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RATE OF GROWTH OF POLYNOMIALS NOT VANISHING INSIDE A CIRCLE

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ABSTRACT. A well known result due to Ankeny and Rivlin [1] states that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n satisfying $p(z) \neq 0$ for $|z| < 1$ then for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|.$$

It was proposed by late Professor R.P. Boas, Jr. to obtain an inequality analogous to this inequality for polynomials having no zeros in $|z| < K,\ K > 0$. In this paper, we obtain some results in this direction, by considering polynomials of the form $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \leq t \leq n$ which have no zeros in $|z| < K,\ K \geq 1$.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n, and let

$$\|p\| = \max_{|z|=1} |p(z)|, \quad M(p, R) = \max_{|z|=R} |p(z)|.$$

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For a polynomial, $p(z) = \sum_{v=0}^{n} a_v z^v$ of degree n, it is well known and is a simple consequence of the Maximum Modulus Principle (see [16] or [13, Vol. 1, p. 137]) that for $R \geq 1$,

$$M(p,R) \leq R^n \|p\|. \tag{1.1}$$

This result is best possible with equality holding for $p(z) = \lambda z^n$, λ being a complex number. Since the extremal polynomial $p(z) = \lambda z^n$ in (1.1) has all its zeros at the origin, it should be possible to improve upon the bound in (1.1) for polynomials not vanishing at the origin. This fact was observed by Ankeny and Rivlin [1], who proved that if a polynomial $p(z)$ has no zeros in $|z| < 1$, then for $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n + 1}{2}\right) \|p\|. \tag{1.2}$$

Inequality (1.2) becomes equality for $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

Govil [7] observed that since equality in (1.2) holds only for polynomials $p(z) = \lambda + \mu z^n,\ \ |\lambda| = |\mu|$, which satisfy

$$|\text{coefficient of } z^n\,| = \frac{1}{2}\,\|p\|, \tag{1.3}$$

one should be able to improve upon the bound in (1.2) for polynomials not satisfying (1.3), and in this connection he therefore proved the following refinement of (1.2).

Theorem A. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $p(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\|p\|}{\|p\| + 2|a_n|} - \ln\left(1 + \frac{(R-1)\|p\|}{\|p\| + 2|a_n|}\right)\right\}. \tag{1.4}$$

The above inequality becomes equality for the polynomial $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

This result of Govil [7] was sharpened by Dewan and Bhat [4], which was then later generalized by Govil and Nyuydinkong [10], where they considered polynomials not vanishing in $|z| < K$, $K \geq 1$. Recently, Gardner, Govil and Weems [5] generalized the result of Govil and Nyuydinkong [10], by considering polynomials of the form $a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \leq t \leq n$. More specifically, the result of Gardner, Govil and Weems [5] is:

Theorem B. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \leq t \leq n$, is a polynomial of degree n and $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) &\leq \left(\frac{R^n + K^t}{1+K^t}\right)\|p\| - \left(\frac{R^n - 1}{1+K^t}\right) m - \frac{n}{1+K^t}\left(\frac{(\|p\| - m)^2 - (1+K^t)^2|a_n|^2}{(\|p\| - m)}\right) \\ &\quad \times \left\{\frac{(R-1)(\|p\| - m)}{(\|p\| - m) + (1+K^t)|a_n|} - \ln\left(1 + \frac{(R-1)(\|p\| - m)}{(\|p\| - m) + (1+K^t)|a_n|}\right)\right\}, \end{aligned} \tag{1.5}$$

where $m = \min\limits_{|z|=K} |p(z)|$.

The result of Govil and Nyundinkong [10] is a special case of Theorem B, when $t = 1$. In this paper, we prove the following generalization and sharpening of Theorem A, and thus as well of inequality (1.2).

