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SOME RESULTS ON L^1 APPROXIMATION OF THE r-TH DERIVATE OF FOURIER SERIES

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Abstract

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Abstract

In this paper we obtain the conditions for L^1-convergence of the r-th derivatives of the cosine and sine trigonometric series. These results are extensions of corresponding Sidon's and Telyakovskii's theorems for trigonometric series (case: $r = 0$).
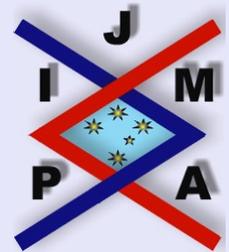
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Some Results on
L^1-**Approximation of the** r-**th**
Derivate of Fourier Series

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1. Introduction

Let

$$(1.1) \qquad f(x) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \,,$$

$$(1.2) \qquad g(x) \;=\; \sum_{n=1}^{\infty} a_n \sin nx$$

be the cosine and sine trigonometric series with real coefficients.

Let $\Delta a_n = a_n - a_{n+1}$, $n \in \{0, 1, 2, 3, \ldots\}$. The Dirichlet's kernel, conjugate Dirichlet's kernel and modified Dirichlet's kernel are denoted respectively by
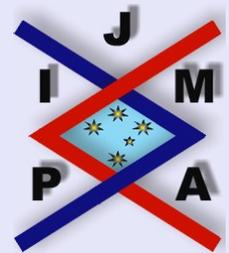
$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}} \,,$$

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}} \,,$$

$$\overline{D}_n(t) = -\frac{1}{2}\operatorname{ctg}\frac{t}{2} + \tilde{D}_n(t) = -\frac{\cos\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}.$$

Let

$$E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt} \ \text{ and } \ E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}.$$

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Then the r-th derivatives $D_n^{(r)}(t)$ and $\tilde{D}_n^{(r)}(t)$ can be written as

$$(1.3) \qquad 2D_n^{(r)}(t) = E_n^{(r)}(t) + E_{-n}^{(r)}(t),$$

$$(1.4) \qquad 2i\tilde{D}_n^{(r)}(t) = E_n^{(r)}(t) - E_{-n}^{(r)}(t).$$

In [2], Sidon proved the following theorem.

Theorem 1.1. *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \leq 1$, for every n and let $\sum_{n=1}^{\infty} |p_n|$ converge. If*

$$(1.5) \qquad a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \ n \in \mathbb{N}$$

then the cosine series (1.1) is the Fourier series of its sum f.

Several authors have studied the problem of L^1−convergence of the series (1.1) and (1.2).

In [4] Telyakovskii defined the following class of L^1-convergence of Fourier series. A sequence $\{a_k\}_{k=0}^{\infty}$ belongs to the class S, or $\{a_k\} \in S$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} A_k < \infty$ and $|\Delta a_k| \leq A_k$ for all k.

The importance of Telyakovskii's contributions are twofold. Firstly, he expressed Sidon's conditions (1.5) in a succinct equivalent form, and secondly, he showed that the class S is also a class of L^1-convergence. Thus, the class S is usually called the Sidon–Telyakovskii class.

In the same paper, Telyakovskii proved the following two theorems.
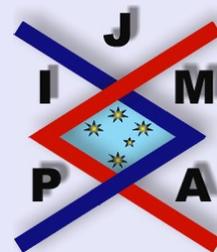
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Theorem 1.2. *[4]. Let the coefficients of the series $f(x)$ belong to the class S. Then the series is a Fourier series and the following inequality holds:*

$$\int_0^\pi |f(x)|\,dx \le M \sum_{n=0}^\infty A_n,$$

where M is a positive constant, independent on f.

Theorem 1.3. *[4]. Let the coefficients of the series $g(x)$ belong to the class S. Then the following inequality holds for $p = 1, 2, 3, \ldots$*

$$\int_{\pi/(p+1)}^\pi |g(x)|\,dx = \sum_{n=1}^p \frac{|a_n|}{n} + O\left(\sum_{n=1}^\infty A_n\right).$$

In particular, $g(x)$ is a Fourier series iff $\sum_{n=1}^\infty \frac{|a_n|}{n} < \infty$.

