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THE DZIOK-SRIVASTAVA OPERATOR AND $k-$UNIFORMLY STARLIKE FUNCTIONS

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ABSTRACT. Inclusion relations for $k-$uniformly starlike functions under the Dziok-Srivastava operator are established. These results are also extended to $k-$uniformly convex functions, close-to-convex, and quasi-convex functions.

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1. INTRODUCTION

Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disc $U = \{z : |z| < 1\}$. A function $f \in A$ is said to be in $UST(k,\gamma)$, the class of $k-$uniformly starlike functions of order $\gamma,\ 0 \le \gamma < 1,$ if f satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \qquad k \ge 0. \tag{1.1}$$

Replacing f in (1.1) by zf' we obtain the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \qquad k \ge 0 \tag{1.2}$$

required for the function f to be in the subclass $UCV(k,\gamma)$ of $k-$uniformaly convex functions of order γ.

Uniformly starlike and convex functions were first introduced by Goodman [5] and then studied by various authors. For a wealth of references, see Ronning [13].

Setting

$$\Omega_{k,\gamma} = \left\{u + iv; u > k\sqrt{(u-1)^2 + v^2} + \gamma\right\},$$

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with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ and considering the functions which map U on to the conic domain $\Omega_{k,\gamma}$, such that $1 \in \Omega_{k,\gamma}$, we may rewrite the conditions (1.1) or (1.2) in the form

$$p(z) \prec q_{k,\gamma}(z). \tag{1.3}$$

We note that the explicit forms of function $q_{k,\gamma}$ for $k = 0$ and $k = 1$ are

$$q_{0,\gamma}(z) = \frac{1 + (1 - 2\gamma)z}{1 - z}, \text{ and } q_{1,\gamma}(z) = 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2.$$

For $0 < k < 1$ we obtain

$$q_{k,\gamma}(z) = \frac{1-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}(\arccos k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2-\gamma}{1-k^2},$$

and if $k > 1$, then $q_{k,\gamma}$ has the form

$$q_{k,\gamma}(z) = \frac{1-\gamma}{k^2-1}\sin\left(\frac{\pi}{2K(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right) + \frac{k^2-\gamma}{k^2-1},$$

where $u(z) = \frac{z-\sqrt{k}}{1-\sqrt{k}z}$ and K is such that $k = \cosh\frac{\pi K'(z)}{4K(z)}$.

By virtue of (1.3) and the properties of the domains $\Omega_{k,\gamma}$ we have

$$\Re(p(z)) > \Re(q_{k,\gamma}(z)) > \frac{k+\gamma}{k+1}. \tag{1.4}$$

Define $UCC(k, \gamma, \beta)$ to be the family of functions $f \in A$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) \geq k\left|\frac{zf'(z)}{g(z)} - 1\right| + \gamma, \quad k\geq 0,\ 0\leq\gamma < 1$$

for some $g \in UST(k, \beta)$.

Similarly, we define $UQC(k, \gamma, \beta)$ to be the family of functions $f \in A$ such that

$$\Re\left(\frac{(zf'(z))'}{g'(z)}\right) \geq k\left|\frac{(zf'(z))'}{g'(z)} - 1\right| + \gamma, \qquad k \geq 0,\ \ 0 \leq \gamma < 1$$

for some $g \in UCV(k, \beta)$.

We note that $UCC(0, \gamma, \beta)$ is the class of close-to-convex functions of order γ and type β and $UQC(0, \gamma, \beta)$ is the class of quasi-convex functions of order γ and type β.

The aim of this paper is to study the inclusion properties of the above mentioned classes under the following linear operator which is defined by Dziok and Srivastava [3].

For $\alpha_j \in C$ $(j = 1, 2, 3, \dots, l)$ and $\beta_j \in C - \{0, -1, -2, \dots\}$ $(j = 1, 2, \dots m)$, the generalized hypergeometric function is defined by

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n}\cdot\frac{z^n}{n!},$$
$$(l \leq m + 1; l, m \in N_0 = \{0, 1, 2, \dots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N} = \{1, 2, \dots\}$ and 1 when $n = 0$.

Corresponding to the function $h(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z) = z\,{}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)$ the Dziok-Srivastava operator [3], $H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)$ is defined by

$$\begin{aligned} H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z) &= h(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_m)_{n-1}} \cdot \frac{a_n z^n}{(n-1)!.} \end{aligned}$$

where "$*$" stands for convolution.

