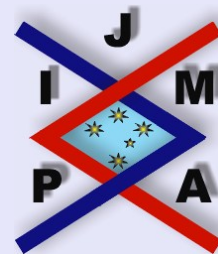
$$\sum_{j=1}^n v_j x_j - x_i = 0, \quad i = 1, \dots, n,$$

we deduce that $x_i = a$, $i = 1, \dots, n$, or $x_i = b$, $i = 1, \dots, n$.

This completes the proof of the theorem. \square

Corollary 2.3. *Let $\mathbf{v} = (v_1, \dots, v_n)$ be a real n -tuple such that $\mathbf{v} \geq \mathbf{0}$, $\sum_{i=1}^n v_i = 1$ and let $\mathbf{x} = (x_1, \dots, x_n)$ be an n -tuple in $[a, b]^n$, $0 < a < b$. Then for any real numbers r and s such that $\frac{s}{r} \geq 2$ we have*

$$\begin{aligned} & \left(\frac{Q_s(a, b; \mathbf{x})}{Q_r(a, b; \mathbf{x})} \right)^s - 1 \\ & \geq \frac{1}{Q_r(a, b; \mathbf{x})^s} \left[\sum_{i=1}^n v_i \left| \sum_{j=1}^n v_j x_j^r - x_i^r \right|^{\frac{s}{r}} \right. \\ (2.9) \quad & \left. + 2 \sum_{i=1}^n v_i \left(\frac{x_i^r - a^r}{b^r - a^r} (b^r - x_i^r)^{\frac{s}{r}} + \frac{b^r - x_i^r}{b^r - a^r} (x_i^r - a^r)^{\frac{s}{r}} \right) \right] \\ & \geq \frac{1}{Q_r(a, b; \mathbf{x})^s} \left[\sum_{i=1}^n v_i \left| \sum_{j=1}^n v_j x_j^r - x_i^r \right|^{\frac{s}{r}} \right] \end{aligned}$$



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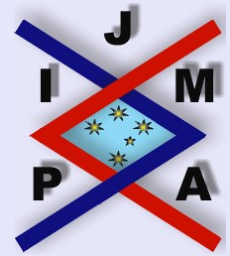
$$+2 \sum_{i=1}^n v_i \left(\frac{2(x_i^r - a^r)(b^r - x_i^r)}{b^r - a^r} \right)^{\frac{s}{r}},$$

where $Q_p(a, b; \mathbf{x}) = (a^p + b^p - \sum_{i=1}^n v_i x_i)^{\frac{1}{p}}$, $p \in \mathbb{R} \setminus \{0\}$.

If $\frac{s}{r} > 2$ and $\mathbf{v} > \mathbf{0}$, the equalities hold in (2.9) iff $x_i = a$, $i = 1, \dots, n$ or $x_i = b$, $i = 1, \dots, n$.

Proof. We define a function $f : (0, \infty) \rightarrow (0, \infty)$ as $f(x) = x^{\frac{s}{r}}$. It can be easily checked that for any real numbers r and s such that $\frac{s}{r} \geq 2$ the function f is superquadratic. We define a new positive n -tuple ξ in $[a^r, b^r]$ as $\xi_i = x_i^r$, $i = 1, \dots, n$. From Theorem 2.2 we have

$$\begin{aligned} & a^s + b^s - \sum_{i=1}^n v_i x_i^s - \left(a^r + b^r - \sum_{i=1}^n v_i x_i^r \right)^{\frac{s}{r}} \\ & \geq \sum_{i=1}^n v_i \left| \sum_{j=1}^n v_j x_j^r - x_i^r \right|^{\frac{s}{r}} \\ & \quad + 2 \sum_{i=1}^n v_i \left[\frac{x_i^r - a^r}{b^r - a^r} (b^r - x_i^r)^{\frac{s}{r}} + \frac{b^r - x_i^r}{b^r - a^r} (x_i^r - a^r)^{\frac{s}{r}} \right] \\ & \geq \sum_{i=1}^n v_i \left| \sum_{j=1}^n v_j x_j^r - x_i^r \right|^{\frac{s}{r}} \\ & \quad + 2 \sum_{i=1}^n v_i \left(\frac{2(x_i^r - a^r)(b^r - x_i^r)}{b^r - a^r} \right)^{\frac{s}{r}} \geq 0. \end{aligned} \tag{2.10}$$



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We have

$$a^s + b^s - \sum_{i=1}^n v_i x_i^s - \left(a^r + b^r - \sum_{i=1}^n v_i x_i^r \right)^{\frac{s}{r}} = Q_s(a, b; \mathbf{x})^s - Q_r(a, b; \mathbf{x})^s$$

so from (2.10) the inequalities in (2.9) follow.

