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# APPROXIMATION BY MODIFIED SZÁSZ-MIRAKYAN OPERATORS

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ABSTRACT. We introduce the modified Szász-Mirakyan operators $S_{n;r}$ related to the Borel methods $B_r$ of summability of sequences. We give theorems on approximation properties of these operators in the polynomial weight spaces.

*Key words and phrases:* Szász-Mirakyan operator, Polynomial weight space, Order of approximation, Voronovskaya type theorem.

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## 1. INTRODUCTION

The approximation of functions by Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0,\ n \in \mathbb{N}, \tag{1.1}$$

$(\mathbb{R}_0 = [0,\infty), \mathbb{N} = \{1,2,...\})$ has been examined in many papers and monographs (e.g. [11], [1], [2], [4], [5]).

The above operators were modified by several authors (e.g. [3], [6], [9], [10], [12]) which showed that new operators have similar or better approximation properties than $S_n$. M. Becker in the paper [1] studied approximation problems for the operators $S_n$ in the polynomial weight space $C_p$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, connected with the weight function $w_p$,

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \quad \text{if} \quad p \in \mathbb{N}, \tag{1.2}$$

for $x \in \mathbb{R}_0$. The $C_p$ is the set of all functions $f : \mathbb{R}_0 \to \mathbb{R}$ $(\mathbb{R} = (-\infty,\infty))$ for which $fw_p$ is uniformly continuous and bounded on $\mathbb{R}_0$ and the norm is defined by

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in\mathbb{R}_0} w_p(x)|f(x)|. \tag{1.3}$$

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The space $C_p^m$, $m \in \mathbb{N}$, $p \in \mathbb{N}_0$, of $m$-times differentiable functions $f \in C_p$ with derivatives $f^{(k)} \in C_p$, $1 \le k \le m$, and the norm (1.3) was considered also in [1].

In [1] it was proved that $S_n$ is a positive linear operator acting from the space $C_p$ to $C_p$ for every $p \in \mathbb{N}_0$. Moreover, for a fixed $p \in \mathbb{N}_0$ there exist $M_k(p) = const. > 0$, $k = 1, 2$, depending only on $p$ such that for every $f \in C_p$ there hold the inequalities:

$$\|S_n(f)\|_p \le M_1(p)\|f\|_p \quad \text{for} \quad n \in \mathbb{N}, \tag{1.4}$$

and

$$w_p(x)\,|S_n(f;x) - f(x)| \le M_2(p)\omega_2\left(f; C_p; \sqrt{\frac{x}{n}}\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{1.5}$$

where $\omega_2(f; C_p; \cdot)$ is the second modulus of continuity of $f$.

In this paper we introduce the following modified Szász-Mirakyan operators

$$S_{n;r}(f;x) := \frac{1}{A_r(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}f\left(\frac{rk}{n}\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{1.6}$$

for $f \in C_p$ and every fixed $r \in \mathbb{N}$, where

$$A_r(t) := \sum_{k=0}^{\infty}\frac{t^{rk}}{(rk)!} \quad \text{for} \quad t \in \mathbb{R}_0. \tag{1.7}$$

Clearly $A_1(t) = e^t$, $A_2(t) = \cosh t \equiv \frac{1}{2}\left(e^t + e^{-t}\right)$ and $S_{n;1}(f;x) \equiv S_n(f;x)$ for $f \in C_p$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. (The operators $S_{n;2}$ were investigated in [9] for functions belonging to exponential weight spaces.)

We mention that the definition of $S_{n;r}$ is related to the Borel method of summability of sequences. It is well known ([7]) that a sequence $(a_n)_0^\infty$, $a_n \in \mathbb{R}$, is summable to $g$ by the Borel method $B_r$, $r \in \mathbb{N}$, if the series $\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}a_k$ is convergent on $\mathbb{R}$ and

$$\lim_{x\to\infty} re^{-x}\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}a_k = g.$$

In Section 2 we shall give some elementary properties of $S_{n;r}$. The approximation theorems will be given in Section 3.