In [5], we extended the Sidon–Telyakovskii class $S = S_0$, i.e., we defined the class S_r, $r = 1, 2, 3, \ldots$ as follows: $\{a_k\}_{k=1}^\infty \in S_r$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty k^r A_k < \infty$ and $|\Delta a_k| \le A_k$ for all k.

We note that by $A_k \downarrow 0$ and $\sum_{k=1}^\infty k^r A_k < \infty$, we get

(1.6) $$k^{r+1} A_k = o(1), \quad k \to \infty.$$

It is trivially to see that $S_{r+1} \subset S_r$ for all $r = 1, 2, 3, \ldots$. Now, let $\{a_n\}_{n=1}^\infty \in S_1$. For arbitrary real number a_0, we shall prove that sequence $\{a_n\}_{n=0}^\infty$ belongs to S_0. We define $A_0 = \max(|\Delta a_0|, A_1)$. Then $|\Delta a_0| \le A_0$, i.e. $|\Delta a_n| \le A_n$, for all $n \in \{0, 1, 2, \ldots\}$ and $\{A_n\}_{n=0}^\infty$ is monotonically decreasing sequence.

On the other hand,

$$\sum_{n=0}^{\infty} A_n \le A_0 + \sum_{n=1}^{\infty} n A_n < \infty.$$

Thus, $\{a_n\}_{n=0}^{\infty} \in S_0$, i.e. $S_{r+1} \subset S_r$, for all $r = 0, 1, 2, \ldots$. The next example verifies that the implication

$$\{a_n\} \in S_{r+1} \Rightarrow \{a_n\} \in S_r, \quad r = 0, 1, 2, \ldots$$

is not reversible.

Example 1.1. *For $n = 0, 1, 2, 3, \ldots$ define $a_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}$. Then $a_n \to 0$ as $n \to \infty$ and for $n = 0, 1, 2, 3, \ldots$, $\Delta a_n = \frac{1}{(n+1)^2}$. Firstly we shall show that $\{a_n\}_{n=1}^{\infty} \notin S_1$.*

Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A \downarrow 0$ and $\Delta a_n = |\Delta a_n| \le A_n$. However, $\sum_{n=1}^{\infty} n A_n \ge \sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$ is divergent, i.e. $\{a_n\} \notin S_1$.

Now, for all $n = 0, 1, 2, \ldots$ let $A_n = \frac{1}{(n+1)^2}$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=0}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{a_n\}_{n=0}^{\infty} \in S_0$.

Our next example will show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty} \in S_r$ but $\{a_n\}_{n=1}^{\infty} \notin S_{r+1}$, for all $r = 1, 2, 3, \ldots$.

Namely, for all $n = 1, 2, 3, \ldots$ let $a_n = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}}$. Then $a_n \to 0$ as $n \to \infty$ and for $n = 1, 2, 3, \ldots$, $\Delta a_n = \frac{1}{n^{r+2}}$. Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A_n \downarrow 0$ and $\Delta a_n = |\Delta a_n| \le A_n$. However,

$$\sum_{n=1}^{\infty} n^{r+1} A_n \ge \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$$
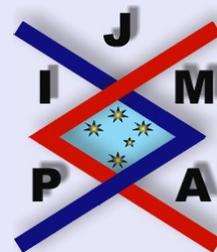
is divergent, i.e. $\{a_n\} \notin S_{r+1}$. On the other hand, for all $n = 1, 2, \ldots$ let $A_n = \frac{1}{n^{r+2}}$. Then $A_n \downarrow 0$, $|\Delta a_n| \leq A_n$ and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{a_n\} \in S_r$.
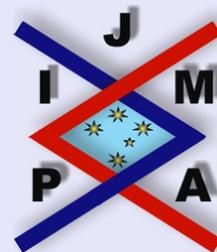
In the same paper [5] we proved the following theorem.

Theorem 1.4. *[5]. Let the coefficients of the series (1.1) belong to the class S_r, $r = 0, 1, 2, \ldots$. Then the $r-$th derivative of the series (1.1) is a Fourier series of some $f^{(r)} \in L^1(0, \pi)$ and the following inequality holds:*

$$\int_0^{\pi} \left| f^{(r)}(x) \right| dx \leq M \sum_{n=1}^{\infty} n^r A_n,$$

where $0 < M = M(r) < \infty$.

This is an extension of the Telyakovskii Theorem 1.2.

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2. Results

In this paper, we shall prove the following main results.

