

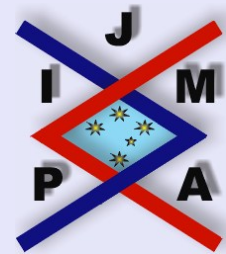
Journal of Inequalities in Pure and Applied Mathematics

ON A BOJANIC-STANOJEVIC TYPE INEQUALITY AND ITS APPLICATIONS

Z. TOMOVSKI

Faculty of Mathematical and Natural Sciences
P.O. Box 162
91000 Skopje
MACEDONIA
EMail: tomovski@iunona.pmf.ukim.edu.mk

©2000 Victoria University
ISSN (electronic): 1443-5756
006-99



volume 1, issue 2, article 13,
2000.

*Received 22 September, 1999;
accepted 7 March, 2000.*

Communicated by: H.M. Srivastava

[Abstract](#)

[Contents](#)

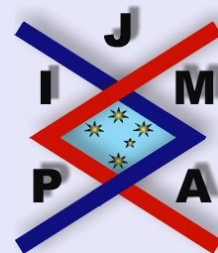


[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

An extension of the Bojanić–Stanojević type inequality [1] is made by considering the r -th derivate of the Dirichlet kernel $D_k^{(r)}$ instead of D_k . Namely, the following inequality is proved

$$\left\| \sum_{k=1}^n \alpha_k D_k^{(r)}(x) \right\|_1 \leq M_p n^{r+1} \left(\frac{1}{n} \sum_{k=1}^n |\alpha_k|^p \right)^{1/p},$$

where $\|\cdot\|_1$ is the L^1 -norm, $\{\alpha_k\}$ is a sequence of real numbers, $1 < p \leq 2$, $r = 0, 1, 2, \dots$ and M_p is an absolute constant dependent only on p . As an application of this inequality, it is shown that the class \mathcal{F}_{pr} is a subclass of $\mathcal{BV} \cap \mathcal{C}_r$, where \mathcal{F}_{pr} is the extension of the Fomin's class, \mathcal{C}_r is the extension of the Garrett–Stanojević class [8] and \mathcal{BV} is the class of all null sequences of bounded variation.

2000 Mathematics Subject Classification: 26D15, 42A20

Key words: Bojanić–Stanojević inequality, Sidon–Fomin's inequality, Bernstein's inequality. L^1 -convergence cosine series.

Contents

1	Introduction	3
2	Main Result	5
3	Application	6
	References	

On a Bojanić–Stanojević Type Inequality and its Applications

Živorad Tomovski

Title Page

Contents



Go Back

Close

Quit

Page 2 of 10

1. Introduction

In 1939, Sidon [5] proved his namesake inequality, which is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications, for instance in L^1 -convergence problems and summation methods with respect to trigonometric series, newer and newer improvements of the original inequality have been proved by several authors.

Fomin [2], by applying the linear method for summing of Fourier series, gave another proof of the inequality and thus it is known as Sidon-Fomin's inequality. In addition, S. A. Telyakovskii in [7] has given an elegant proof of Sidon-Fomin's inequality.

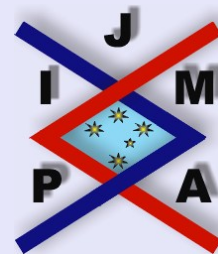
Lemma 1.1. (Sidon-Fomin). *Let $\{\alpha_k\}_{k=0}^n$ be a sequence of real numbers such that $|\alpha_k| \leq 1$ for all k . Then there exists a positive constant M such that for any $n \geq 0$,*

$$(1.1) \quad \left\| \sum_{k=0}^n \alpha_k D_k(x) \right\|_1 \leq M(n+1).$$

In [9] we extended this result and we gave two different proofs of the following lemma.

Lemma 1.2. [9]. *Let $\{\alpha_j\}_{j=0}^k$ be a sequence of real numbers such that $|\alpha_k| \leq 1$ for all k . Then there exists a positive constant M , such that for any $n \geq 0$,*

$$(1.2) \quad \left\| \sum_{k=0}^n \alpha_k D_k^{(r)}(x) \right\|_1 \leq M(n+1)^{r+1}.$$



Title Page

Contents



Go Back

Close

Quit

Page 3 of 10

However, Bojanić and Stanojević [1] proved the following more general inequality of (1.1).