It is well known [3] that

$$\begin{aligned} \alpha_1 H_m^l(\alpha_1+1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z) &= z[H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z)]' \\ &\quad + (\alpha_1-1)H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z). \end{aligned} \tag{1.5}$$

To make the notation simple, we write,

$$H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z).$$

We note that many subclasses of analytic functions, associated with the Dziok-Srivastava operator $H_m^l[\alpha_1]$ and many special cases, were investigated recently by Dziok-Srivastava [3], Liu [7], Liu and Srivastava [9], [10] and others. Also we note that special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [6], the Carlson-Shaffer operator [2], the Ruscheweyh derivative operator [14], the generalized Bernardi-Libera-Livingston linear operator (cf. [1]) and the Srivastava-Owa fractional derivative operators (cf. [11], [12]).

2. MAIN RESULTS

In this section we prove some results on the linear operator $H_m^l[\alpha_1]$. First is the inclusion theorem.

Theorem 2.1. *Let* $\Re\alpha_1 > \frac{1-\gamma}{k+1}$, *and* $f \in A$. *If* $H_m^l[\alpha_1+1]f \in UST(k,\gamma)$ *then* $H_m^l[\alpha_1]f \in UST(k,\gamma)$.

In order to prove the above theorem we shall need the following lemma which is due to Eenigenburg, Miller, Mocanu, and Read [4].

Lemma A. *Let* β,γ *be complex constants and* h *be univalently convex in the unit disk* U *with* $h(0)=c$ *and* $\Re(\beta h(z)+\gamma)>0$. *Let* $g(z) = c + \sum_{n=1}^{\infty} p_n z^n$ *be analytic in* U. *Then*

$$g(z) + \frac{zg'(z)}{\beta g(z)+\gamma} \prec h(z) \Rightarrow g(z) \prec h(z).$$

Proof of Theorem 2.1. Setting $p(z) = z(H_m^l[\alpha_1]f(z))'/(H_m^l[\alpha_1]f(z))$ in (1.5) we can write

$$\alpha_1 \frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} = \frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} + (\alpha_1-1) = p(z) + (\alpha_1-1). \tag{2.1}$$

Differentiating (2.1) yields

$$\frac{z(H_m^l[\alpha_1+1]f(z))'}{H_m^l[\alpha_1+1]} = p(z) + \frac{zp'(z)}{p(z)+(\alpha_1-1)}. \tag{2.2}$$

From this and the argument given in Section 1 we may write

$$p(z) + \frac{zp'(z)}{p(z)+(\alpha_1-1)} \prec q_{k,\gamma}(z).$$

Therefore the theorem follows by Lemma A and the condition (1.4) since $q_{k,\gamma}$ is univalent and convex in U and $\Re(q_{k,\gamma}) > \frac{k+\gamma}{k+1}$. □

Theorem 2.2. *Let* $\Re\alpha_1 > \frac{1-\gamma}{k+1}$*, and* $f \in A$*. If* $H_m^l[\alpha_1+1]f \in UCV(k,\gamma)$ *then* $H_m^l[\alpha_1]f \in UCV(k,\gamma)$.

Proof. By virtue of (1.1), (1.2) and Theorem 2.1 we have

$$\begin{aligned} H_m^l[\alpha_1+1]f \in UCV(k,\gamma) &\Leftrightarrow z\left(H_m^l[\alpha_1+1]f\right)' \in UST(k,\gamma) \\ &\Leftrightarrow H_m^l[\alpha_1+1]zf' \in UST(k,\gamma) \\ &\Rightarrow H_m^l[\alpha_1]zf' \in UST(k,\gamma) \\ &\Leftrightarrow H_m^l[\alpha_1]f \in UCV(k,\gamma). \end{aligned}$$

and the proof is complete. □

We next prove

Theorem 2.3. *Let* $\Re\alpha_1 > \frac{1-\gamma}{k+1}$*, and* $f \in A$*. If* $H_m^l[\alpha_1+1]f \in UCC(k,\gamma,\beta)$ *then* $H_m^l[\alpha_1]f \in UCC(k,\gamma,\beta)$.

To prove the above theorem, we shall need the following lemma which is due to Miller and Mocanu [10].