## 2. Auxiliary Results

It is known ([1]) that for $e_k(x) = x^k$, $k = 0, 1, 2$, there holds: $S_n(e_0;x) = 1$, $S_n(e_1;x) = x$ and $S_n(e_2;x) = x^2 + \frac{x}{n}$, which imply that

$$S_n\left((e_1(t) - e_1(x))^2; x\right) = \frac{x}{n} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{2.1}$$

Moreover, for every fixed $q \in \mathbb{N}$, there exists a polynomial $\mathcal{P}_q(x) = \sum_{k=0}^{q} a_k x^k$ with real coefficients $a_k$, $a_q \ne 0$, depending only on $q$ such that

$$S_n\left((e_1(t) - e_1(x))^{2q}; x\right) \le \mathcal{P}_q(x)n^{-q} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{2.2}$$

From (1.1) – (1.4), (1.6) and (1.7) we deduce that $S_{n;r}$ is a positive linear operator well defined on every space $C_p$, $p \in \mathbb{N}_0$, and

$$S_{n;r}(e_0;x) = 1, \tag{2.3}$$

$$S_{n;r}(e_1;x) = \frac{x}{n}\frac{A_r'(nx)}{A_r(nx)}, \tag{2.4}$$

$$S_{n;r}(e_2;x) = \frac{x^2}{n^2}\frac{A_r''(nx)}{A_r(nx)} + \frac{x}{n^2}\frac{A_r'(nx)}{A_r(nx)}, \tag{2.5}$$

for $x \in \mathbb{R}_0$ and $n, r \in \mathbb{N}$, and

$$S_{n;r}(f;0) = f(0) \quad \text{for} \quad f \in C_p,\ n, r \in \mathbb{N}. \tag{2.6}$$

Here we derive a simpler formula for $A_r$.

**Lemma 2.1.** *Let $r \in \mathbb{N}$ be a fixed number. Then $A_r$ defined by (1.7) can be rewritten in the form: $A_1(t) = e^t$, $A_2(t) = \cosh t$,*

$$A_{2m}(t) = \frac{1}{m}\left[\cosh t + \sum_{k=1}^{m-1} \exp\left(t\cos\frac{k\pi}{m}\right)\cos\left(t\sin\frac{k\pi}{m}\right)\right], \tag{2.7}$$

*for $2 \le m \in \mathbb{N}$, and*

$$A_{2m+1}(t) = \frac{1}{2m+1}\left[e^t + 2\sum_{k=1}^{m} \exp\left(t\cos\frac{2k\pi}{2m+1}\right)\cos\left(t\sin\frac{2k\pi}{2m+1}\right)\right], \tag{2.8}$$

*for $m \in \mathbb{N}$ and $t \in \mathbb{R}_0$.*

*Proof.* The formulas for $A_1$ and $A_2$ are obvious by (1.7). For $r \ge 3$ and $t \in \mathbb{R}_0$ we have

$$e^t = \sum_{k=0}^{\infty}\frac{t^{rk}}{(rk)!} + \sum_{k=0}^{\infty}\frac{t^{rk+1}}{(rk+1)!} + \cdots + \sum_{k=0}^{\infty}\frac{t^{rk+r-1}}{(rk+r-1)!}$$

which by (1.7) can be written in the form

$$e^t = A_r(t) + \int_0^t A_r(u)\,du + \int_0^t \int_0^{v_1} A_r(u)\,du\,dv_1 + \cdots + \int_0^t \int_0^{v_1} \ldots \int_0^{v_{r-2}} A_r(u)\,du\,dv_{r-2}\ldots dv_1.$$

