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SOME INEQUALITIES FOR THE EXPECTATION AND VARIANCE OF A RANDOM VARIABLE WHOSE PDF IS n-TIME DIFFERENTIABLE

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ABSTRACT. Some inequalities for the expectation and variance of a random variable whose p.d.f. is n-time differentiable are given.

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1. INTRODUCTION

Let $f : [a, b] \to \mathbb{R}_+$ be the p.d.f. of the random variable X and

$$E(X) := \int_a^b t f(t)\, dt$$

its *expectation* and

$$\sigma(X) = \left[\int_a^b (t - E(X))^2 f(t)\, dt\right]^{\frac{1}{2}} = \left[\int_a^b t^2 f(t)\, dt - [E(X)]^2\right]^{\frac{1}{2}}$$

its dispersion or standard deviation.

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In [1], using the identity

$$[x-E(X)]^2+\sigma^2(X)=\int_a^b (x-t)^2 f(t)\,dt \tag{1.1}$$

and applying a variety of inequalities such as: Hölder's inequality, pre-Grüss, pre-Chebychev, pre-Lupaş, or Ostrowski type inequalities, a number of results concerning the expectation and variance of the random variable X were obtained.

For example,

$$\begin{aligned}&\sigma^2(X)+[x-E(X)]^2\\ &\le\begin{cases}(b-a)\left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right]\|f\|_\infty, & \text{if } f\in L_\infty[a,b];\\ \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}}\|f\|_p, & \text{if } f\in L_p[a,b],\\ & p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2,\end{cases}\end{aligned} \tag{1.2}$$

for all $x\in[a,b]$, which imply, amongst other things, that

$$\begin{aligned}0&\le\sigma(X)\\ &\le\begin{cases}(b-a)^{\frac{1}{2}}\left[\frac{(b-a)^2}{12}+\left[E(X)-\frac{a+b}{2}\right]^2\right]^{\frac{1}{2}}\|f\|_\infty^{\frac{1}{2}}, & \text{if } f\in L_\infty[a,b];\\ \left\{\frac{[b-E(X)]^{2q+1}+[E(X)-a]^{2q+1}}{2q+1}\right\}^{\frac{1}{2q}}\|f\|_p^{\frac{1}{2}}, & \text{if } f\in L_p[a,b],\\ & p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \frac{b-a}{2}+\left|E(X)-\frac{a+b}{2}\right|,\end{cases}\end{aligned} \tag{1.3}$$

and

$$0\le\sigma^2(X)\le[b-E(X)][E(X)-a]\le\frac{1}{4}(b-a)^2. \tag{1.4}$$

In this paper more accurate inequalities are obtained by assuming that the p.d.f. of X is n-time differentiable and that $f^{(n)}$ is absolutely continuous on $[a,b]$. For other recent results on the application of Ostrowski type inequalities in Probability Theory, see [2]-[4].

2. Some Preliminary Integral Identities

The following lemma, which is interesting in itself, holds.

Lemma 2.1. *Let X be a random variable whose probability distribution function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned}\sigma^2(X)+[E(X)-x]^2&=\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{(k+3)\,k!}f^{(k)}(x)\\ &\quad+\frac{1}{n!}\int_a^b(t-x)^2\left(\int_x^t(t-s)^n f^{(n+1)}(s)\,ds\right)dt\end{aligned} \tag{2.1}$$

for all $x\in[a,b]$.

Proof. Is by Taylor's formula with integral remainder. Recall that

$$f(t)=\sum_{k=0}^{n}\frac{(t-x)^{k}}{k!}f^{(k)}(x)+\frac{1}{n!}\int_{x}^{t}(t-s)^{n}f^{(n+1)}(s)\,ds \tag{2.2}$$

for all $t,x\in[a,b]$.