Theorem 2.1. *A null sequence* $\{a_n\}$ *belongs to the class* S_r, $r = 0, 1, 2, \ldots$ *if and only if it can be represented as*

$$(2.1) \qquad a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \ n \in \mathbb{N}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{p_n\}_{n=1}^{\infty}$ *are sequences such that* $|\alpha_n| \leq 1,$ *for all* n *and*

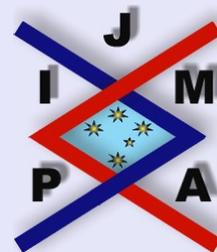$$\sum_{n=1}^{\infty} n^r \, |p_n| < \infty.$$

Corollary 2.2. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{p_n\}_{n=1}^{\infty}$ *be sequences such that* $|\alpha_n| \leq 1,$ *for every* n *and let* $\sum_{n=1}^{\infty} n^r \, |p_n| < \infty$, $r = 0, 1, 2, \ldots$. *If*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \ n \in \mathbb{N}$$

then the $r-th$ *derivate of the series* (1.1) *is a Fourier series of some* $f^{(r)} \in L^1.$

Theorem 2.3. *Let the coefficients of the series* $g(x)$ *belong to the class* S_r, $r = 0, 1, 2, \ldots$ *Then the* r-*th derivate of the series* (1.2) *converges to a function and for* $m = 1, 2, 3, \ldots$ *the following inequality holds:*

$$(*) \qquad \int_{\pi/(m+1)}^{\pi} \left| g^{(r)}(x) \right| dx \leq M \left(\sum_{n=1}^{m} |a_n| \cdot n^{r-1} + \sum_{n=1}^{\infty} n^r A_n \right),$$

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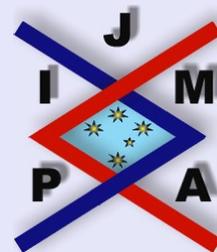
where

$$0 < M = M(r) < \infty\,.$$

Moreover, if $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$, then the r-th derivate of the series (1.2) is a Fourier series of some $g^{(r)} \in L^1(0, \pi)$ and

$$\int_0^\pi \left| g^{(r)}(x) \right| dx \le M \left(\sum_{n=1}^\infty |a_n| \cdot n^{r-1} + \sum_{n=1}^\infty n^r A_n \right)$$

Corollary 2.4. *Let the coefficients of the series $g(x)$ belong to the class S_r, $r \ge 1$. Then the following inequality holds:*

$$\int_0^\pi \left| g^{(r)}(x) \right| dx \le M \sum_{n=1}^\infty n^r A_n,$$

where $0 < M = M(r) < \infty$.

3. Lemmas

For the proof of our new theorems we need the following lemmas.

The following lemma proved by Sheng, can be reformulated in the following way.

Lemma 3.1. *[1] Let r be a nonnegative integer and $x \in (0, \pi]$, where $n \geq 1$. Then*

$$D_n^{(r)}(x) = \sum_{k=0}^{r} \frac{\left(n + \frac{1}{2}\right)^k \sin\left[\left(n + \frac{1}{2}\right) x + \frac{k\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \varphi_k(x),$$
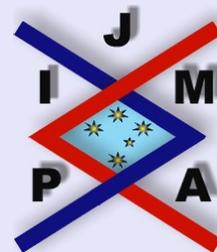
where $\varphi_r \equiv \frac{1}{2}$ and φ_k, $k = 0, 1, 2, \ldots, r - 1$ denotes various entire 4π – periodic functions of x, independent of n. More precisely, φ_k, $k = 0, 1, 2, \ldots, r$ are trigonometric polynomials of $\frac{x}{2}$.