By $(r-1)$-times differentiation we get the equality

$$e^t = A_r^{(r-1)}(t) + A_r^{(r-2)}(t) + \cdots + A_r'(t) + A_r(t) \quad \text{for} \quad t \in \mathbb{R}_0,$$

which shows that $y = A_r(t)$ is the solution of the differential equation

$$y^{(r-1)} + y^{(r-2)} + \cdots + y' + y = e^t \tag{2.9}$$

satisfying the initial conditions

$$y(0) = 1, \quad y'(0) = y''(0) = \cdots = y^{(r-2)}(0) = 0. \tag{2.10}$$

Using now the Laplace transformation

$$\mathcal{L}[y(t)] = Y(s) := \int_0^\infty y(t)e^{-st}dt, \quad s = x + iy,$$

we have by (2.10)

$$\mathcal{L}\left[y^{(k)}(t)\right] = s^kY(s) - s^{k-1} \quad \text{for} \quad k = 1, \ldots, r-1,$$

and consequently we get from (2.7)

$$\left(s^{r-1}+s^{r-2}+\cdots+s+1\right)Y(s)=\frac{1}{s-1}+s^{r-2}+s^{r-3}+\cdots+s+1,$$

and

$$Y(s)=\frac{s^{r-1}}{s^r-1}. \tag{2.11}$$

By the inverse Laplace transformation we get

$$y(t)=\mathcal{L}^{-1}\left[\frac{s^{r-1}}{s^r-1}\right] \quad \text{for} \quad t\in\mathbb{R}_0, \tag{2.12}$$

and this $\mathcal{L}^{-1}$ transform can be calculated by the residues of $Y$.

It is known that the inverse transform of a rational function $\frac{P(s)}{Q(s)}$ with the simple poles $s_k$ can be written as follows

$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right]=\sum_{s_k}{}^{*}\frac{P(s_k)e^{s_k t}}{Q'(s_k)}+2re\sum_{s_k}{}^{**}\frac{P(s_k)e^{s_k t}}{Q'(s_k)}, \tag{2.13}$$

where $\sum^{*}$ denotes the sum for all real $s_k$ and $\sum^{**}$ denotes the sum for all complex $s_k=x_k+iy_k$ with a positive $y_k$.

The function $Y$ defined by (2.11) has the simple poles $s_k=\sqrt[r]{1}=e^{2k\pi i/r}$ for $k=0,1,\dots,r-1$. From this and (2.12) and (2.13) for $r=2m$, $2\le m\in\mathbb{N}$, we get

$$\begin{aligned} y(t)&=\frac{1}{2m}\left(\sum_{s_k}{}^{*}e^{s_k t}+2re\sum_{s_k}{}^{**}e^{s_k t}\right)\\ &=\frac{1}{m}\left[\cosh t+\sum_{k=1}^{m-1}\exp\left(t\cos\frac{k\pi}{m}\right)\cos\left(t\sin\frac{k\pi}{m}\right)\right]. \end{aligned}$$

This shows that the formula (2.7) is proved.

Analogously by (2.12) and (2.13) we obtain (2.8). □

From (2.7) and (2.8) we have that

$$\begin{aligned} A_3(t)&=\frac{1}{3}\left(e^t+2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right),\\ A_4(t)&=\frac{1}{2}(\cosh t+\cos t),\\ A_6(t)&=\frac{1}{3}\left(\cosh t+2\cosh\frac{t}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right), \quad \text{for} \quad t\in\mathbb{R}_0. \end{aligned}$$

Applying the formula (1.7) and Lemma 2.1, we immediately obtain the following:

**Lemma 2.2.** *For every fixed $r\in\mathbb{N}$ there exists a positive constant $M_3(r)$ depending only on $r$ such that*

$$1\le\frac{e^{nx}}{A_r(nx)}\le M_3(r) \quad \textit{for} \quad x\in\mathbb{R}_0, n\in\mathbb{N}. \tag{2.14}$$