Together with

$$\sigma^{2}(X)+[E(X)-x]^{2}=\int_{a}^{b}(t-x)^{2}f(t)\,dt, \tag{2.3}$$

where f is the p.d.f. of the random variable X, we obtain

$$\begin{aligned}&\sigma^{2}(X)+[E(X)-x]^{2}\\&=\int_{a}^{b}(t-x)^{2}\left[\sum_{k=0}^{n}\frac{(t-x)^{k}}{k!}f^{(k)}(x)+\frac{1}{n!}\int_{x}^{t}(t-s)^{n}f^{(n+1)}(s)\,ds\right]dt\\&=\sum_{k=0}^{n}f^{(k)}(x)\int_{a}^{b}\frac{(t-x)^{k+2}}{k!}dt+\frac{1}{n!}\int_{a}^{b}(t-x)^{2}\left(\int_{x}^{t}(t-s)^{n}f^{(n+1)}(s)\,ds\right)dt\end{aligned} \tag{2.4}$$

and since

$$\int_{a}^{b}\frac{(t-x)^{k+2}}{k!}dt=\frac{(b-x)^{k+3}+(-1)^{k}(x-a)^{k+3}}{(k+3)\,k!},$$

the identity (2.4) readily produces (2.1) □

Corollary 2.2. *Under the above assumptions, we have*

$$\begin{aligned}\sigma^{2}(X)+\left[E(X)-\frac{a+b}{2}\right]^{2}&=\sum_{k=0}^{n}\frac{\left[1+(-1)^{k}\right](b-a)^{k+3}}{2^{k+3}(k+3)\,k!}f^{(k)}\left(\frac{a+b}{2}\right)\\&\quad+\frac{1}{n!}\int_{a}^{b}\left(t-\frac{a+b}{2}\right)^{2}\left(\int_{\frac{a+b}{2}}^{t}(t-s)^{n}f^{(n+1)}(s)\,ds\right)dt.\end{aligned} \tag{2.5}$$

The proof follows by using (2.4) with $x=\frac{a+b}{2}$.

Corollary 2.3. *Under the above assumptions,*

$$\begin{aligned}&\sigma^{2}(X)+\frac{1}{2}\left[(E(X)-a)^{2}+(E(X)-b)^{2}\right]\\&=\sum_{k=0}^{n}\frac{(b-a)^{k+3}}{(k+3)\,k!}\left[\frac{f^{(k)}(a)+(-1)^{k}f^{(k)}(b)}{2}\right]\\&\quad+\frac{1}{n!}\int_{a}^{b}\int_{a}^{b}K(t,s)(t-s)^{n}f^{(n+1)}(s)\,dsdt,\end{aligned} \tag{2.6}$$

where

$$K(t,s):=\begin{cases}\frac{(t-a)^{2}}{2} & \text{if}\quad a\le s\le t\le b,\\ -\frac{(t-b)^{2}}{2} & \text{if}\quad a\le t<s\le b.\end{cases}$$

Proof. In (2.1), choose $x = a$ and $x = b$, giving

$$\sigma^2(X) + [E(X) - a]^2 = \sum_{k=0}^{n} \frac{(b-a)^{k+3}}{(k+3)\,k!} f^{(k)}(a) + \frac{1}{n!}\int_a^b (t-a)^2 \left(\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt \tag{2.7}$$

and

$$\sigma^2(X) + [E(X) - b]^2 = \sum_{k=0}^{n} \frac{(-1)^k (b-a)^{k+3}}{(k+3)\,k!} f^{(k)}(b) + \frac{1}{n!}\int_a^b (t-b)^2 \left(\int_b^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt. \tag{2.8}$$

Adding these and dividing by 2 gives (2.6). □

Taking into account that $\mu = E(X) \in [a,b]$, then we also obtain the following.