Theorem 1.1. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \le t \le n$, *is a polynomial of degree* n *and* $p(z) \ne 0$ *in* $|z| < K$, $K \ge 1$, *then for* $R \ge 1$,

$$\begin{aligned} M(p,R) &\le \left(\frac{R^n + s_0}{1+s_0}\right)\|p\| - \left(\frac{R^n - 1}{1+s_0}\right)m - \frac{n}{1+s_0}\left(\frac{(\|p\|-m)^2 - (1+s_0)^2|a_n|^2}{(\|p\|-m)}\right) \\ &\quad \times \left\{\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+s_0)|a_n|} - \ln\left(1 + \frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+s_0)|a_n|}\right)\right\}, \end{aligned} \tag{1.6}$$

where $m = \min_{|z|=K} |p(z)|$, *and*

$$s_0 = K^{t+1}\frac{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t-1} + 1}{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t+1} + 1}. \tag{1.7}$$

For $K = 1$, the above theorem reduces to the result of Dewan and Bhat [4, p. 131], which is a sharpening of Theorem A. Note that by Lemma 2.7 (stated in Section 2), we have $s_0 \ge K^t$, and therefore if we combine this with the fact that $\left(\frac{R^n+x}{1+x}\right)\|p\| - \left(\frac{R^n-1}{1+x}\right)m$ is a decreasing function of x, we obtain from the above theorem the following:

Corollary 1.2. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \le t \le n$, *is a polynomial of degree* n *and* $p(z) \ne 0$ *in* $|z| < K$, $K \ge 1$, *then for* $R \ge 1$,

$$M(p,R) \le \left(\frac{R^n + K^t}{1+K^t}\right)\|p\| - \left(\frac{R^n - 1}{1+K^t}\right)m, \tag{1.8}$$

where $m = \min_{|z|=K} |p(z)|$.

The special case of the above corollary with $K = 1$, and $t = 1$, was proved by Aziz and Dawood [2]. If in (1.6), we divide both the sides by R^n, and make $R \to \infty$, we will get:

Corollary 1.3. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \le t \le n$, *is a polynomial of degree* n *and* $p(z) \ne 0$ *in* $|z| < K$, $K \ge 1$, *then*

$$|a_n| \le \frac{1}{1+s_0}\Big(\|p\| - m\Big), \tag{1.9}$$

where again $m = \min_{|z|=K} |p(z)|$.

In case one does not have knowledge of $m = \min_{|z|=K} |p(z)|$, one could use the following result which does not depend on m, but is a generalization and refinement of inequality (1.2). It is easy to see that the following theorem also generalizes Theorem A.

Theorem 1.4. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \le t \le n$, *is a polynomial of degree* n *and* $p(z) \ne 0$ *in* $|z| < K$, $K \ge 1$, *then for* $R \ge 1$,

$$\begin{aligned} M(p,R) &\le \left(\frac{R^n + s_1}{1+s_1}\right)\|p\| - \frac{n}{1+s_1}\left(\frac{\|p\|^2 - (1+s_1)^2|a_n|^2}{\|p\|}\right) \\ &\quad \times \left\{\frac{(R-1)\|p\|}{\|p\|+(1+s_1)|a_n|} - \ln\left(1 + \frac{(R-1)\|p\|}{\|p\|+(1+s_1)|a_n|}\right)\right\}, \end{aligned} \tag{1.10}$$

where $s_1 = K^{t+1}\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|}K^{t-1}+1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|}K^{t+1}+1}$.

If in the above theorem, we divide both sides of (1.10) by R^n and make $R \to \infty$, we will get

Corollary 1.5. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \le t \le n$, *is a polynomial of degree* n *and* $p(z) \neq 0$ *in* $|z| < K$, $K \geq 1$, *then*

$$|a_n| \le \frac{1}{1+s_1}\|p\|. \tag{1.11}$$

Remark 1.6. Both Corollaries 1.2 and 1.3 generalize and sharpen the well known inequality, obtainable by an application of Visser's Inequality [17], that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $p(z) \neq 0$ in $|z| < 1$ then $|a_n| \le \frac{n}{2}\|p\|$.

Remark 1.7. Since by Lemma 2.8 (stated in Section 2), we have $s_1 \geq s_0$, the bounds in Corollaries 1.2 and 1.3 are not comparable, and depending on the value of m, either one of these corollaries may give the sharper bound.