Lemma 3.2. *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then the following relation holds for $\nu = 0, 1, 2, \ldots, r$ and $r = 0, 1, 2, \ldots$*

$$\begin{aligned} U_k &= \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j + \frac{1}{2}\right)^\nu \sin\left[\left(j + \frac{1}{2}\right) x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \right| dx \\ &= O\left((k+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2\nu}\right)^{1/2}\right). \end{aligned}$$

Proof. Applying first Cauchy–Buniakowski inequality, yields

$$U_k \leq \left[\int_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-\nu)}}\right]^{1/2}$$

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$$\times \left\{ \int_{\pi/(k+1)}^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j + \frac{1}{2} \right)^{\nu} \sin \left[\left(j + \frac{1}{2} \right) x + \frac{(\nu+3)\pi}{2} \right] \right]^2 dx \right\}^{1/2}.$$
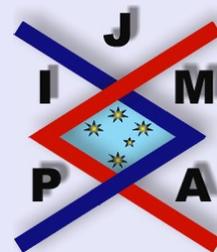
Since

$$
\begin{aligned}
\int_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin \left(\frac{x}{2} \right) \right)^{2(r+1-\nu)}} & \leq \pi^{2(r+1-\nu)} \int_{\pi/(k+1)}^{\pi} \frac{dx}{x^{2(r+1-\nu)}} \\
& \leq \frac{\pi (k+1)^{2(r+1-\nu)-1}}{2(r+1-\nu)-1} \\
& \leq \pi (k+1)^{2(r+1-\nu)-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
U_k & \leq \left[\pi (k+1)^{2(r+1-\nu)-1} \right]^{1/2} \\
& \quad \times \left\{ \int_{0}^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j + \frac{1}{2} \right)^{\nu} \sin \left[\left(j + \frac{1}{2} \right) x + \frac{\nu+3}{2}\pi \right] \right]^2 dx \right\}^{1/2} \\
& \leq \left[2\pi (k+1)^{2(r+1-\nu)-1} \right]^{1/2} \\
& \quad \times \left\{ \int_{0}^{2\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j + \frac{1}{2} \right)^{\nu} \sin \left[(2j+1)t + \frac{\nu+3}{2}\pi \right] \right]^2 dt \right\}^{1/2}.
\end{aligned}
$$

Then, applying Parseval's equality, we obtain:

$$U_k \leq \left[2\pi (k+1)^{2(r+1-\nu)-1} \right]^{1/2} \left[\sum_{j=0}^{k} |\alpha_j|^2 (j+1)^{2\nu} \right]^{1/2}.$$

Finally,

$$U_k = O\left((k+1)^{r-\nu+\frac{1}{2}}\left(\sum_{j=0}^{k}\alpha_j^2(j+1)^{2\nu}\right)^{1/2}\right).$$

$\square$

Lemma 3.3. *Let* $r \in \{0,1,2,3,\dots\}$ *and* $\{\alpha_k\}_{k=0}^{n}$ *be a sequence of real numbers such that* $|\alpha_k| \leq 1,$ *for all* k. *Then there exists a finite constant* $M = M(r) > 0$ *such that for any* $n \geq 0$

$(**)$
$$\int_{\pi/(n+1)}^{\pi}\left|\sum_{k=0}^{n}\alpha_k\overline{D}_k^{(r)}(x)\right|dx \leq M \cdot (n+1)^{r+1}.$$

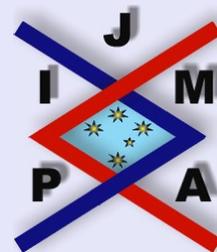*Proof.* Similar to Lemma 3.1 it is not difficult to proof the following equality

$$\overline{D}_n^{(r)}(x) = \sum_{k=0}^{r}\frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}}\varphi_k(x),$$

where φ_k denotes the same various 4π-periodic functions of x, independent of n.

Now, we have:

$$\int_{\pi/(n+1)}^{\pi}\left|\sum_{k=0}^{n}\alpha_k\overline{D}_k^{(r)}(x)\right|dx$$
$$\leq \int_{\pi/(n+1)}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\left(\sum_{\nu=0}^{r}\frac{\left(j+\frac{1}{2}\right)^\nu \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}}\varphi_\nu(x)\right)\right|dx.$$

Since φ_ν are bounded, we have:

$$\int_{\pi/(n+1)}^{\pi} \left| \sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^\nu \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \varphi_\nu(x) \right| dx \leq K\, U_n,$$

where U_n is the integral as in Lemma 3.2, and $K = K(r)$ is a positive constant.