Corollary 2.4. *With the above assumptions,*

$$\sigma^2(X) = \sum_{k=0}^{n} \frac{(b-\mu)^{k+3} + (-1)^k (\mu-a)^{k+3}}{(k+3)\,k!} f^{(k)}(\mu) + \frac{1}{n!}\int_a^b (t-\mu)^2 \left(\int_\mu^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt. \tag{2.9}$$

Proof. The proof follows from (2.1) with $x = \mu \in [a,b]$. □

Lemma 2.5. *Let the conditions of Lemma 2.1 relating to f hold. Then the following identity is valid.*

$$\sigma^2(X) + [E(X) - x]^2 = \sum_{k=0}^{n} \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{k+3} \cdot \frac{f^{(k)}(x)}{k!} + \frac{1}{n!}\int_a^b K_n(x,s) f^{(n+1)}(s)\,ds, \tag{2.10}$$

where

$$K(x,s) = \begin{cases} (-1)^{n+1} \psi_n(s-a, x-s), & a \le s \le x \\ \psi_n(b-s, s-x), & x < s \le b \end{cases} \tag{2.11}$$

with

$$\psi_n(u,v) = \frac{u^{n+1}}{(n+3)(n+2)(n+1)} \cdot \left[(n+2)(n+1)u^2 + 2(n+3)(n+1)uv + (n+3)(n+2)v^2\right]. \tag{2.12}$$

Proof. From (2.1), an interchange of the order of integration gives

$$\begin{aligned}\frac{1}{n!}\int_a^b (t-x)^2\,dt\int_x^t (t-s)^n f^{(n+1)}(s)\,ds &= \frac{1}{n!}\left\{-\int_a^x\int_a^s (t-x)^2 (t-s)^n f^{(n+1)}(s)\,dtds\right.\\ &\quad\left.+\int_x^b\int_s^b (t-x)^2 (t-s)^n f^{(n+1)}(s)\,dtds\right\}\\ &= \frac{1}{n!}\int_a^b \tilde{K}_n(x,s)\,f^{(n+1)}(s)\,ds,\end{aligned}$$

where

$$\tilde{K}_n(x,s)=\begin{cases} p_n(x,s)=-\int_a^s (t-x)^2 (t-s)^n\,dt, & a\le s\le x\\ q_n(x,s)=\int_s^b (t-x)^2 (t-s)^n\,dt, & x<s<b.\end{cases}$$

To prove the lemma it is sufficient to show that $K\equiv\tilde{K}$.

Now,

$$\begin{aligned}\tilde{p}_n(x,s) &= -\int_a^s (t-x)^2 (t-s)^n\,dt=(-1)^{n+1}\int_0^{s-a}(u+x-s)^2 u^n du\\ &= (-1)^{n+1}\int_0^{s-a}\left[u^2+2(x-s)u+(x-s)^2\right]u^n du\\ &= (-1)^{n+1}\psi_n(s-a,x-s),\end{aligned}$$

where $\psi(\cdot,\cdot)$ is as given by (2.12). Further,

$$\tilde{q}_n(x,s)=\int_s^b (t-x)^2 (t-s)^n\,dt=\int_0^{b-s}[u+(s-x)]^2 u^n du=\psi_n(b-s,s-x),$$

where, again, $\psi(\cdot,\cdot)$ is as given by (2.12). Hence $K\equiv\tilde{K}$ and the lemma is proved. □

3. SOME INEQUALITIES

We are now able to obtain the following inequalities.

Theorem 3.1. *Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then*

$$\left|\sigma^2(X)+[E(X)-x]^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{(k+3)\,k!}f^{(k)}(x)\right| \tag{3.1}$$

$$\le\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}\left[(x-a)^{n+4}+(b-x)^{n+4}\right], & \textit{if}\quad f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{n!\left(n+3+\frac{1}{q}\right)}\frac{\left[(x-a)^{n+3+\frac{1}{q}}+(b-x)^{n+3+\frac{1}{q}}\right]}{(nq+1)^{\frac{1}{q}}}, & \textit{if}\quad f^{(n+1)}\in L_p[a,b],\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}\left[(x-a)^{n+3}+(b-x)^{n+3}\right],\end{cases}$$

for all $x \in [a,b]$*, where* $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ *are the usual Lebesque norms on* $[a,b]$*, i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)| \quad \textit{and} \quad \|g\|_p := \left(\int_a^b |g(t)|^p \, dt\right)^{\frac{1}{p}}, \;\; p \geq 1.$$