**Lemma 2.3.** *Let $r\in\mathbb{N}$. Then for $e_1(x)=x$ there holds*

$$\lim_{n\to\infty} nS_{n;r}\left(e_1(t)-e_1(x);x\right)=0 \tag{2.15}$$

*and*

$$\lim_{n\to\infty} nS_{n;r}\left((e_1(t)-e_1(x))^2;x\right)=x,$$

*at every* $x\in\mathbb{R}_0$. *Moreover, we have*

$$S_{n;r}\left((e_1(t)-e_1(x))^{2q};x\right)\le M_3(r)S_n\left((e_1(t)-e_1(x))^{2q};x\right) \tag{2.16}$$

*for* $x\in\mathbb{R}_0$, $n\in\mathbb{N}$ *and every fixed* $q\in\mathbb{N}$.

*Proof.* The inequality (2.16) is obvious by (1.1), (1.6) and (2.14).

We shall prove only (2.15) for $r=2m$, $m\in\mathbb{N}$.

If $r=2$ then $A_2(t)=\cosh t$ and by (2.4) we have

$$\begin{aligned} S_{n;2}\left(e_1(t)-e_1(x);x\right) &= x\left(\frac{\sinh nx}{\cosh nx}-1\right)\\ &=\frac{-2x}{e^{2nx}+1}\quad \text{for}\quad x\in\mathbb{R}_0, n\in\mathbb{N}, \end{aligned}$$

which implies (2.15).

If $r=2m$ with $2\le m\in\mathbb{N}$, then by (2.4), (2.7) and (2.14) we get

$$\begin{aligned} |S_{n;2m}\left(e_1(t)-e_1(x);x\right)| &= \frac{x}{A_{2m}(nx)}\left|\frac{1}{n}A'_{2m}(nx)-A_{2m}(nx)\right|\\ &= \frac{x}{mA_{2m}(nx)}\Bigg|\sinh nx-\cosh nx\\ &\qquad +\sum_{k=1}^{m-1}\exp\left(nx\cos\frac{k\pi}{m}\right)\left[\cos\frac{k\pi}{m}\cos\left(nx\sin\frac{k\pi}{m}\right)\right.\\ &\qquad \left.-\sin\frac{k\pi}{m}\sin\left(nx\sin\frac{k\pi}{m}\right)-\cos\left(nx\sin\frac{k\pi}{m}\right)\right]\Bigg|\\ &\le M_3(2m)\frac{x}{m}\left[e^{-2nx}+3\sum_{k=1}^{m-1}\exp\left(-2nx\sin^2\frac{k\pi}{m}\right)\right] \end{aligned}$$

and from this we immediately obtain (2.15). □

From (1.6), (1.1) – (1.4) and (2.14) the following lemma results.

**Lemma 2.4.** *The operator* $S_{n;r}$, $n,r\in\mathbb{N}$, *is linear and positive, and acts from the space* $C_p$ *to* $C_p$ *for every* $p\in\mathbb{N}_0$. *For* $f\in C_p$

$$\begin{aligned} \|S_{n;r}(f)\|_p &\le \|f\|_p\|S_{n;r}(1/w_p)\|_p\\ &\le M_3(r)\|f\|_p\cdot\|S_n(1/w_p)\|_p\le M_4(p,r)\|f\|_p \quad for \quad n,r,\in\mathbb{N}, \end{aligned}$$

*where* $M_4(p,r)=M_1(p)M_3(r)$ *and* $M_1(p)$, $M_3(r)$ *are positive constants given in (1.4) and (2.14).*

## 3. THEOREMS

First we shall prove two theorems on the order of approximation of $f\in C_p$ by $S_{n;r}$, $r\ge 2$.