Applying Lemma 3.2, to the last integral, we obtain:

$$\int_{\pi/(n+1)}^{\pi} \left| \sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^\nu \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \varphi_\nu(x) \right| dx$$

$$= O\left((n+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{n} \alpha_j^2 (j+1)^{2\nu} \right)^{1/2} \right)$$

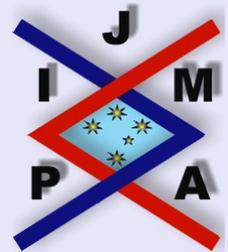$$= O\left((n+1)^{r-\nu+\frac{1}{2}} (n+1)^{\nu+\frac{1}{2}} \right) = O\left((n+1)^{r+1} \right).$$

Finally the inequality $(\ast\ast)$ is satisfied. $\qquad\square$

Remark 3.1. *For $r = 0$, we obtain the Telyakovskii type inequality, proved in [4].*

Lemma 3.4. *Let r be a non-negative integer. Then for all $0 < |t| \leq \pi$ and all $n \geq 1$ the following estimates hold:*

(i) $\left| E_{-n}^{(r)}(t) \right| \leq \frac{4n^r \pi}{|t|}$,

(ii) $\left| \tilde{D}_n^{(r)}(t) \right| \leq \frac{4n^r \pi}{|t|}$,

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(iii) $\left|\overline{D}_n^{(r)}(t)\right| \leq \frac{4n^r\pi}{|t|} + O\left(\frac{1}{|t|^{r+1}}\right).$

Proof. (i) The case $r = 0$ is trivial. Really,

$$|E_n(t)| \leq |D_n(t)| + |\tilde{D}_n(t)| \leq \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{2|t|} < \frac{4\pi}{|t|},$$

$$|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}.$$

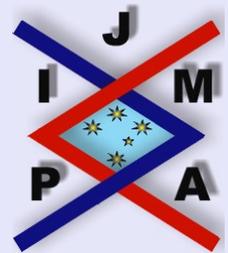Let $r \geq 1$. Applying the Abel's transformation, we have:

$$E_n^{(r)}(t) = i^r \sum_{k=1}^{n} k^r e^{ikt} = i^r \left[\sum_{k=1}^{n-1} \Delta(k^r)\left(E_k(t) - \frac{1}{2}\right) + n^r\left(E_n(t) - \frac{1}{2}\right) \right]$$

$$
\begin{aligned}
|E_n^{(r)}(t)| &\leq \sum_{k=1}^{n-1}[(k+1)^r - k^r]\left(\frac{1}{2} + |E_k(t)|\right) + n^r\left(\frac{1}{2} + |E_n(t)|\right) \\
&\leq \left(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|}\right)\left\{\sum_{k=1}^{n-1}[(k+1)^r - k^r] + n^r\right\} = \frac{4\pi n^r}{|t|}.
\end{aligned}
$$

Since $E_{-n}^{(r)}(t) = E_n^{(r)}(-t)$, we obtain $\left|E_{-n}^{(r)}(t)\right| \leq \frac{4n^r\pi}{|t|}$.

(ii) Applying the inequality (i), we obtain

$$\left|\tilde{D}_n^{(r)}(t)\right| = \left|i\tilde{D}_n^{(r)}(t)\right| \leq \frac{1}{2}\left|E_n^{(r)}(t)\right| + \frac{1}{2}\left|E_{-n}^{(r)}(t)\right| \leq \frac{4n^r\pi}{|t|}.$$
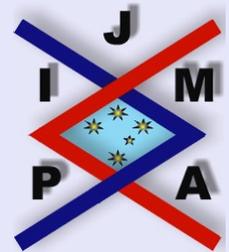
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(iii) We note that $\left|\left(\text{ctg}\frac{t}{2}\right)^{(r)}\right| = O\left(\frac{1}{|t|^{r+1}}\right)$. Applying the inequality (ii), we obtain

$$|\overline{D}_n^{(r)}(t)| \leq |\tilde{D}_n^{(r)}(t)| + \frac{1}{2}\left|\left(\text{ctg}\frac{t}{2}\right)^{(r)}\right| \leq \frac{4n^r\pi}{|t|} + O\left(\frac{1}{|t|^{r+1}}\right) .$$

$\square$

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4.   Proofs of the Main Results