Remark 1.8. From the results used in the proofs of Theorem B, and Theorem 1.1, it appears that the bound obtained by Theorem 1.1 should in general be sharper than the bound obtained from Theorem B, but we are not able to prove this. However, we produce the following two examples, where the bounds obtained by Theorem 1.1 and Theorem 1.4 are considerably sharper than the bounds obtained from Theorem B. Also, in Example 1.1, the bound obtained by Theorem 1.1 is quite close to the actual bound.

Example 1.1. Consider $p(z) = 1000 + z^2 + z^3 + z^4$. Clearly, here $t = 2$ and $n = 4$. We take $K = 5.4$, since we find numerically that $p(z) \neq 0$ for $|z| < 5.4483$. For this polynomial, the bound for $M(p, 2)$ by Theorem B comes out to be 1447.503, and by Theorem 1.1, it comes out to be 1101.84, which is a significant improvement over the bound obtained from Theorem B. Numerically, we find that for this polynomial $M(p, 2) \approx 1028$, which is quite close to the bound 1101.84, that we obtained by Theorem 1.1. The bound for $M(p, 2)$ obtained by Theorem 1.4 is 1105.05, which is also quite close to the actual bound ≈ 1028. However, in this case Theorem 1.1 gives the best bound.

Example 1.2. Now, consider $p(z) = 1000 + z^2 - z^3 - z^4$. Here also, $t = 2$ and $n = 4$. We found numerically that $p(z) \neq 0$ for $|z| < 5.43003$, and thus we take $K = 5.4$. If we take $R = 3$, then for this polynomial the bound for $M(p, 3)$ obtained by Theorem B comes out to be 3479.408, while by Theorem 1.4 it comes out to be 1545.3, and by Theorem 1.1 it comes out to be 1534.5, a considerable improvement. Thus again the bounds obtained from Theorem 1.1 and Theorem 1.4 are considerably smaller than the bound obtained from Theorem B, and the bound 1534.5 obtained by Theorem 1.1 is much closer to the actual bound $M(p, 3) \approx 1100.6$, than the bound 3479.408, obtained from Theorem B.

2. LEMMAS

We need the following lemmas.

Lemma 2.1. *Let* $f(z)$ *be analytic inside and on the circle* $|z| = 1$ *and let* $\|f\| = \max_{|z|=1} |f(z)|$. *If* $f(0) = a$, *where* $|a| < \|f\|$, *then for* $|z| < 1$,

$$|f(z)| \le \left(\frac{\|f\||z| + |a|}{\|f\| + |a||z|}\right)\|f\|. \tag{2.1}$$

This is a well-known generalization of Schwarz's lemma (see for example [13, p. 167]).

Lemma 2.2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n, *then for* $|z| = R \geq 1$,

$$|p(z)| \le \left(\frac{\|p\| + R|a_n|}{R\|p\| + |a_n|}\right)\|p\|R^n. \tag{2.2}$$

The proof of this lemma follows easily by applying Lemma 2.1 to $T(z) = z^n p(\frac{1}{z})$ and noting that $\|T\| = \|p\|$ (see Rahman [14, Lemma 2] for details).

From Lemma 2.2, one immediately gets (see Govil [7, Lemma 3]):

Lemma 2.3. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then for $|z| = R \geq 1$,*

$$|p(z)| \leq R^n \left(1 - \frac{(\|p\| - |a_n|)(R-1)}{(R\|p\| + |a_n|)}\right) \|p\|. \tag{2.3}$$

Lemma 2.4. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $R \geq 1$, then*

$$\left(1 - \frac{(x - n|a_n|)(R-1)}{(Rx + n|a_n|)}\right) x \tag{2.4}$$

is an increasing function of x, for $x > 0$.

The above lemma which follows by the derivative test is also due to Govil [7, Lemma 5].