Lemma B. *Let* h *be convex in the unit disk* U *and let* $E \geq 0$*. Suppose* $B(z)$ *is analytic in* U *with* $\Re B(z) \geq E$*. If* g *is analytic in* U *and* $g(0) = h(0)$*. Then*

$$Ez^2g''(z) + B(z)zg'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

Proof of Theorem 2.3. Since $H_m^l[\alpha_1+1]f \in UCC(k,\gamma,\beta)$, by definition, we can write

$$\frac{z(H_m^l[\alpha_1+1]f)'(z)}{k(z)} \prec q_{k,\gamma}(z)$$

for some $k(z) \in UST(k,\beta)$. For g such that $H_m^l[\alpha_1+1]g(z) = k(z)$, we have

$$\frac{z(H_m^l[\alpha_1+1]f)'(z)}{H_m^l[\alpha_1+1]g(z)} \prec q_{k,\gamma}(z). \tag{2.3}$$

Letting $h(z) = \frac{z(H_m^l[\alpha_1]f)'(z)}{(H_m^l[\alpha_1]g)(z)}$ and $H(z) = \frac{z(H_m^l[\alpha_1]g)'(z)}{H_m^l[\alpha_1]g(z)}$ we observe that h and H are analytic in U and $h(0) = H(0) = 1$. Now, by Theorem 2.1, $H_m^l[\alpha_1]g \in UST(k,\beta)$ and so $\Re H(z) > \frac{k+\beta}{k+1}$. Also, note that

$$z(H_m^l[\alpha_1]f)'(z) = (H_m^l[\alpha_1]g(z))h(z). \tag{2.4}$$

Differentiating both sides of (2.4) yields

$$\frac{z(H_m^l[\alpha_1](zf'))'(z)}{H_m^l[\alpha_1]g(z)} = \frac{z(H_m^l[\alpha_1]g)'(z)}{H_m^l[\alpha_1]g(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z).$$

Now using the identity (1.5) we obtain

$$\begin{aligned} \frac{z(H_m^l[\alpha_1+1]f)'(z)}{H_m^l[\alpha_1+1]g(z)} &= \frac{H_m^l[\alpha_1+1](zf')(z)}{H_m^l[\alpha_1+1]g(z)} \\ &= \frac{z(H_m^l[\alpha_1](zf'))'(z)+(\alpha_1-1)H_m^l[\alpha_1](zf')(z)}{z(H_m^l[\alpha_1]g)'(z)+(\alpha_1-1)H_m^l[\alpha_1]g(z)} \\ &= \frac{\frac{z(H_m^l[\alpha_1](zf'))'(z)}{H_m^l[\alpha_1]g(z)}+(\alpha_1-1)\frac{H_m^l[\alpha_1](zf')(z)}{H_m^l[\alpha_1]g(z)}}{\frac{z(H_m^l[\alpha_1]g)'(z)}{H_m^l[\alpha_1]g(z)}+(\alpha_1-1)} \\ &= \frac{H(z)h(z)+zh'(z)+(\alpha_1-1)h(z)}{H(z)+(\alpha_1-1)} \\ &= h(z)+\frac{1}{H(z)+(\alpha_1-1)}zh'(z). \end{aligned} \tag{2.5}$$

From (2.3), (2.4), and (2.5) we conclude that

$$h(z)+\frac{1}{H(z)+(\alpha_1-1)}zh'(z) \prec q_{k,\gamma}(z).$$

On letting $E=0$ and $B(z)=\frac{1}{H(z)+(\alpha_1-1)}$, we obtain

$$\Re(B(z))=\frac{1}{|(\alpha_1-1)+H(z)|^2}\Re((\alpha_1-1)+H(z))>0.$$

The above inequality satisfies the conditions required by Lemma B. Hence $h(z)\prec q_{k\gamma}(z)$ and so the proof is complete. □

Using a similar argument to that in Theorem 2.2 we can prove

Theorem 2.4. *Let* $\Re\alpha_1>\frac{1-\gamma}{k+1}$*, and* $f\in A$*. If* $H_m^l[\alpha_1+1]f\in UQC(k,\gamma,\beta)$*, then* $H_m^l[\alpha_1]f\in UQC(k,\gamma,\beta)$*.*

Finally, we examine the closure properties of the above classes of functions under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt, \quad c>-1.$$

Theorem 2.5. *Let* $c>\frac{-(k+\gamma)}{k+1}$*. If* $H_m^l[\alpha_1]f\in UST(k,\gamma)$ *so is* $L_c(H_m^l[\alpha_1]f)$*.*

Proof. From definition of $L_c(f)$ and the linearity of operator $H_m^l[\alpha_1]$ we have

$$z(H_m^l[\alpha_1]L_c(f))'(z)=(c+1)H_m^l[\alpha_1]f(z)-c(H_m^l[\alpha_1]L_c(f))(z). \tag{2.6}$$