Proof. By Lemma 2.1,

$$\begin{aligned}
&\sigma^2(X) + [E(X) - x]^2 - \sum_{k=0}^{n} \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{k!\,(k+3)} f^{(k)}(x) \\
&= \frac{1}{n!}\int_a^b (t-x)^2 \left(\int_x^t (t-s)^n f^{(n+1)}(s)\, ds\right) dt \\
&:= M(a,b;x). \qquad (3.2)
\end{aligned}$$

Clearly,

$$\begin{aligned}
|M(a,b;x)| &\leq \frac{1}{n!}\int_a^b (t-x)^2 \left|\int_x^t (t-s)^n f^{(n+1)}(s)\, ds\right| dt \\
&\leq \frac{1}{n!}\int_a^b (t-x)^2 \left[\sup_{s\in[x,t]} \left|f^{(n+1)}(s)\right| \left|\int_x^t |t-s|^n \, ds\right|\right] dt \\
&\leq \frac{\left\|f^{(n+1)}\right\|_\infty}{n!}\int_a^b \frac{(t-x)^2 |t-x|^{n+1}}{n+1} dt \\
&= \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}\int_a^b |t-x|^{n+3}\, dt \\
&= \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}\left[\int_a^x (x-t)^{n+3}\, dt + \int_x^b (t-x)^{n+3}\, dt\right] \\
&= \frac{\left\|f^{(n+1)}\right\|_\infty \left[(x-a)^{n+4} + (b-x)^{n+4}\right]}{(n+1)!\,(n+4)}
\end{aligned}$$

and the first inequality in (3.1) is obtained.

For the second, we use Hölder's integral inequality to obtain

$$\begin{aligned}
|M(a,b;x)| &\leq \frac{1}{n!}\int_a^b (t-x)^2 \left|\int_x^t |t-s|^{nq}\, ds\right|^{\frac{1}{q}} \left|\int_x^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} dt \\
&\leq \frac{1}{n!}\left(\int_a^b \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}} \int_a^b (t-x)^2 |t-x|^{\frac{nq+1}{q}}\, dt \\
&= \frac{1}{n!}\frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}}\int_a^b |t-x|^{n+2+\frac{1}{q}}\, dt \\
&= \frac{1}{n!}\frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}}\left[\frac{(b-x)^{n+3+\frac{1}{q}} + (x-a)^{n+3+\frac{1}{q}}}{n+3+\frac{1}{q}}\right].
\end{aligned}$$

Finally, note that

$$\begin{aligned}|M(a,b;x)| &\le \frac{1}{n!}\int_a^b (t-x)^2|t-x|^n \left|\int_x^t \left|f^{(n+1)}(s)\right| ds\right| dt \\ &\le \frac{\left\|f^{(n+1)}\right\|_1}{n!}\int_a^b |t-x|^{n+2}\,dt \\ &= \frac{\left\|f^{(n+1)}\right\|_1}{n!}\left[\frac{(x-a)^{n+3}+(b-x)^{n+3}}{n+3}\right]\end{aligned}$$

and the third part of (3.1) is obtained. □

It is obvious that the best inequality in (3.1) is when $x=\frac{a+b}{2}$, giving Corollary 3.2.

Corollary 3.2. *With the above assumptions on X and f,*

$$(3.3)\quad \left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\sum_{k=0}^{n}\frac{\left[1+(-1)^k\right](b-a)^{k+3}}{2^{k+3}(k+3)\,k!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\le\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{2^{n+3}(n+1)!(n+4)}(b-a)^{n+4}, & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{2^{n+2+\frac{1}{q}}n!\left(n+3+\frac{1}{q}\right)}\frac{(b-a)^{n+3+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\ p>1,\\ & \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{2^{n+2}n!(n+3)}(b-a)^{n+3}. & \end{cases}$$

The following corollary is interesting as it provides the opportunity to approximate the variance when the values of $f^{(k)}(\mu)$ are known, $k=0,\ldots,n$.