Lemma 2.5. *Let $p_n(z) = \prod_{\nu=1}^{n}(1 - z_\nu z)$ be a polynomial of degree n not vanishing in $|z| < 1$ and let $p_n'(0) = p_n''(0) = \cdots = p_n^{(l)}(0) = 0$. If $\Phi(z) = \{p_n(z)\}^\epsilon = \sum_{n=0}^{\infty} b_{k,\epsilon} z^k$, where $\epsilon = 1$ or -1, then*

$$|b_{k,\epsilon}| \leq \frac{n}{k}, \qquad (l + 1 \leq k \leq 2l + 1) \tag{2.5}$$

and

$$|b_{2l+2,1}| \leq \frac{n}{2(l+1)^2}(n + l - 1), \qquad |b_{2l+2,-1}| \leq \frac{n}{2(l+1)^2}(n + l + 1). \tag{2.6}$$

The above result is due to Rahman and Stankiewicz [15, Theorem $2'$, p. 180].

Lemma 2.6. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $p(z) \neq 0$ in $|z| < K$ then $|p(z)| > m$ for $|z| < K$, and in particular*

$$|a_0| > m, \tag{2.7}$$

where $m = \min_{|z|=K} |p(z)|$.

Proof. We can assume without loss of generality that $p(z)$ has no zeros on $|z| = K$, for otherwise the result holds trivially. Since $p(z)$, being a polynomial, is analytic in $|z| \leq K$ and has no zeros in $|z| \leq K$, by the Minimum Modulus Principle,

$$|p(z)| \geq m \text{ for } |z| \leq K,$$

which in particular implies $|a_0| = |p(0)| > m$, which is (2.7). □

Lemma 2.7. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$ is a polynomial of degree n, $p(z) \neq 0$ for $|z| < K$, $K \geq 1$, and if $m = \min_{|z|=K} |p(z)|$, then*

$$s_0 = K^{t+1} \frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0| - m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0| - m} K^{t+1} + 1} \geq K^t, \quad t \geq 1. \tag{2.8}$$

Proof. The above lemma is due to Gardner, Govil and Weems [6, Lemma 3], however for the sake of completeness we present the brief outline of its proof. Without loss of generality we can assume $a_0 > 0$ for otherwise we can consider the polynomial $P(z) = e^{-\arg a_0} p(z)$, which clearly also has no zeros in $|z| < K$ and $M(P, R) = M(p, R)$. Since the polynomial $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v \neq 0$ for $|z| < K$, hence, by Lemma 2.6, the polynomial $p(z) - m \neq 0$

for $|z| < K$, implying that the polynomial $P(z) = p(Kz) - m \neq 0$ for $|z| < 1$. If we now apply Lemma 2.5 to the polynomial $\frac{P(z)}{a_0 - m}$, which clearly satisfies its hypotheses, we get

$$\frac{|a_t|K^t}{a_0 - m} \leq \frac{n}{t},$$

which is clearly equivalent to

$$\frac{t}{n}\left(\frac{|a_t|K^{t+1}}{a_0 - m}\right) + 1 \leq \frac{t}{n}\left(\frac{|a_t|K^t}{a_0 - m}\right) + K,$$

and from which (2.8) follows. □

Lemma 2.8. *The function*

$$s(x) = K^{t+1}\left(\frac{(t/n)(|a_t|/x)K^{t-1} + 1}{(t/n)(|a_t|/x)K^{t+1} + 1}\right)$$

is an increasing function of x. Since $|a_0| > |a_0| - m$, in particular this lemma implies that $s_1 > s_0$.

Proof. The proof follows by considering the first derivative of $s(x)$. □

The following lemma, which is again due to Gardner, Govil and Weems [6, Lemma 10], is of independent interest, because besides proving a generalization and refinement of the Erdös-Lax Theorem [11], it also provides generalizations and refinements of the results of Aziz and Dawood [2], Chan and Malik [3], Govil [8, p. 31], Govil [9, Lemma 6] and Malik [12].

