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SOME RESULTS ON L^1-APPROXIMATION OF THE r-TH DERIVATE OF FOURIER SERIES

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ABSTRACT. In this paper we obtain the conditions for L^1-convergence of the r-th derivatives of the cosine and sine trigonometric series. These results are extensions of corresponding Sidon's and Telyakovskii's theorems for trigonometric series (case: $r = 0$).

Key words and phrases: L^1-approximation, Fourier series, Sidon-Telyakovskii class, Telyakovskii inequality.

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1. INTRODUCTION

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \,, \tag{1.1}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

be the cosine and sine trigonometric series with real coefficients.

Let $\Delta a_n = a_n - a_{n+1}$, $n \in \{0, 1, 2, 3, \ldots\}$. The Dirichlet's kernel, conjugate Dirichlet's kernel and modified Dirichlet's kernel are denoted respectively by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}} \,,$$

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}} \,,$$

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$$\overline{D}_n(t) = -\frac{1}{2}\text{ctg}\frac{t}{2} + \tilde{D}_n(t) = -\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}.$$

Let

$$E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt} \text{ and } E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}.$$

Then the r-th derivatives $D_n^{(r)}(t)$ and $\tilde{D}_n^{(r)}(t)$ can be written as

$$2D_n^{(r)}(t) = E_n^{(r)}(t) + E_{-n}^{(r)}(t), \tag{1.3}$$

$$2i\tilde{D}_n^{(r)}(t) = E_n^{(r)}(t) - E_{-n}^{(r)}(t). \tag{1.4}$$

In [2], Sidon proved the following theorem.

Theorem 1.1. *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \le 1$, for every n and let $\sum_{n=1}^{\infty}|p_n|$ converge. If*

$$a_n = \sum_{k=n}^{\infty}\frac{p_k}{k}\sum_{l=n}^{k}\alpha_l,\ n \in \mathbb{N} \tag{1.5}$$

then the cosine series (1.1) is the Fourier series of its sum f.

Several authors have studied the problem of L^1-convergence of the series (1.1) and (1.2).

In [4] Telyakovskii defined the following class of L^1-convergence of Fourier series. A sequence $\{a_k\}_{k=0}^{\infty}$ belongs to the class S, or $\{a_k\} \in S$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty}A_k < \infty$ and $|\Delta a_k| \le A_k$ for all k.

The importance of Telyakovskii's contributions are twofold. Firstly, he expressed Sidon's conditions (1.5) in a succinct equivalent form, and secondly, he showed that the class S is also a class of L^1-convergence. Thus, the class S is usually called the Sidon–Telyakovskii class.

In the same paper, Telyakovskii proved the following two theorems.

Theorem 1.2. [4]. *Let the coefficients of the series $f(x)$ belong to the class S. Then the series is a Fourier series and the following inequality holds:*

$$\int_0^{\pi}|f(x)|\,dx \le M\sum_{n=0}^{\infty}A_n,$$

where M is a positive constant, independent on f.

Theorem 1.3. [4]. *Let the coefficients of the series $g(x)$ belong to the class S. Then the following inequality holds for $p = 1, 2, 3, \dots$*

$$\int_{\pi/(p+1)}^{\pi}|g(x)|\,dx = \sum_{n=1}^{p}\frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty}A_n\right).$$

In particular, $g(x)$ is a Fourier series iff $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$.

In [5], we extended the Sidon–Telyakovskii class $S = S_0$, i.e., we defined the class S_r, $r = 1, 2, 3, \dots$ as follows: $\{a_k\}_{k=1}^{\infty} \in S_r$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty}k^rA_k < \infty$ and $|\Delta a_k| \le A_k$ for all k.

We note that by $A_k \downarrow 0$ and $\sum_{k=1}^{\infty}k^rA_k < \infty$, we get

$$k^{r+1}A_k = o(1), \quad k \to \infty. \tag{1.6}$$

It is trivially to see that $S_{r+1} \subset S_r$ for all $r = 1, 2, 3, \dots$. Now, let $\{a_n\}_{n=1}^{\infty} \in S_1$. For arbitrary real number a_0, we shall prove that sequence $\{a_n\}_{n=0}^{\infty}$ belongs to S_0. We define $A_0 =$

$\max(|\Delta a_0|, A_1)$. Then $|\Delta a_0| \le A_0$, i.e. $|\Delta a_n| \le A_n$, for all $n \in \{0, 1, 2, ...\}$ and $\{A_n\}_{n=0}^{\infty}$ is monotonically decreasing sequence.