Corollary 3.3. *With the above assumptions and $\mu=\frac{a+b}{2}$, we have*

$$(3.4)\quad \left|\sigma^2(X)-\sum_{k=0}^{n}\frac{(b-\mu)^{k+3}+(-1)^k(\mu-a)^{k+3}}{(k+3)\,k!}f^{(k)}(\mu)\right|$$
$$\le\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}\left[(\mu-a)^{n+4}+(b-\mu)^{n+4}\right], & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{n!\left(n+3+\frac{1}{q}\right)}\frac{\left[(\mu-a)^{n+3+\frac{1}{q}}+(b-\mu)^{n+3+\frac{1}{q}}\right]}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}\left[(\mu-a)^{n+3}+(b-\mu)^{n+3}\right]. & \end{cases}$$

The following result also holds.

Theorem 3.4. *Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then*

(3.5)
$$\left|\sigma^2(X)+\frac{1}{2}\left[(E(X)-a)^2+(E(X)-b)^2\right]-\sum_{k=0}^{n}\frac{(b-a)^{k+3}}{(k+3)\,k!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|$$
$$\leq\begin{cases}\frac{1}{(n+4)(n+1)!}\left\|f^{(n+1)}\right\|_\infty (b-a)^{n+4}, & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{2^{1/q-1}}{n!(qn+1)^{\frac{1}{q}}[(n+2)q+2]^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_p\frac{(b-a)^{n+3+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2n!}\left\|f^{(n+1)}\right\|_1 (b-a)^{n+3}, & \end{cases}$$

where $\|\cdot\|_p$ $(1\leq p\leq\infty)$ are the usual Lebesque $p-$norms.

Proof. Using Corollary 2.3,

$$\left|\sigma^2(X)+\frac{1}{2}\left[(E(X)-a)^2+(E(X)-b)^2\right]-\sum_{k=0}^{n}\frac{(b-a)^{k+3}}{(k+3)\,k!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|$$
$$\leq\frac{1}{n!}\int_a^b\int_a^b|K(t,s)|\,|t-s|^n\left|f^{(n+1)}(s)\right|dsdt$$
$$=: N(a,b).$$

It is obvious that

$$\begin{aligned}N(a,b) &\leq \left\|f^{(n+1)}\right\|_\infty\frac{1}{n!}\int_a^b\int_a^b|K(t,s)|\,|t-s|^n\,dsdt\\ &= \left\|f^{(n+1)}\right\|_\infty\frac{1}{n!}\int_a^b\left(\int_a^t|K(t,s)|\,|t-s|^n\,ds+\int_t^b|K(t,s)|\,|t-s|^n\,ds\right)dt\\ &= \frac{1}{n!}\left\|f^{(n+1)}\right\|_\infty\int_a^b\left[\frac{(t-a)^2}{2}\cdot\frac{(t-a)^{n+1}}{n+1}+\frac{(t-b)^2}{2}\cdot\frac{(b-t)^{n+1}}{n+1}\right]dt\\ &= \frac{1}{2(n+1)!}\left\|f^{(n+1)}\right\|_\infty\int_a^b\left[(t-a)^{n+3}+(b-t)^{n+3}\right]dt\\ &= \frac{1}{2(n+1)!}\left\|f^{(n+1)}\right\|_\infty\left[\frac{(b-a)^{n+4}}{n+4}+\frac{(b-a)^{n+4}}{n+4}\right]\\ &= \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+4)(n+1)!}(b-a)^{n+4}\end{aligned}$$

so the first part of (3.5) is proved.