Proof of Theorem 2.1. Let (2.1) hold. Then

$$\Delta a_k = \alpha_k \sum_{m=k}^{\infty} \frac{p_m}{m}\,,$$

and we denote

$$A_k = \sum_{m=k}^{\infty} \frac{|p_m|}{m}\,.$$

Since $|\alpha_k| \le 1$, we get

$$|\Delta a_k| \le |\alpha_k| \sum_{m=k}^{\infty} \frac{|p_m|}{m} \le A_k\,,\text{ for all }k\,.$$
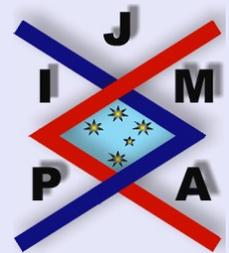
However,

$$\sum_{k=1}^{\infty} k^r A_k = \sum_{k=1}^{\infty} k^r \sum_{m=k}^{\infty} \frac{|p_m|}{m} = \sum_{m=1}^{\infty} \frac{|p_m|}{m} \sum_{k=1}^{m} k^r \le \sum_{m=1}^{\infty} m^r |p_m| < \infty\,,$$

and $A_k \downarrow 0$ i.e. $\{a_k\} \in S_r$.

Now, if $\{a_k\} \in S_r$, we put $\alpha_k = \frac{\Delta a_k}{A_k}$ and $p_k = k\,(A_k - A_{k+1})$.
Hence $|\alpha_k| \le 1$, and by (1.6) we get:

$$\sum_{k=1}^{\infty} k^r |p_k| = \sum_{k=1}^{\infty} k^{r+1} (A_k - A_{k+1}) \le \sum_{k=1}^{\infty} (r+1)\,k^r A_k < \infty\,.$$

Some Results on
L^1-Approximation of the r-th
Derivate of Fourier Series

Živorad Tomovski

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Finally,

$$a_k = \sum_{i=k}^{\infty} \Delta a_i = \sum_{i=k}^{\infty} \alpha_i A_i = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \Delta A_m$$

$$= \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \frac{p_m}{m} = \sum_{m=k}^{\infty} \frac{p_m}{m} \sum_{i=k}^{m} \alpha_i \, ,$$

i.e. (2.1) holds. □

Proof of Corollary 2.2. The proof of this corollary follows from Theorems 1.4 and 2.1. □

Proof of Theorem 2.3. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$.
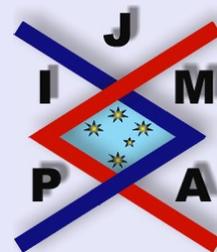
Applying the Abel's transformation, we have:

(4.1) $$g(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k(x), \quad x \in (0, \pi].$$

Applying Lemma 3.4 (iii), we have that the series $\sum_{k=1}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon, \pi]$, where $\varepsilon > 0$.

Thus, representation (4.1) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \, .$$

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Then,

$$
\int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|dx \leq \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx
$$
$$
+ O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \right) .
$$

Let

$$
I_1 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx ,
$$
$$
I_2 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx .
$$

Since $\operatorname{ctg}\frac{x}{2} = \frac{2}{x} + \sum_{n=1}^{\infty} \frac{4x}{x^2-4n^2\pi^2}$ (see [3]) it is not difficult to proof the following estimate

$$
\left(\operatorname{ctg}\frac{x}{2} \right)^{(r)} = \frac{2(-1)^r r!}{x^{r+1}} + O(1), \ x \in (0,\pi] .
$$

Thus

$$
\overline{D}_n^{(r)}(x) = \frac{(-1)^{r+1} r!}{x^{r+1}} + O\left((n+1)^{r+1} \right), \ x \in (0,\pi]
$$

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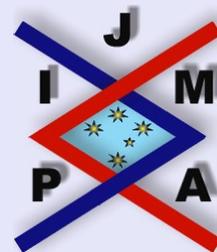
Hence,

$$
I_1 = r! \sum_{j=1}^{m} \left| \sum_{k=0}^{j-1} \Delta a_k \right| \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}}
$$
$$
+ O\left(\sum_{j=1}^{m} \left[\sum_{k=0}^{j-1} |\Delta a_k| (k+1)^{r+1} \right] \int_{\pi/(j+1)}^{\pi/j} dx \right)
$$
$$
= O_r\left(\sum_{j=1}^{m} |a_j| j^{r-1} \right) + O\left(\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)} \right),
$$

where O_r depends on r. However,

$$
\begin{aligned}
\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)} &= \sum_{j=1}^{m} \frac{1}{j(j+1)} \sum_{k=0}^{j-1} (k+1)^{r+1}|\Delta a_k| \\
&\leq \sum_{k=0}^{\infty} (k+1)^{r+1}|\Delta a_k| \sum_{j=k+1}^{\infty} \frac{1}{j(j+1)} \\
&= \sum_{k=0}^{\infty} (k+1)^{r}|\Delta a_k| \\
&= |\Delta a_0| + \sum_{k=1}^{\infty} (k+1)^{r}|\Delta a_k| \\
&\leq |a_1| + 2^r \sum_{k=1}^{\infty} k^{r}|\Delta a_k|
\end{aligned}
$$