On the other hand,

$$\sum_{n=0}^{\infty} A_n \le A_0 + \sum_{n=1}^{\infty} nA_n < \infty .$$

Thus, $\{a_n\}_{n=0}^{\infty} \in S_0$, i.e. $S_{r+1} \subset S_r$, for all $r = 0, 1, 2, \dots$. The next example verifies that the implication

$$\{a_n\} \in S_{r+1} \Rightarrow \{a_n\} \in S_r\,, \quad r = 0, 1, 2, \dots$$

is not reversible.

Example 1.1. For $n = 0, 1, 2, 3, ...$ define $a_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}$. Then $a_n \to 0$ as $n \to \infty$ and for $n = 0, 1, 2, 3, \dots$, $\Delta a_n = \frac{1}{(n+1)^2}$. Firstly we shall show that $\{a_n\}_{n=1}^{\infty} \notin S_1$.

Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A \downarrow 0$ and $\Delta a_n = |\Delta a_n| \le A_n$. However, $\sum_{n=1}^{\infty} nA_n \ge \sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$ is divergent, i.e. $\{a_n\} \notin S_1$.

Now, for all $n = 0, 1, 2, \dots$ let $A_n = \frac{1}{(n+1)^2}$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=0}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{a_n\}_{n=0}^{\infty} \in S_0$.

Our next example will show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty} \in S_r$ but $\{a_n\}_{n=1}^{\infty} \notin S_{r+1}$, for all $r = 1, 2, 3, \dots$.

Namely, for all $n = 1, 2, 3, \dots$ let $a_n = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}}$. Then $a_n \to 0$ as $n \to \infty$ and for $n = 1, 2, 3, \dots$, $\Delta a_n = \frac{1}{n^{r+2}}$. Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A_n \downarrow 0$ and $\Delta a_n = |\Delta a_n| \le A_n$. However,

$$\sum_{n=1}^{\infty} n^{r+1} A_n \ge \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent, i.e. $\{a_n\} \notin S_{r+1}$. On the other hand, for all $n = 1, 2, \dots$ let $A_n = \frac{1}{n^{r+2}}$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{a_n\} \in S_r$.

In the same paper [5] we proved the following theorem.

Theorem 1.4. [5]. *Let the coefficients of the series (1.1) belong to the class* $S_r,\ r = 0, 1, 2, \dots$. *Then the* $r-$*th derivative of the series (1.1) is a Fourier series of some* $f^{(r)} \in L^1(0, \pi)$ *and the following inequality holds:*

$$\int_0^{\pi} \left| f^{(r)}(x) \right| dx \le M \sum_{n=1}^{\infty} n^r A_n,$$

where $0 < M = M(r) < \infty$.

This is an extension of the Telyakovskii Theorem 1.2.

2. RESULTS

In this paper, we shall prove the following main results.

Theorem 2.1. *A null sequence* $\{a_n\}$ *belongs to the class* S_r, $r = 0, 1, 2, \dots$ *if and only if it can be represented as*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l,\ n \in \mathbb{N} \tag{2.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ are sequences such that $|\alpha_n| \leq 1$, for all n and

$$\sum_{n=1}^{\infty} n^r |p_n| < \infty.$$

Corollary 2.2. *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \leq 1$, for every n and let $\sum_{n=1}^{\infty} n^r |p_n| < \infty$, $r = 0, 1, 2, \dots$. If*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \; n \in \mathbb{N}$$

then the $r-$th derivate of the series (1.1) is a Fourier series of some $f^{(r)} \in L^1$.

Theorem 2.3. *Let the coefficients of the series $g(x)$ belong to the class S_r, $r = 0, 1, 2, \dots$ Then the r-th derivate of the series (1.2) converges to a function and for $m = 1, 2, 3, \dots$ the following inequality holds:*

$$(*) \qquad \int_{\pi/(m+1)}^{\pi} \left| g^{(r)}(x) \right| dx \leq M \left(\sum_{n=1}^{m} |a_n| \cdot n^{r-1} + \sum_{n=1}^{\infty} n^r A_n \right),$$

where

$$0 < M = M(r) < \infty .$$

Moreover, if $\sum_{n=1}^{\infty} n^{r-1} |a_n| < \infty$, then the r-th derivate of the series (1.2) is a Fourier series of some $g^{(r)} \in L^1(0, \pi)$ and

$$\int_0^{\pi} \left| g^{(r)}(x) \right| dx \leq M \left(\sum_{n=1}^{\infty} |a_n| \cdot n^{r-1} + \sum_{n=1}^{\infty} n^r A_n \right)$$