**Theorem 3.1.** *Let* $p\in\mathbb{N}_0$ *and* $2\le r\in\mathbb{N}$ *be fixed numbers. Then there exists* $M_5(p,r)=const.>0$ *(depending only on* $p$ *and* $r$*) such that for every* $f\in C_p^1$ *there holds the inequality*

$$w_p(x)\left|S_{n;r}(f;x)-f(x)\right|\le M_5(p,r)\|f'\|_p\sqrt{\frac{x}{n}}\,, \tag{3.1}$$

*for* $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$.

*Proof.* Let $f \in C_p^1$. Then by (1.6), (1.7) and (2.14) it follows that

$$\begin{aligned} |S_{n;r}(f;x) - f(x)| &\le S_{n;r}\left(|f(t) - f(x)|\,;x\right) \\ &\le M_3(r) S_n\left(|f(t) - f(x)|;x\right) \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \end{aligned}$$

and for $t, x \in \mathbb{R}_0$

$$|f(t) - f(x)| = \left| \int_x^t f'(u)\,du \right| \le \|f'\|_p \left( \frac{1}{w_p(t)} + \frac{1}{w_p(x)} \right) |t - x|.$$

Using now the operator $S_n$, (1.1) – (1.4) and (2.1), we get

$$\begin{aligned} w_p(x) S_n\left(|f(t) - f(x)|;x\right) &\le \|f'\|_p \left\{ w_p(x) S_n\left( \frac{|t-x|}{w_p(t)};x \right) + S_n(|t-x|;x) \right\} \\ &\le \|f'\|_p \left( S_n\left((t-x)^2;x\right)\right)^{1/2} \left\{ 2\|S_n(1/w_{2p})\|_{2p}^{1/2} + 1 \right\} \\ &\le \left( 2\sqrt{M_1(2p)} + 1 \right) \|f'\|_p \sqrt{\frac{x}{n}} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \end{aligned}$$

Combining the above, we obtain the estimation (3.1). □

**Theorem 3.2.** *Let* $p \in \mathbb{N}_0$ *and* $2 \le r \in \mathbb{N}$ *be fixed. Then there exists* $M_6(p,r) = const. > 0$ *(depending only on* $p$ *and* $r$*) such that for every* $f \in C_p$, $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$ *there holds*

$$w_p(x)\,|S_{n;r}(f;x) - f(x)| \le M_6(p,r)\,\omega_1\left( f; C_p; \sqrt{\frac{x}{n}} \right), \tag{3.2}$$

*where* $\omega_1(f;C_p;\cdot)$ *is the modulus of continuity of* $f \in C_p$*, i.e.*

$$\omega_1(f;C_p;t) := \sup_{0 \le u \le t} \|\Delta_u f(\cdot)\|_p \quad \textit{for} \quad t \ge 0, \tag{3.3}$$

*and* $\Delta_u f(x) = f(x+u) - f(x)$.

*Proof.* The inequality (3.2) for $x = 0$ follows by (1.2), (2.6) and (3.3).

Let $f \in C_p$ and $x > 0$. We use the Steklov function $f_h$,

$$f_h(x) := \frac{1}{h} \int_0^h f(x+t)\,dt \quad \text{for} \quad x \in \mathbb{R}_0, h > 0.$$

This $f_h$ belongs to the space $C_p^1$ and by (3.3) it follows that

$$\|f - f_h\|_p \le \omega_1(f;C_p;h) \tag{3.4}$$

and

$$\|f_h'\|_p \le h^{-1}\omega_1(f;C_p;h), \quad \text{for} \quad h > 0. \tag{3.5}$$

By the above properties of $f_h$ and (2.3) we can write

$$\left|S_{n;r}\big(f(t);x\big) - f(x)\right| \le \left|S_{n;r}\left(f(t) - f_h(t);x\right)\right| + \left|S_{n;r}\big(f_h(t);x\big) - f_h(x)\right| + |f_h(x) - f(x)|\,,$$

for $n \in \mathbb{N}$ and $h > 0$. Next, by Lemma 2.4 and (3.4) we get

$$w_p(x)\left|S_{n;r}\big(f(t) - f_h(t);x\big)\right| \le M_4(p,r)\|f - f_h\|_p \le M_4(p,r)\omega_1(f;C_p;h).$$