Lemma 2.9. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ t \geq 1$ is a polynomial of degree n having no zeros in $|z| < K$, where $K \geq 1$, then*

$$M(p', 1) \leq \frac{n}{1 + s_0}(\|p\| - m), \tag{2.9}$$

where $m = \min_{|z|=K} |p(z)|$ and

$$s_0 = K^{t+1}\left(\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}K^{t-1} + 1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}K^{t+1} + 1}\right).$$

Since in view of Lemma 2.7 and Lemma 2.8, we have $s_1 \geq K^t$, the following lemma which is also due to Gardner, Govil and Weems [6, Lemma 11], provides a generalization of the Erdös-Lax Theorem [11], and sharpens results of Chan and Malik [3], and Malik [12].

Lemma 2.10. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ t \geq 1$, is a polynomial of degree n having no zeros in $|z| < K$, where $K \geq 1$, then*

$$M(p', 1) \leq \frac{n}{1 + s_1}\|p\|, \tag{2.10}$$

where $m = \min_{|z|=K} |p(z)|$ and

$$s_1 = K^{t+1}\left(\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|}K^{t-1} + 1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|}K^{t+1} + 1}\right).$$

Lemma 2.11. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \leq t \leq n$, is a polynomial of degree n having no zeros in $|z| < K,\ K \geq 1$, then*

$$|a_n| \leq \frac{1}{1 + s_0}(\|p\| - m), \tag{2.11}$$

and

$$|a_n| \leq \frac{1}{1+s_1}\|p\|, \tag{2.12}$$

where s_0 and s_1 are as defined in Theorem 1.1 and Theorem 1.4 respectively.

Proof. If $p(z) = \sum_{v=0}^{n} a_v z^v$, then $p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$. Hence Cauchy's inequality when applied to $p'(z)$ gives

$$|na_n| \leq \|p'\|. \tag{2.13}$$

On the other hand, by Lemma 2.9,

$$\|p'\| \leq \frac{n}{1+s_0}(\|p\| - m). \tag{2.14}$$

Combining (2.13) and (2.14), we obtain

$$|na_n| \leq \frac{n}{1+s_0}(\|p\| - m), \tag{2.15}$$

from which (2.11) follows. To prove (2.12), simply use Lemma 2.10 instead of Lemma 2.9 in the above proof. □

3. PROOF OF THEOREM 1.1

To prove Theorem 1.1, first note that for each θ, $0 \leq \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(e^{i\theta}) = \int_1^R p'(re^{i\theta})e^{i\theta}dr.$$

Hence

$$\begin{aligned} |p(Re^{i\theta}) - p(e^{i\theta})| &\leq \int_1^R |p'(re^{i\theta})|dr \\ &\leq \int_1^R r^{n-1}\left(1 - \frac{(\|p'\| - n|a_n|)(r-1)}{(r\|p'\| + n|a_n|)}\right)\|p'\|dr, \end{aligned} \tag{3.1}$$

by applying Lemma 2.3 to $p'(z)$, which is a polynomial of degree $(n-1)$.

By Lemma 2.4, the integrand in (3.1) is an increasing function of $\|p'\|$, hence applying Lemma 2.9 to (3.1), we get for $0 \leq \theta < 2\pi$,

$$\begin{aligned} &|p(Re^{i\theta}) - p(e^{i\theta})| \\ &\leq \int_1^R r^{n-1}\left(1 - \frac{\{\frac{n}{1+s_0}(\|p\| - m) - n|a_n|\}(r-1)}{r\frac{n}{1+s_0}(\|p\| - m) + n|a_n|}\right)\frac{n}{1+s_0}(\|p\| - m)dr \\ &= \frac{n}{1+s_0}(\|p\| - m)\int_1^R r^{n-1}\left(1 - \frac{\{(\|p\| - m) - (1+s_0)|a_n|\}(r-1)}{r(\|p\| - m) + (1+s_0)|a_n|}\right)dr \\ &= \frac{n}{1+s_0}(\|p\| - m)\int_1^R r^{n-1}dr - \frac{n}{1+s_0}\Big((\|p\| - m) - (1+s_0)|a_n|\Big) \\ &\qquad \times \int_1^R \left(\frac{r^{n-1}(r-1)(\|p\| - m)}{r(\|p\| - m) + (1+s_0)|a_n|}\right)dr. \end{aligned} \tag{3.2}$$