Corollary 2.4. *Let the coefficients of the series $g(x)$ belong to the class S_r, $r \geq 1$. Then the following inequality holds:*

$$\int_0^{\pi} \left| g^{(r)}(x) \right| dx \leq M \sum_{n=1}^{\infty} n^r A_n,$$

where $0 < M = M(r) < \infty$.

3. Lemmas

For the proof of our new theorems we need the following lemmas.

The following lemma proved by Sheng, can be reformulated in the following way.

Lemma 3.1. [1] *Let r be a nonnegative integer and $x \in (0, \pi]$, where $n \geq 1$. Then*

$$D_n^{(r)}(x) = \sum_{k=0}^{r} \frac{\left(n + \frac{1}{2}\right)^k \sin\left[\left(n + \frac{1}{2}\right) x + \frac{k\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \varphi_k(x),$$

where $\varphi_r \equiv \frac{1}{2}$ and φ_k, $k = 0, 1, 2, \dots, r-1$ denotes various entire 4π – periodic functions of x, independent of n. More precisely, φ_k, $k = 0, 1, 2, \dots, r$ are trigonometric polynomials of $\frac{x}{2}$.

Lemma 3.2. *Let* $\{\alpha_j\}_{j=0}^{k}$ *be a sequence of real numbers. Then the following relation holds for* $\nu = 0, 1, 2, \dots, r$ *and* $r = 0, 1, 2, \dots$

$$\begin{aligned} U_k &= \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \right| dx \\ &= O\left((k+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2\nu} \right)^{1/2} \right). \end{aligned}$$

Proof. Applying first Cauchy–Buniakowski inequality, yields

$$\begin{aligned} U_k \le & \left[\int_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-\nu)}} \right]^{1/2} \\ & \times \left\{ \int_{\pi/(k+1)}^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{(\nu+3)\pi}{2}\right] \right]^2 dx \right\}^{1/2}. \end{aligned}$$

Since

$$\begin{aligned} \int_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-\nu)}} &\le \pi^{2(r+1-\nu)} \int_{\pi/(k+1)}^{\pi} \frac{dx}{x^{2(r+1-\nu)}} \\ &\le \frac{\pi(k+1)^{2(r+1-\nu)-1}}{2(r+1-\nu)-1} \\ &\le \pi(k+1)^{2(r+1-\nu)-1}, \end{aligned}$$

we have

$$\begin{aligned} U_k \le & \left[\pi(k+1)^{2(r+1-\nu)-1}\right]^{1/2} \\ & \times \left\{ \int_0^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right] \right]^2 dx \right\}^{1/2} \\ \le & \left[2\pi(k+1)^{2(r+1-\nu)-1}\right]^{1/2} \left\{ \int_0^{2\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^{\nu} \sin\left[(2j+1)t + \frac{\nu+3}{2}\pi\right] \right]^2 dt \right\}^{1/2}. \end{aligned}$$

Then, applying Parseval's equality, we obtain:

$$U_k \le \left[2\pi (k+1)^{2(r+1-\nu)-1}\right]^{1/2} \left[\sum_{j=0}^{k} |\alpha_j|^2 (j+1)^{2\nu} \right]^{1/2}.$$

Finally,

$$U_k = O\left((k+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2\nu} \right)^{1/2} \right).$$

□

Lemma 3.3. *Let $r \in \{0,1,2,3,\dots\}$ and $\{\alpha_k\}_{k=0}^n$ be a sequence of real numbers such that $|\alpha_k| \le 1$, for all k. Then there exists a finite constant $M = M(r) > 0$ such that for any $n \ge 0$*

$$(**) \qquad \int_{\pi/(n+1)}^{\pi} \left| \sum_{k=0}^{n} \alpha_k \overline{D}_k^{(r)}(x) \right| dx \le M \cdot (n+1)^{r+1}.$$

Proof. Similar to Lemma 3.1 it is not difficult to proof the following equality

$$\overline{D}_n^{(r)}(x) = \sum_{k=0}^{r} \frac{\left(n+\frac{1}{2}\right)^{k} \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \varphi_k(x),$$

where φ_k denotes the same various 4π-periodic functions of x, independent of n.