In view of Theorem 3.1 and (3.5) we have

$$w_p(x)\left|S_{n;r}\big(f_h;x\big) - f_h(x)\right| \le M_5(p,r)\|f_h'\|_p\sqrt{\frac{x}{n}} \le M_5(p,r)h^{-1}\sqrt{\frac{x}{n}}\,\omega_1(f;C_p;h).$$

Consequently,

$$w_p(x)\left|S_{n;r}(f;x)-f(x)\right| \leq \omega_1(f;C_p;h)\left(M_4(p,r)+M_5(p,r)h^{-1}\sqrt{\frac{x}{n}}+1\right), \tag{3.6}$$

for $x>0$, $n\in\mathbb{N}$ and $h>0$. Putting $h=\sqrt{x/n}$ in (3.6) for given $x$ and $n$, we obtain the desired estimation (3.2). ☐

Theorem 3.2 implies the following:

**Corollary 3.3.** *If $f\in C_p$, $p\in\mathbb{N}_0$, and $2\leq r\in\mathbb{N}$, then*

$$\lim_{n\to\infty} S_{n;r}(f;x)=f(x) \quad \textit{at every} \quad x\in\mathbb{R}_0.$$

*This convergence is uniform on every interval $[x_1,x_2]$, $x_1\geq 0$.*

The Voronovskaya type theorem given in [1] for the operators $S_n$ can be extended to $S_{n;r}$ with $r\geq 2$.

**Theorem 3.4.** *Suppose that $f\in C_p^2$, $p\in\mathbb{N}_0$, and $2\leq r\in\mathbb{N}$. Then*

$$\lim_{n\to\infty} n\left(S_{n;r}(f;x)-f(x)\right)=\frac{x}{2}f''(x) \tag{3.7}$$

*at every $x\in\mathbb{R}_0$.*

*Proof.* The statement (3.7) for $x=0$ is obvious by (2.6). Choosing $x>0$, we can write the Taylor formula for $f\in C_p^2$:

$$f(t)=f(x)+f'(x)+\frac{1}{2}f''(x)(t-x)^2+\varphi(t,x)(t-x)^2 \quad \text{for} \quad t\in\mathbb{R}_0,$$

where $\varphi(t)\equiv\varphi(t,x)$ is a function belonging to $C_p$ and $\lim_{t\to x}\varphi(t)=\varphi(x)=0$.

Using now the operator $S_{n;r}$ and (2.3), we get

$$S_{n;r}(f(t);x)=f(x)+f'(x)S_{n;r}(t-x;x)+\frac{1}{2}f''(x)S_{n;r}((t-x)^2;x)+S_{n;r}\left(\varphi(t)(t-x)^2;x\right),$$

for $n\in\mathbb{N}$, which by Lemma 2.3 implies that

$$\lim_{n\to\infty} n\left(S_{n;r}(f(t);x)-f(x)\right)=\frac{x}{2}f''(x)+\lim_{n\to\infty} nS_{n;r}\left(\varphi(t)(t-x)^2;x\right). \tag{3.8}$$

It is clear that

$$\left|S_{n;r}\left(\varphi(t)(t-x)^2;x\right)\right| \leq \left(S_{n;r}(\varphi^2(t);x)S_{n;r}((t-x)^4;x)\right)^{1/2}, \tag{3.9}$$

and by Corollary 3.3

$$\lim_{n\to\infty} S_{n;r}(\varphi^2(t);x)=\varphi^2(x)=0. \tag{3.10}$$

Moreover, by (2.16) and (2.2) we deduce that the sequence $(n^2S_{n;r}((t-x)^4;x))_1^\infty$ is bounded at every fixed $x\in\mathbb{R}_0$. From this and (3.9) and (3.10) we get