Using Hölder's integral inequality for double integrals,

$$
\begin{aligned}
&N(a,b) \\
&\le \frac{1}{n!}\left(\int_a^b\int_a^b \left|f^{(n+1)}(s)\right|^p dsdt\right)^{\frac{1}{p}} \times \left(\int_a^b\int_a^b |K(t,s)|^q\,|t-s|^{qn}\,dsdt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\left[\int_a^b\left(\int_a^t |K(t,s)|^q\,|t-s|^{qn}\,ds + \int_t^b |K(t,s)|^q\,|t-s|^{qn}\,ds\right)dt\right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\left[\int_a^b\left[\frac{(t-a)^{2q}}{2^q}\int_a^t |t-s|^{qn}\,ds + \frac{(t-b)^{2q}}{2^q}\int_t^b |t-s|^{qn}\,ds\right]dt\right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\left[\int_a^b\left[\frac{(t-a)^{2q}(t-a)^{qn+1}}{2^q(qn+1)} + \frac{(t-b)^{2q}(b-t)^{qn+1}}{2^q(qn+1)}\right]dt\right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\frac{1}{2^q(qn+1)}\right]^{\frac{1}{q}}\left[\int_a^b (t-a)^{(n+2)q+1}\,dt + \int_a^b (b-t)^{(n+2)q+1}\,dt\right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\frac{1}{2^q(qn+1)}\right]^{\frac{1}{q}}\left[\frac{(b-a)^{(n+2)q+2}}{(n+2)q+2} + \frac{(b-a)^{(n+2)q+2}}{(n+2)q+2}\right]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{1/q}\left\|f^{(n+1)}\right\|_p (b-a)^{n+2+\frac{1}{p}+\frac{2}{q}}}{n!2\,(qn+1)^{\frac{1}{q}}\,((n+2)q+2)^{\frac{1}{q}}} \\
&= \frac{2^{1/q-1}\left\|f^{(n+1)}\right\|_p\left[(b-a)^{n+3+\frac{1}{q}}\right]}{n!\,(qn+1)^{\frac{1}{q}}\,[(n+2)q+2]^{\frac{1}{q}}}
\end{aligned}
$$

and the second part of (3.5) is proved.

Finally, we observe that

$$
\begin{aligned}
N(a,b) &\le \frac{1}{n!}\sup_{(t,s)\in[a,b]^2}|K(t,s)|\,|t-s|^n\int_a^b\int_a^b\left|f^{(n+1)}(s)\right|dsdt \\
&= \frac{1}{n!}\frac{(b-a)^2}{2}\cdot(b-a)^n(b-a)\int_a^b\left|f^{(n+1)}(s)\right|ds \\
&= \frac{1}{2n!}(b-a)^{n+3}\left\|f^{(n+1)}\right\|_1,
\end{aligned}
$$

which is the final result of (3.5). □

The following particular case can be useful in practical applications. For $n = 0$, (3.1) becomes

$$\left|\sigma^2(X) + [E(X) - x]^2 - (b-a)\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right] f(x)\right| \tag{3.6}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{4}\left[(x-a)^4 + (b-x)^4\right], & \text{if } f' \in L_\infty[a,b]; \\ \frac{q\|f'\|_p}{3q+1}\left[(x-a)^{3+\frac{1}{q}} + (b-x)^{3+\frac{1}{q}}\right], & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f'\|_1\left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2\right], \end{cases}$$

for all $x \in [a,b]$. In particular, for $x = \frac{a+b}{2}$,

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^3}{12} f\left(\frac{a+b}{2}\right)\right| \tag{3.7}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{32}(b-a)^4, & \text{if } f' \in L_\infty[a,b]; \\ \frac{q\|f'\|_p (b-a)^{3+\frac{1}{q}}}{2^{2+\frac{1}{q}}(3q+1)}, & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{12}(b-a)^3, \end{cases}$$

which is, in a sense, the best inequality that can be obtained from (3.6). If in (3.6) $x = \mu = E(X)$, then

$$\left|\sigma^2(X) - (b-a)\left[\left(E(X) - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right] f(E(X))\right| \tag{3.8}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{4}\left[(E(X)-a)^4 + (b-E(X))^4\right], & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{\left(3+\frac{1}{q}\right)}\left[(E(X)-a)^4 + (b-E(X))^4\right], & \text{if } f' \in L_p[a,b],\ p>1, \\ \|f'\|_1\left[\frac{(b-a)^2}{12} + \left(E(X) - \frac{a+b}{2}\right)^2\right]. \end{cases}$$

In addition, from (3.5),

$$\left|\sigma^2(X) + \frac{1}{2}\left[(E(X)-a)^2 + (E(X)-b)^2\right] - \frac{(b-a)^3}{3}\left[\frac{f(a)+f(b)}{2}\right]\right| \tag{3.9}$$

$$\leq \begin{cases} \frac{1}{4}\|f'\|_\infty (b-a)^4, & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{n!2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\|f'\|_p (b-a)^{3+\frac{1}{q}}, & \text{if } f' \in L_p[a,b],\ p>1, \\ \frac{1}{2}\|f'\|_1 (b-a)^3, \end{cases}$$

which provides an approximation for the variance in terms of the expectation and the values of f at the end points a and b.