Now, we have:

$$\int_{\pi/(n+1)}^{\pi} \left| \sum_{k=0}^{n} \alpha_k \overline{D}_k^{(r)}(x) \right| dx$$
$$\le \int_{\pi/(n+1)}^{\pi} \left| \sum_{j=0}^{n} \alpha_j \left(\sum_{\nu=0}^{r} \frac{\left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \varphi_\nu(x) \right) \right| dx.$$

Since φ_ν are bounded, we have:

$$\int_{\pi/(n+1)}^{\pi} \left| \sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \varphi_\nu(x) \right| dx \le K\, U_n\,,$$

where U_n is the integral as in Lemma 3.2, and $K = K(r)$ is a positive constant.

Applying Lemma 3.2, to the last integral, we obtain:

$$\int_{\pi/(n+1)}^{\pi} \left| \sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \varphi_\nu(x) \right| dx$$
$$= O\left((n+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{n} \alpha_j^2 (j+1)^{2\nu} \right)^{1/2} \right)$$
$$= O\left((n+1)^{r-\nu+\frac{1}{2}} (n+1)^{\nu+\frac{1}{2}} \right) = O\left((n+1)^{r+1} \right).$$

Finally the inequality $(**)$ is satisfied. □

Remark 3.4. For $r = 0$, we obtain the Telyakovskii type inequality, proved in [4].

Lemma 3.5. *Let r be a non-negative integer. Then for all $0 < |t| \le \pi$ and all $n \ge 1$ the following estimates hold:*

(i) $\left| E_{-n}^{(r)}(t) \right| \le \frac{4n^r\pi}{|t|}$,

(ii) $\left| \tilde{D}_n^{(r)}(t) \right| \le \frac{4n^r\pi}{|t|}$,

(iii) $\left| \overline{D}_n^{(r)}(t) \right| \le \frac{4n^r\pi}{|t|} + O\left(\frac{1}{|t|^{r+1}}\right)$.

Proof. (i) The case $r = 0$ is trivial. Really,

$$|E_n(t)| \le |D_n(t)| + |\tilde{D}_n(t)| \le \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{2|t|} < \frac{4\pi}{|t|},$$

$$|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}.$$

Let $r \geq 1$. Applying the Abel's transformation, we have:

$$E_n^{(r)}(t) = i^r \sum_{k=1}^{n} k^r e^{ikt} = i^r \left[\sum_{k=1}^{n-1} \Delta(k^r) \left(E_k(t) - \frac{1}{2} \right) + n^r \left(E_n(t) - \frac{1}{2} \right) \right]$$

$$\begin{aligned} |E_n^{(r)}(t)| &\leq \sum_{k=1}^{n-1} [(k+1)^r - k^r] \left(\frac{1}{2} + |E_k(t)| \right) + n^r \left(\frac{1}{2} + |E_n(t)| \right) \\ &\leq \left(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|} \right) \left\{ \sum_{k=1}^{n-1} [(k+1)^r - k^r] + n^r \right\} = \frac{4\pi n^r}{|t|} . \end{aligned}$$

Since $E_{-n}^{(r)}(t) = E_n^{(r)}(-t)$, we obtain $\left| E_{-n}^{(r)}(t) \right| \leq \frac{4n^r\pi}{|t|}$.

(ii) Applying the inequality (i), we obtain

$$\left| \tilde{D}_n^{(r)}(t) \right| = \left| i\tilde{D}_n^{(r)}(t) \right| \leq \frac{1}{2} \left| E_n^{(r)}(t) \right| + \frac{1}{2} \left| E_{-n}^{(r)}(t) \right| \leq \frac{4n^r\pi}{|t|} .$$

(iii) We note that $\left| \left(\mathrm{ctg} \frac{t}{2} \right)^{(r)} \right| = O\left(\frac{1}{|t|^{r+1}} \right)$. Applying the inequality (ii), we obtain

$$|\overline{D}_n^{(r)}(t)| \leq |\tilde{D}_n^{(r)}(t)| + \frac{1}{2} \left| \left(\mathrm{ctg} \frac{t}{2} \right)^{(r)} \right| \leq \frac{4n^r\pi}{|t|} + O\left(\frac{1}{|t|^{r+1}} \right) .$$