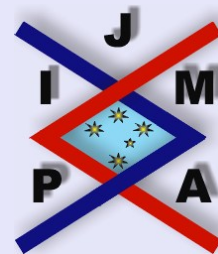
The equality case follows from the equality case in Theorem 2.2, as the function $f(x) = x^{\frac{s}{r}}$ is strictly superquadratic for $\frac{s}{r} > 2$. \square

Remark 1. *It is an immediate result of Corollary 2.3 that if $\frac{s}{r} > 2$ and there is at least one $j \in \{1, \dots, n\}$ such that*

$$v_j (x_j^r - a^r) (b^r - x_j^r) > 0,$$

then for this j we have

$$\left(\frac{Q_s(a, b; \mathbf{x})}{Q_r(a, b; \mathbf{x})} \right)^s - 1 > \frac{2v_j}{Q_r(a, b; \mathbf{x})^s} \left(\frac{2(x_j^r - a^r)(b^r - x_j^r)}{b^r - a^r} \right)^{\frac{s}{r}} > 0.$$



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3. An Alternative Proof of Theorem B

In this section we give an interesting alternative proof of Theorem B based on Theorem E. To carry out that proof we need the following technical lemma.

Lemma 3.1. *Let $\mathbf{y} = (y_1, \dots, y_m)$ be a decreasing real m -tuple and $\mathbf{p} = (p_1, \dots, p_m)$ a nonnegative real m -tuple with $\sum_{i=1}^m p_i = 1$. We define*

$$\bar{y} = \sum_{i=1}^m p_i y_i$$

and the m -tuple

$$\bar{\mathbf{y}} = (\bar{y}, \bar{y}, \dots, \bar{y}).$$

Then the m -tuples $\boldsymbol{\eta} = \mathbf{y}$, $\boldsymbol{\xi} = \bar{\mathbf{y}}$ and \mathbf{p} satisfy conditions (1.8) and (1.9).

Proof. Note that \bar{y} is a convex combination of y_1, y_2, \dots, y_m , so we know that

$$y_m \leq \bar{y} \leq y_1.$$

From the definitions of the m -tuples $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ we have

$$\sum_{i=1}^m p_i \xi_i = \bar{y} \sum_{i=1}^m p_i = \bar{y} = \sum_{i=1}^m p_i y_i = \sum_{i=1}^m p_i \eta_i.$$

Hence, condition (1.9) is satisfied. Furthermore, for $k = 1, 2, \dots, m - 1$ we



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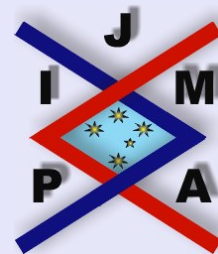
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have

$$\begin{aligned} \sum_{i=1}^k p_i \eta_i - \sum_{i=1}^k p_i \xi_i &= \sum_{i=1}^k p_i y_i - \bar{y} \sum_{i=1}^k p_i \\ &= \sum_{i=1}^k p_i y_i - \sum_{j=1}^m p_j y_j \sum_{i=1}^k p_i . \end{aligned}$$

Since $\sum_{i=1}^m p_i = 1$, we can write

$$\begin{aligned} &\sum_{i=1}^k p_i \eta_i - \sum_{i=1}^k p_i \xi_i \\ &= \left(\sum_{j=1}^k p_j + \sum_{j=k+1}^m p_j \right) \sum_{i=1}^k p_i y_i - \left(\sum_{j=1}^k p_j y_j + \sum_{j=k+1}^m p_j y_j \right) \sum_{i=1}^k p_i \\ &= \sum_{j=k+1}^m p_j \sum_{i=1}^k p_i y_i - \sum_{i=1}^k p_i \sum_{j=k+1}^m p_j y_j \\ &= \sum_{i=1}^k p_i \left(\sum_{j=k+1}^m p_j y_i - \sum_{j=k+1}^m p_j y_j \right) \\ &= \sum_{i=1}^k p_i \sum_{j=k+1}^m p_j (y_i - y_j) . \end{aligned}$$



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Since p is nonnegative and y is decreasing, we obtain

$$\sum_{i=1}^k p_i \eta_i - \sum_{i=1}^k p_i \xi_i \geq 0, \quad k = 1, 2, \dots, m-1,$$

which means that condition (1.8) is satisfied as well. \square

Now we can give an alternative proof of Theorem B which is mainly based on Theorem E.