Theorem 3.5. *Let X be a random variable whose p.d.f. $f:[a,b]\to\mathbb{R}_+$ is $n-$time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. Then*

$$\left|\sigma^2(X)+(E(X)-x)^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k+3}\cdot\frac{f^{(k)}(x)}{k!}\right| \tag{3.10}$$

$$\leq\begin{cases}\left[(x-a)^{n+4}+(b-x)^{n+4}\right]\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}, & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ C^{\frac{1}{q}}\left[(x-a)^{(n+3)q+1}+(b-x)^{(n+3)q+1}\right]^{\frac{1}{q}}\frac{\left\|f^{(n+1)}\right\|_p}{n!}, & \text{if } f^{(n+1)}\in L_p[a,b],\ p>1;\\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n+3}\cdot\frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}, & \end{cases}$$

where

$$C=\int_0^1\left[\frac{u^{n+3}}{n+3}+2(1-u)\frac{u^{n+2}}{n+2}+(1-u)^2\frac{u^{n+1}}{n+1}\right]^q du. \tag{3.11}$$

Proof. From (2.10),

$$\left|\sigma^2(X)+(E(X)-x)^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k+3}\cdot\frac{f^{(k)}(x)}{k!}\right| \tag{3.12}$$
$$=\left|\frac{1}{n!}\int_a^b K_n(x,s)f^{(n+1)}(s)\,ds\right|.$$

Now, on using the fact that from (2.11), (2.12), $\psi_n(u,v)\geq 0$ for $u,v\geq 0$,

$$\left|\frac{1}{n!}\int_a^b K_n(x,s)f^{(n+1)}(s)\,ds\right| \tag{3.13}$$
$$\leq\frac{\left\|f^{(n+1)}\right\|_\infty}{n!}\left\{\int_a^x\psi_n(s-a,x-s)\,ds+\int_x^b\psi_n(b-s,s-x)\,ds\right\}.$$

Further,

$$\psi_n(u,v)=\frac{u^{n+3}}{n+3}+2v\frac{u^{n+2}}{n+2}+v^2\frac{u^{n+1}}{n+1} \tag{3.14}$$

and so

$$\int_a^x\psi_n(s-a,x-s)\,ds \tag{3.15}$$
$$=\int_a^x\left[\frac{(s-a)^{n+3}}{n+3}+2(x-s)\frac{(s-a)^{n+2}}{n+2}+(x-s)^2\frac{(s-a)^{n+1}}{n+1}\right]ds$$
$$=(x-a)^{n+4}\int_0^1\left[\frac{\lambda^{n+3}}{n+3}+2(1-\lambda)\frac{\lambda^{n+2}}{n+2}+(1-\lambda)^2\frac{\lambda^{n+1}}{n+1}\right]d\lambda,$$

where we have made the substitution $\lambda=\frac{s-a}{x-a}$.

Collecting powers of λ gives

$$\lambda^{n+3}\left[\frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1}\right]-\frac{2\lambda^{n+2}}{(n+2)(n+1)}+\frac{\lambda^{n+1}}{n+1}$$

and so, from (3.15),

$$\int_a^x \psi_n(s-a,x-s)\,ds = (x-a)^{n+4}\left\{\frac{1}{n+4}\left[\frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1}\right] - \frac{2}{(n+3)(n+2)(n+1)}+\frac{1}{(n+2)(n+1)}\right\} = \frac{(x-a)^{n+4}}{(n+4)(n+1)}. \tag{3.16}$$