□

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1. Let (2.1) hold. Then

$$\Delta a_k = \alpha_k \sum_{m=k}^{\infty} \frac{p_m}{m} ,$$

and we denote

$$A_k = \sum_{m=k}^{\infty} \frac{|p_m|}{m} .$$

Since $|\alpha_k| \leq 1$, we get

$$|\Delta a_k| \leq |\alpha_k| \sum_{m=k}^{\infty} \frac{|p_m|}{m} \leq A_k \text{, for all } k .$$

However,

$$\sum_{k=1}^{\infty} k^r A_k = \sum_{k=1}^{\infty} k^r \sum_{m=k}^{\infty} \frac{|p_m|}{m} = \sum_{m=1}^{\infty} \frac{|p_m|}{m} \sum_{k=1}^{m} k^r \leq \sum_{m=1}^{\infty} m^r |p_m| < \infty ,$$

and $A_k \downarrow 0$ i.e. $\{a_k\} \in S_r$.

Now, if $\{a_k\} \in S_r$, we put $\alpha_k = \frac{\Delta a_k}{A_k}$ and $p_k = k(A_k - A_{k+1})$.

Hence $|\alpha_k| \leq 1$, and by (1.6) we get:

$$\sum_{k=1}^{\infty} k^r |p_k| = \sum_{k=1}^{\infty} k^{r+1} (A_k - A_{k+1}) \leq \sum_{k=1}^{\infty} (r+1) k^r A_k < \infty .$$

Finally,

$$a_k = \sum_{i=k}^{\infty} \Delta a_i = \sum_{i=k}^{\infty} \alpha_i A_i = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \Delta A_m = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \frac{p_m}{m} = \sum_{m=k}^{\infty} \frac{p_m}{m} \sum_{i=k}^{m} \alpha_i \,,$$

i.e. (2.1) holds. □

Proof of Corollary 2.2. The proof of this corollary follows from Theorems 1.4 and 2.1. □

Proof of Theorem 2.3. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$.

Applying the Abel's transformation, we have:

$$g(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k(x) \,, \quad x \in (0, \pi] \,. \tag{4.1}$$

Applying Lemma 3.5 (iii), we have that the series $\sum_{k=1}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon, \pi]$, where $\varepsilon > 0$.

Thus, representation (4.1) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \,.$$

Then,

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| dx \leq \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx + O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \right) .$$

Let

$$I_1 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \,, \quad I_2 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \,.$$

Since $\operatorname{ctg}\frac{x}{2} = \frac{2}{x} + \sum_{n=1}^{\infty} \frac{4x}{x^2 - 4n^2\pi^2}$ (see [3]) it is not difficult to proof the following estimate

$$\left(\operatorname{ctg}\frac{x}{2} \right)^{(r)} = \frac{2(-1)^r r!}{x^{r+1}} + O(1), \ \ x \in (0, \pi] \ .$$

Thus

$$\overline{D}_n^{(r)}(x) = \frac{(-1)^{r+1} r!}{x^{r+1}} + O\left((n+1)^{r+1} \right), \ \ x \in (0, \pi]$$

Hence,

$$\begin{aligned} I_1 &= r! \sum_{j=1}^{m} \left| \sum_{k=0}^{j-1} \Delta a_k \right| \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}} + O\left(\sum_{j=1}^{m} \left[\sum_{k=0}^{j-1} |\Delta a_k| (k+1)^{r+1} \right] \int_{\pi/(j+1)}^{\pi/j} dx \right) \\ &= O_r\left(\sum_{j=1}^{m} |a_j| j^{r-1} \right) + O\left(\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)^{r+1} |\Delta a_k|}{j(j+1)} \right), \end{aligned}$$

where O_r depends on r. But

$$\begin{aligned}\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)} &= \sum_{j=1}^{m}\frac{1}{j(j+1)}\sum_{k=0}^{j-1}(k+1)^{r+1}|\Delta a_k| \\ &\le \sum_{k=0}^{\infty}(k+1)^{r+1}|\Delta a_k|\sum_{j=k+1}^{\infty}\frac{1}{j(j+1)} \\ &= \sum_{k=0}^{\infty}(k+1)^{r}|\Delta a_k| \\ &= |\Delta a_0| + \sum_{k=1}^{\infty}(k+1)^{r}|\Delta a_k| \\ &\le |a_1| + 2^r\sum_{k=1}^{\infty}k^r|\Delta a_k| \\ &\le \sum_{k=1}^{\infty}|\Delta a_k| + 2^r\sum_{k=1}^{\infty}k^rA_k \\ &\le (1+2^r)\sum_{k=1}^{\infty}k^rA_k\,.\end{aligned}$$

Thus,

$$\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{|\Delta a_k|(k+1)^{r+1}}{j(j+1)} = O_r\left(\sum_{k=1}^{\infty}k^rA_k\right),$$

where O_r depends on r.