Proof of Theorem B. Since $\bar{x} = \sum_{i=1}^n w_i x_i$ is a convex combination of x_1, x_2, \dots, x_n it is clear that there is an $s \in \{1, 2, \dots, n-1\}$ such that

$$x_1 \leq \dots \leq x_s \leq \bar{x} \leq x_{s+1} \leq \dots \leq x_n,$$

that is,

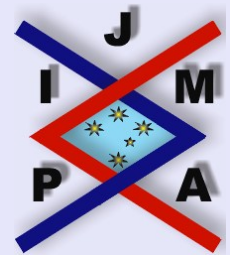
$$(3.1) \quad -x_1 \geq \dots \geq -x_s \geq -\bar{x} \geq -x_{s+1} \geq \dots \geq -x_n.$$

Adding $x_1 + x_n$ to all the inequalities in (3.1) we obtain

$$x_n \geq \dots \geq x_1 + x_n - x_s \geq x_1 + x_n - \bar{x} \geq x_1 + x_n - x_{s+1} \geq \dots \geq x_1,$$

which gives us

$$(3.2) \quad x_1 + x_n - \bar{x} = x_1 + x_n - \sum_{i=1}^n w_i x_i \in [x_1, x_n].$$



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We use (1.10) to prove the theorem. For this, we define the $(n + 2)$ -tuples ξ , η and p as follows:

$$\begin{aligned} \eta_1 &= x_n, \eta_2 = x_n, \eta_3 = x_{n-1}, \dots, \eta_n = x_2, \eta_{n+1} = x_1, \eta_{n+2} = x_1, \\ p_1 &= 1, p_2 = -w_n, p_3 = -w_{n-1}, \dots, p_n = -w_2, p_{n+1} = -w_1, p_{n+2} = 1, \\ \xi_1 &= \xi_2 = \dots = \xi_{n+2} = \bar{\eta}, \quad \bar{\eta} = \sum_{i=1}^{n+2} p_i \eta_i = x_1 + x_n - \sum_{j=1}^n w_j x_j. \end{aligned}$$

It is easily verified that ξ and η are decreasing and that $\sum_{i=1}^{n+2} p_i = 1$. It remains to see that ξ , η and p satisfy conditions (1.8) and (1.9).

Condition (1.9) is trivially fulfilled since

$$\sum_{i=1}^{n+2} p_i \xi_i = \bar{\eta} \sum_{i=1}^{n+2} p_i = \bar{\eta} = \sum_{i=1}^{n+2} p_i \eta_i.$$

Further, we have $\xi_i = \bar{\eta}$, $i = 1, 2, \dots, n + 2$. To prove (1.8), we need to demonstrate that

$$(3.3) \quad \bar{\eta} \sum_{i=1}^k p_i \leq \sum_{i=1}^k p_i \eta_i, \quad k = 1, 2, \dots, n + 1.$$

For $k = 1$, (3.3) becomes $\bar{\eta} \leq x_n$, and this holds because of (3.2). On the other hand, for $k = n + 1$, (3.3) becomes

$$\bar{\eta} \left(1 - \sum_{i=1}^n w_i \right) \leq x_n - \sum_{i=1}^n w_i x_i,$$



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that is,

$$0 \leq x_n - \bar{x},$$

and this holds because of (3.2).