Similarly, on using (3.14),

$$\int_x^b \psi_n(b-s,s-x)\,ds = \int_x^b\left[\frac{(b-s)^{n+3}}{n+3}+2(s-x)\frac{(b-s)^{n+2}}{n+2}+(s-x)^2\frac{(b-s)^{n+1}}{n+1}\right]ds$$

and making the substitution $\nu=\frac{b-s}{b-x}$ gives

$$\begin{aligned}\int_x^b \psi_n(b-s,s-x)\,ds &= (b-x)^{n+4}\int_0^1\left[\frac{\nu^{n+3}}{n+3}+2(1-\nu)\frac{\nu^{n+2}}{n+2}+(1-\nu)^2\frac{\nu^{n+1}}{n+1}\right]d\nu\\ &= \frac{(b-x)^{n+4}}{(n+4)(n+1)}, \end{aligned}\tag{3.17}$$

where we have used (3.15) and (3.16). Combining (3.16) and (3.17) gives the first inequality in (3.10). For the second inequality in (3.10), we use Hölder's integral inequality to obtain

$$\left|\frac{1}{n!}\int_a^b K_n(x,s)\,f^{(n+1)}(s)\,ds\right| \le \frac{\left\|f^{(n+1)}(s)\right\|_p}{n!}\left(\int_a^b |K_n(x,s)|^q\,ds\right)^{\frac{1}{q}}. \tag{3.18}$$

Now, from (2.11) and (3.14)

$$\begin{aligned}\int_a^b |K_n(x,s)|^q\,ds &= \int_a^x \psi^q(s-a,x-s)\,ds+\int_x^b \psi^q(b-s,s-x)\,ds\\ &= C\left[(x-a)^{(n+3)q+1}+(b-x)^{(n+3)q+1}\right],\end{aligned}$$

where C is as defined in (3.11) and we have used (3.15) and (3.16). Substitution into (3.18) gives the second inequality in (3.10).

Finally, for the third inequality in (3.10). From (3.12),

$$\begin{aligned}&\left|\frac{1}{n!}\int_a^b K_n(x,s)\,f^{(n+1)}(s)\,ds\right|\\ &\le \frac{1}{n!}\left\{\int_a^x \psi_n(s-a,x-s)\left|f^{(n+1)}(s)\right|ds+\int_x^b \psi_n(b-s,s-x)\left|f^{(n+1)}(s)\right|ds\right\}\\ &\le \frac{1}{n!}\left\{\psi_n(x-a,0)\int_a^x\left|f^{(n+1)}(s)\right|ds+\psi_n(b-x,0)\int_x^b\left|f^{(n+1)}(s)\right|ds\right\},\end{aligned}\tag{3.19}$$

where, from (3.14),

$$\psi_n(u,0)=\frac{u^{n+3}}{n+3}. \tag{3.20}$$

Hence, from (3.19) and (3.20)

$$\begin{aligned}\left|\frac{1}{n!}\int_a^b K_n(x,s)\, f^{(n+1)}(s)\,ds\right| &\le \frac{1}{n!}\max\left\{\frac{(x-a)^{n+3}}{n+3},\frac{(b-x)^{n+3}}{n+3}\right\}\left\|f^{(n+1)}(\cdot)\right\|_1 \\ &= \frac{1}{n!\,(n+3)}\left[\max\{x-a,b-x\}\right]^{n+3}\left\|f^{(n+1)}(\cdot)\right\|_1,\end{aligned}$$

which, on using the fact that for $X, Y \in \mathbb{R}$

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$$

gives, from (3.12), the third inequality in (3.10). The theorem is now completely proved. □

Remark 3.6. The results of Theorem 3.5 may be compared with those of Theorem 3.1. Theorem 3.5 is based on the single integral identity developed in Lemma 2.5, while Theorem 3.1 is based on the double integral identity representation for the bound. It may be noticed from (3.1) and (3.10) that the bounds are the same for $f^{(n+1)} \in L_\infty[a,b]$, that for $f^{(n+1)} \in L_1[a,b]$ (which is always true since $f^{(n)}$ is absolutely continuous) the bound obtained in (3.1) is better and for $f^{(n+1)} \in L_p[a,b]$, $p>1$, the result is inconclusive.

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