Substituting $\frac{z(H_m^l[\alpha_1]L_c(f))'(z)}{H_m^l[\alpha_1]L_c(f)(z)}=p(z)$ in (2.6) we may write

$$p(z)=(c+1)\frac{H_m^l[\alpha_1]f(z)}{(H_m^l[\alpha_1]L_c(f))(z)}-c. \tag{2.7}$$

Differentiating (2.7) gives

$$\frac{z(H_m^l[\alpha_1]f)'(z)}{(H_m^l[\alpha_1]f)(z)}=p(z)+\frac{zp'(z)}{p(z)+c}.$$

Now, the theorem follows by Lemma A, since $\Re(q_{k,\gamma}(z)+c)>0$. □

A similar argument leads to

Theorem 2.6. *Let* $c>\frac{-(k+\gamma)}{k+1}$*. If* $H_m^l[\alpha_1]f\in UCV(k,\gamma)$ *so is* $L_c(H_m^l[\alpha_1]f)$*.*

Theorem 2.7. *Let $c > \frac{-(k+\gamma)}{k+1}$. If $H_m^l[\alpha_1]f \in UCC(k,\gamma,\beta)$ so is $L_c(H_m^l[\alpha_1]f)$.*

Proof. By definition, there exists a function $k(z) = (H_m^l[\alpha_1]g)(z) \in UST(k,\beta)$ such that

$$\frac{z(H_m^l[\alpha_1]f)'(z)}{(H_m^l[\alpha_1]g)(z)} \prec q_{k,\gamma}(z) \qquad (z \in U). \tag{2.8}$$

Now from (2.6) we have

$$\frac{z(H_m^l[\alpha_1]f)'(z)}{(H_m^l[\alpha_1]g)(z)} = \frac{z(H_m^l[\alpha_1]L_c(zf'))'(z) + cH_m^l[\alpha_1]L_c(zf')(z)}{z(H_m^l[\alpha_1]L_c(g(z)))'(z) + c(H_m^l[\alpha_1]L_c(g))(z)}$$
$$= \frac{\frac{z(H_m^l[\alpha_1]L_c(zf'))'(z)}{(H_m^l[\alpha_1]L_c(g))(z)} + \frac{c(H_m^l[\alpha_1]L_c(zf'))(z)}{(H_m^l[\alpha_1]L_c(g))(z)}}{\frac{z(H_m^l[\alpha_1]L_c(g))'(z)}{(H_m^l[\alpha_1]L_c(g))(z)} + c}. \tag{2.9}$$

Since $H_m^l[\alpha_1]g \in UST(k,\beta)$, by Theorem 2.5, we have $L_c(H_m^l[\alpha_1]g) \in UST(k,\beta)$.

Letting $\frac{z(H_m^l[\alpha_1]L_c(g))'}{H_m^l[\alpha_1]L_c(g)} = H(z)$, we note that $\Re(H(z)) > \frac{k+\beta}{k+1}$. Now, let h be defined by

$$z(H_m^l[\alpha_1]L_c(f))' = h(z)H_m^l[\alpha_1]L_c(g). \tag{2.10}$$

Differentiating both sides of (2.10) yields

$$\frac{z(H_m^l[\alpha_1](zL_c(f))')'(z)}{(H_m^l[\alpha_1]L_c(g))(z)} = zh'(z) + h(z)\frac{z(H_m^l[\alpha_1]L_c(g))'(z)}{(H_m^l[\alpha_1]L_c(g))(z)} = zh'(z) + H(z)h(z). \tag{2.11}$$

Therefore from (2.9) and (2.11) we obtain

$$\frac{z(H_m^l[\alpha_1]f)'(z)}{(H_m^l[\alpha_1]g)(z)} = \frac{zh'(z) + h(z)H(z) + ch(z)}{H(z)+c}.$$

This in conjunction with (2.8) leads to

$$h(z) + \frac{zh'(z)}{H(z)+c} \prec q_{k,\gamma}(z). \tag{2.12}$$

Letting $B(z) = \frac{1}{H(z)+c}$ in (2.12) we note that $\Re(B(z)) > 0$ if $c > -\frac{k+\beta}{k+1}$. Now for $E = 0$ and B as described we conclude the proof since the required conditions of Lemma B are satisfied. □

A similar argument yields

Theorem 2.8. *Let $c > \frac{-(k+\gamma)}{k+1}$. If $H_m^l[\alpha_1]f \in UQC(k,\gamma,\beta)$ so is $L_c(H_m^l[\alpha_1]f)$.*

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