$$\lim_{n\to\infty} nS_{n;r}\left(\varphi(t)(t-x)^2;x\right)=0$$

which with (3.8) yields the statement (3.7). ☐

## 4. REMARKS

**Remark 1.** We observe that the estimation (1.5) for the operators $S_n$ is better than (3.2) obtained for $S_{n;r}$ with $r \geq 2$. It is generated by formulas (2.3) – (2.5) and Lemma 2.1 which show that the operators $S_{n;r}$, $r \geq 2$, preserve only the function $e_0(x) = 1$. The operators $S_n$ preserve the function $e_k(x) = x^k$, $k = 0, 1$.

**Remark 2.** In the paper [2], the approximation properties of the Szász-Mirakyan operators $S_n$ in the exponential weight spaces $C_q^*$, $q > 0$, with the weight function $v_q(x) = e^{-qx}$, $x \in \mathbb{R}_0$ were examined. Obviously the operators $S_{n;r}$, $r \geq 2$, can be investigated also in these spaces.

**Remark 3.** G. Kirov in [8] defined the new Bernstein polynomials for $m$-times differentiable functions and showed that these operators have better approximation properties than classical Bernstein polynomials.

The Kirov idea was applied to the operators $S_n$ in [10].

We mention that the Kirov method can be extended to the operators $S_{n;r}$ with $r \geq 2$, i.e. for functions $f \in C_p^m$, $m \in \mathbb{N}$, $p \in \mathbb{N}_0$, and a fixed $2 \leq r \in \mathbb{N}$ we can consider the operators

$$S_{n;r}^*(f;x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} \sum_{j=0}^{m} \frac{f^{(j)}\left(\frac{rk}{n}\right)}{j!} \left(\frac{rk}{n} - x\right)^j,$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

In [10] it was proved that the $S_{n;1}^*$ have better approximation properties for $f \in C_p^m$, $m \geq 2$, than $S_{n;1}$.

## REFERENCES

[1] M. BECKER, Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.*, **27**(1) (1978), 127–142.

[2] M. BECKER, D. KUCHARSKI AND R.J. NESSEL, Global approximation theorems for the Szász-Mirakyan operators in exponential weight spaces, *Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1977)*, Birkhäuser Verlag, Basel, Internat. Series of Num. Math., **40** (1978), 319–333.

[3] A. CIUPA, Approximation by a generalized Szász type operator, *J. Comput. Anal. and Applic.*, **5**(4) (2003), 413–424.

[4] R.A. DE VORE AND G.G. LORENTZ, *Constructive Approximation*, Springer-Verlag, Berlin, New York, 1993.

[5] Z. DITZIAN AND V. TOTIK, *Moduli of Smoothness*, Springer-Verlag, New-York, 1987.

[6] P. GUPTA AND V. GUPTA, Rate of convergence on Baskakov-Szász type operators, *Fasc. Math.*, **31** (2001), 37–44.

[7] G.H. HARDY, *Divergent Series*, Oxford Univ. Press, Oxford, 1949.

[8] G.H. KIROV, A generalization of the Bernstein polynomials, *Mathematica (Balcanica)*, **2**(2) (1992), 147–153.

[9] M. LEŚNIEWICZ AND L. REMPULSKA, Approximation by some operators of the Szász-Mirakyan type in exponential weight spaces, *Glas. Mat. Ser.* III, **32**(52)(1) (1997), 57–69.

[10] L. REMPULSKA AND Z. WALCZAK, Modified Szász-Mirakyan operators, *Mathematica (Balcanica)*, **18** (2004), 53–63.

[11] O. SZÁSZ, Generalizations of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. Standards, Sect.* B, **45** (1950), 239–245.

[12] Z. WALCZAK, On the convergence of the modified Szász-Mirakyan operators, *Yokohama Math. J.*, **51**(1) (2004), 11–18.