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$$\leq \sum_{k=1}^{\infty} |\Delta a_k| + 2^r \sum_{k=1}^{\infty} k^r A_k \leq (1 + 2^r) \sum_{k=1}^{\infty} k^r A_k \,.$$

Thus,

$$\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{|\Delta a_k|(k+1)^{r+1}}{j(j+1)} = O_r \left(\sum_{k=1}^{\infty} k^r A_k \right) \,,$$

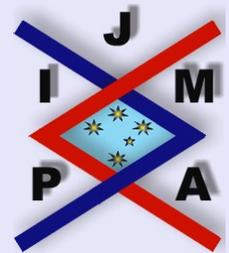where O_r depends on r.

Therefore,

$$I_1 = O_r \left(\sum_{j=1}^{m} |a_j| j^{r-1} \right) + O_r \left(\sum_{k=1}^{\infty} k^r A_k \right) \,.$$

Application of Abel's transformation, yields

$$\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) = \sum_{k=j}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) - A_j \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \,.$$

Let us estimate the second integral:

$$I_2 \leq \sum_{j=1}^{m} \left[\sum_{k=j}^{\infty} (\Delta A_k) \int_{\pi/(j+1)}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \right| \right.$$
$$\left. + A_j \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \right| dx \right] \,.$$

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Applying the Lemma 3.3, we have:

$$(4.2) \qquad J_k = \int_{\pi/(j+1)}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \right| dx = O_r\left((k+1)^{r+1}\right),$$
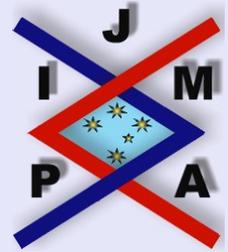
where O_r depends on r. Then, by Lemma 3.4(iii),

$$\int_{\pi/(j+1)}^{\pi/j} \left| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \right| dx|$$

$$= O\left(j^r \left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x} \right) \right) + O\left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}} \right)$$

$$(4.3) \qquad = O(j^r) + O_r(j^r) = O_r(j^r)$$

where O_r depends on r. However, by (4.2), (4.3) and (1.6), we have

$$\begin{aligned} I_2 &\leq \sum_{k=1}^{\infty} (\Delta A_k) J_k + O_r\left(\sum_{j=1}^{\infty} j^r A_j \right) \\ &= O_r(1) \sum_{k=1}^{\infty} (\Delta A_k)(k+1)^{r+1} + O_r\left(\sum_{j=1}^{\infty} j^r A_j \right) \\ &= O_r\left(\sum_{j=1}^{\infty} j^r A_j \right). \end{aligned}$$

Finally, the inequality $(*)$ is satisfied. $\qquad\square$

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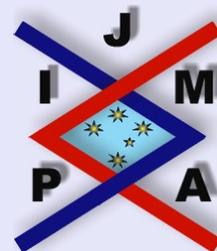Contents

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Proof of Corollary 2.4. By the inequalities

$$
\begin{aligned}
\sum_{n=1}^{m} |a_n| \cdot n^{r-1} &\leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} |\Delta a_k| \\
&\leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} A_k \\
&= \sum_{k=1}^{\infty} A_k \sum_{n=1}^{k} n^{r-1} \leq \sum_{k=1}^{\infty} k^r A_k,
\end{aligned}
$$

and Theorem 2.3, we obtain:

$$
\int_0^{\pi} |g^{(r)}(x)| dx \leq M \left(\sum_{n=1}^{\infty} n^r A_n \right),
$$

where $0 < M = M(r) < \infty$. $\qquad\square$

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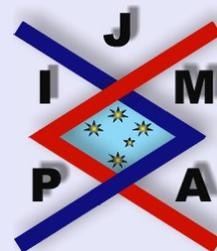
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