Therefore,

$$I_1 = O_r\left(\sum_{j=1}^{m}|a_j|j^{r-1}\right) + O_r\left(\sum_{k=1}^{\infty}k^rA_k\right).$$

Application of Abel's transformation, yields

$$\sum_{k=j}^{\infty}\Delta a_k\overline{D}_k^{(r)}(x) = \sum_{k=j}^{\infty}\Delta A_k\sum_{i=0}^{k}\frac{\Delta a_i}{A_i}\overline{D}_i^{(r)}(x) - A_j\sum_{i=0}^{j-1}\frac{\Delta a_i}{A_i}\overline{D}_i^{(r)}(x)\,.$$

Let us estimate the second integral:

$$I_2 \le \sum_{j=1}^{m}\left[\sum_{k=j}^{\infty}(\Delta A_k)\int_{\pi/(j+1)}^{\pi}\left|\sum_{i=0}^{k}\frac{\Delta a_i}{A_i}\overline{D}_i^{(r)}(x)\right| + A_j\int_{\pi/(j+1)}^{\pi/j}\left|\sum_{i=0}^{j-1}\frac{\Delta a_i}{A_i}\overline{D}_i^{(r)}(x)\right|dx\right].$$

Applying the Lemma 3.3, we have:

$$J_k = \int_{\pi/(j+1)}^{\pi}\left|\sum_{i=0}^{k}\frac{\Delta a_i}{A_i}\overline{D}_i^{(r)}(x)\right|dx = O_r\left((k+1)^{r+1}\right), \tag{4.2}$$

where O_r depends on r. Then, by Lemma 3.5(iii),

$$\begin{aligned}
&\int_{\pi/(j+1)}^{\pi/j}\left|\sum_{i=0}^{j-1}\frac{\Delta a_i}{A_i}\overline{D}_i^{(r)}(x)\right|dx|\\
&\qquad = O\left(j^r\left(\sum_{i=0}^{j-1}\frac{|\Delta a_i|}{A_i}\int_{\pi/(j+1)}^{\pi/j}\frac{dx}{x}\right)\right)+O\left(\sum_{i=0}^{j-1}\frac{|\Delta a_i|}{A_i}\int_{\pi/(j+1)}^{\pi/j}\frac{dx}{x^{r+1}}\right)\\
&\qquad = O(j^r)+O_r(j^r)=O_r(j^r) \qquad (4.3)
\end{aligned}$$

where O_r depends on r. However, by (4.2), (4.3) and (1.6), we have

$$\begin{aligned}
I_2 &\leq \sum_{k=1}^{\infty}(\Delta A_k)\,J_k+O_r\left(\sum_{j=1}^{\infty}j^rA_j\right)\\
&= O_r(1)\sum_{k=1}^{\infty}(\Delta A_k)(k+1)^{r+1}+O_r\left(\sum_{j=1}^{\infty}j^rA_j\right)\\
&= O_r\left(\sum_{j=1}^{\infty}j^rA_j\right).
\end{aligned}$$

Finally, the inequality $(*)$ is satisfied. □

Proof of Corollary 2.4. By the inequalities

$$\begin{aligned}
\sum_{n=1}^{m}|a_n|\cdot n^{r-1} &\leq \sum_{n=1}^{\infty}n^{r-1}\sum_{k=n}^{\infty}|\Delta a_k|\\
&\leq \sum_{n=1}^{\infty}n^{r-1}\sum_{k=n}^{\infty}A_k\\
&= \sum_{k=1}^{\infty}A_k\sum_{n=1}^{k}n^{r-1}\\
&\leq \sum_{k=1}^{\infty}k^rA_k,
\end{aligned}$$

and Theorem 2.3, we obtain:

$$\int_0^{\pi}|g^{(r)}(x)|dx\leq M\left(\sum_{n=1}^{\infty}n^rA_n\right),$$

where $0<M=M(r)<\infty$. □

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