Since by (2.11) in Lemma 2.11, we have $(\|p\| - m) - (1 + s_0)|a_n| \geq 0$, we get for $0 \leq \theta \leq 2\pi$ and $R \geq 1$,

$$\begin{aligned}
&|p(Re^{i\theta}) - p(e^{i\theta})| \\
&\quad \leq \frac{(R^n - 1)}{1 + s_0}(\|p\| - m) - \frac{n}{1 + s_0}\Big((\|p\| - m) - (1 + s_0)|a_n|\Big) \\
&\qquad \times \int_1^R \left(\frac{(r-1)(\|p\| - m)}{r(\|p\| - m) + (1 + s_0)|a_n|}\right) dr \\
&\quad = \frac{(R^n - 1)}{1 + s_0}(\|p\| - m) - \frac{n}{1 + s_0}\Big((\|p\| - m) - (1 + s_0)|a_n|\Big) \\
&\qquad \times \int_1^R \left(1 - \frac{(\|p\| - m) + (1 + s_0)|a_n|}{r(\|p\| - m) + (1 + s_0)|a_n|}\right) dr \\
&\quad = \frac{(R^n - 1)}{1 + s_0}(\|p\| - m) - \frac{n}{1 + s_0}\Big((\|p\| - m) - (1 + s_0)|a_n|\Big) \\
&\qquad \times \left\{(R-1) - \left(\frac{(\|p\| - m) + (1 + s_0)|a_n|}{(\|p\| - m)}\right) \ln\left(\frac{R(\|p\| - m) + (1 + s_0)|a_n|}{(\|p\| - m) + (1 + s_0)|a_n|}\right)\right\} \\
&\quad = \frac{(R^n - 1)}{1 + s_0}(\|p\| - m) - \frac{n}{1 + s_0}\Big((\|p\| - m) - (1 + s_0)|a_n|\Big) \\
&\qquad \times \left(\frac{(\|p\| - m) + (1 + s_0)|a_n|}{(\|p\| - m)}\right) \\
&\qquad \times \left\{\left(\frac{(R-1)(\|p\| - m)}{(\|p\| - m) + (1 + s_0)|a_n|}\right) - \ln\left(\frac{R(\|p\| - m) + (1 + s_0)|a_n|}{(\|p\| - m) + (1 + s_0)|a_n|}\right)\right\} \\
&\quad = \frac{(R^n - 1)}{1 + s_0}(\|p\| - m) - \frac{n}{1 + s_0}\left(\frac{(\|p\| - m)^2 - (1 + s_0)^2|a_n|^2}{(\|p\| - m)}\right) \\
&\qquad \times \left\{\left(\frac{(R-1)(\|p\| - m)}{(\|p\| - m) + (1 + s_0)|a_n|}\right) - \ln\left(\frac{R(\|p\| - m) + (1 + s_0)|a_n|}{(\|p\| - m) + (1 + s_0)|a_n|}\right)\right\},
\end{aligned}$$

which clearly gives

$$\begin{aligned}
M(p,R) \leq \left(\frac{R^n + s_0}{1 + s_0}\right)\|p\| - \left(\frac{R^n - 1}{1 + s_0}\right) m - \frac{n}{1 + s_0}\left(\frac{(\|p\| - m)^2 - (1 + s_0)^2|a_n|^2}{(\|p\| - m)}\right) \\
\times \left\{\left(\frac{(R-1)(\|p\| - m)}{(\|p\| - m) + (1 + s_0)|a_n|}\right) - \ln\left(1 + \frac{(R-1)(\|p\| - m)}{(\|p\| - m) + (1 + s_0)|a_n|}\right)\right\},
\end{aligned}$$

and the proof of the Theorem 1.1 is complete. □

The proof of Theorem 1.4 follows along the same lines as Theorem 1.1, but by using Lemma 2.10 instead of Lemma 2.9, and (2.12) instead of (2.11). We omit the details.

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