If $k \in \{2, \dots, n\}$, (3.3) can be rewritten and in its stead we have to prove that

$$(3.4) \quad \bar{\eta} \left(1 - \sum_{i=n+2-k}^n w_i \right) \leq x_n - \sum_{i=n+2-k}^n w_i x_i.$$

Let us consider the decreasing n -tuple \mathbf{y} , where

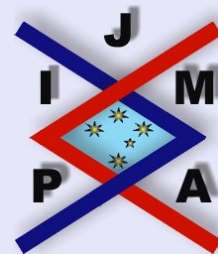
$$y_i = x_1 + x_n - x_i, \quad i = 1, 2, \dots, n.$$

We have

$$\begin{aligned} \bar{y} &= \sum_{i=1}^n w_i y_i \\ &= \sum_{i=1}^n w_i (x_1 + x_n - x_i) \\ &= x_1 + x_n - \sum_{i=1}^n w_i x_i = x_1 + x_n - \bar{x} = \bar{\eta}. \end{aligned}$$

If we apply Lemma 3.1 to the n -tuple \mathbf{y} and to the weights \mathbf{w} , then $m = n$ and for all $l \in \{1, 2, \dots, n-1\}$ the inequality

$$\bar{y} \sum_{i=1}^l w_i \leq \sum_{i=1}^l w_i (x_1 + x_n - x_i)$$



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holds. Taking into consideration that $\bar{y} = \bar{\eta}$, $\sum_{i=1}^l w_i = 1 - \sum_{i=l+1}^n w_i$ and changing indices as $l = n + 1 - k$, we deduce that

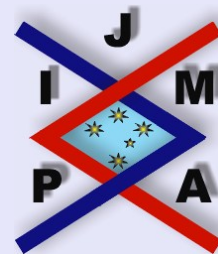
$$(3.5) \quad \bar{\eta} \left(1 - \sum_{i=n+2-k}^n w_i \right) \leq \sum_{i=1}^{n+1-k} w_i (x_1 + x_n - x_i),$$

for all $k \in \{2, \dots, n\}$. The difference between the right side of (3.4) and the right side of (3.5) is

$$\begin{aligned} x_n - \sum_{i=n+2-k}^n w_i x_i - \sum_{i=1}^{n+1-k} w_i (x_1 + x_n - x_i) \\ &= x_n - \sum_{i=n+2-k}^n w_i x_i - x_n \sum_{i=1}^{n+1-k} w_i - \sum_{i=1}^{n+1-k} w_i (x_1 - x_i) \\ &= x_n \left(1 - \sum_{i=1}^{n+1-k} w_i \right) - \sum_{i=n+2-k}^n w_i x_i - \sum_{i=1}^{n+1-k} w_i (x_1 - x_i) \\ &= x_n \sum_{i=n+2-k}^n w_i - \sum_{i=n+2-k}^n w_i x_i - \sum_{i=1}^{n+1-k} w_i (x_1 - x_i) \\ &= \sum_{i=n+2-k}^n w_i (x_n - x_i) + \sum_{i=1}^{n+1-k} w_i (x_i - x_1) \geq 0, \end{aligned}$$

since w is nonnegative and x is increasing. Therefore, the inequality

$$(3.6) \quad \sum_{i=1}^{n+1-k} w_i (x_1 + x_n - x_i) \leq x_n - \sum_{i=n+2-k}^n w_i x_i$$



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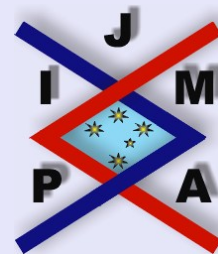
holds for all $k \in \{2, \dots, n\}$. From (3.5) and (3.6) we obtain (3.4). This completes the proof that the m -tuples ξ , η and p satisfy conditions (1.8) and (1.9) and we can apply Theorem E to obtain

$$\sum_{i=1}^{n+2} p_i f(\bar{\eta}) \leq f(x_n) - \sum_{i=1}^n w_i f(x_i) + f(x_1).$$

Taking into consideration that $\sum_{i=1}^{n+2} p_i = 1$ and $\bar{\eta} = x_1 + x_n - \sum_{j=1}^n w_j x_j$ we finally get

$$f\left(x_1 + x_n - \sum_{i=1}^n w_i x_i\right) \leq f(x_1) + f(x_n) - \sum_{i=1}^n w_i f(x_i).$$

□



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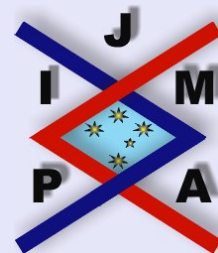
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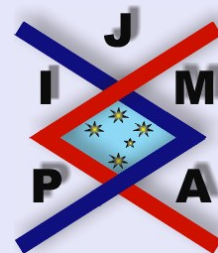
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