Lemma 1.3. [1]. Let $\{\alpha_i\}_{i=0}^n$ be a sequence of real numbers. Then for any $1 < p \leq 2$ and $n \geq 0$

$$(1.3) \quad \left\| \sum_{k=0}^n \alpha_k D_k(x) \right\|_1 \leq M_p(n+1) \left(\frac{1}{n+1} \sum_{k=0}^n |\alpha_k|^p \right)^{1/p},$$

where the constant M_p depends only on p .

We note that this estimate is essentially contained (case $p = 2$) in Fomin [2]. Sidon-Fomin's inequality is a special case of the Bojanić-Stanojević inequality, i.e., it can easily be deduced from Lemma 1.3.

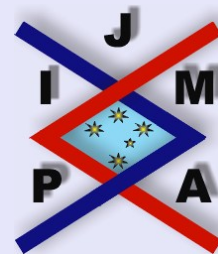
It is easy to see that the Bojanić-Stanojević inequality is not valid for $p = 1$. Indeed, if $\alpha_n = 1$ and $\alpha_k = 0$ ($k \neq n, k \in \mathbb{N}$) then the left side is of order $\log n/n$ while the right side is of order $1/n$ as $n \rightarrow \infty$.

In order to prove our new results we need the following lemma.

Lemma 1.4. [10]. If $T_n(x)$ is a trigonometrical polynomial of order n , then

$$\|T_n^{(r)}\| \leq n^r \|T_n\|.$$

This is S. Bernstein's inequality in the $L^1(0, \pi)$ -metric (see [10, Vol. 2, p.11]).



On a Bojanić–Stanojević Type Inequality and its Applications

Živorad Tomovski

Title Page

Contents



Go Back

Close

Quit

Page 4 of 10

2. Main Result

Now we will prove a counterpart of inequality (1.3) in the case where the r -th derivate of the Dirichlet's kernel $D_k^{(r)}$ is used instead of $D(x)$.

Theorem 2.1. *Let $\{\alpha_k\}_{k=1}^n$ be a sequence of real numbers. Then for any $1 < p \leq 2$ and $r = 0, 1, 2, \dots, n \in \mathbb{N}$ the following inequality holds:*

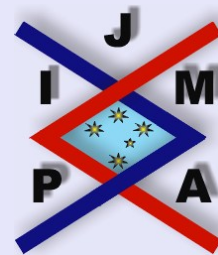
$$(2.1) \quad \left\| \sum_{k=1}^n \alpha_k D_k^{(r)}(x) \right\|_1 \leq M_p n^{r+1} \left(\frac{1}{n} \sum_{k=1}^n |\alpha_k|^p \right)^{1/p},$$

where the constant M_p depends only on p .

Proof. Applying first the Bernstein inequality and then the Bojanić-Stanojević inequality, we have

$$\left\| \sum_{k=1}^n \alpha_k D_k^{(r)}(x) \right\| \leq n^r \left\| \sum_{k=1}^n \alpha_k D_k^{(r)}(x) \right\| \leq M_p n^{r+1} \left(\frac{1}{n} \sum_{k=1}^n |\alpha_k|^p \right)^{1/p}.$$

It is easy to see that the inequality (1.2) is a special case of the inequality (2.1), i.e. it can easily be deduced from Theorem 2.1. \square



On a Bojanić–Stanojević Type Inequality and its Applications

Živorad Tomovski

Title Page

Contents



Go Back

Close

Quit

Page 5 of 10

3. Application

The problem of L^1 -convergence via Fourier coefficients consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for $\|S_n - f\| = o(1)$, $n \rightarrow \infty$ is given in the form $a_n \lg n = o(1)$, $n \rightarrow \infty$. Here S_n denotes the partial sums of the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

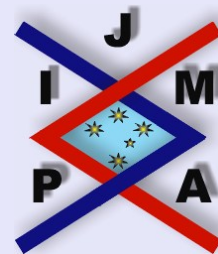
The Sidon-Telyakovskii class \mathcal{S} [7] is a classical example for which the condition $a_n \lg n = o(1)$, $n \rightarrow \infty$ is equivalent to $\|S_n - f\| = o(1)$, $n \rightarrow \infty$. Later Fomin [3] extended the Sidon-Telyakovskii class by defining a class \mathcal{F}_p , $p > 1$ of Fourier coefficients as follows: a sequence $\{a_k\}$ belongs to \mathcal{F}_p , $p > 1$ if $a_k \rightarrow 0$ as $k \rightarrow \infty$ and

$$(3.1) \quad \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=k}^{\infty} |\Delta a_i|^p \right)^{1/p} < \infty.$$

We note that Fomin [3] has given an equivalent form of the condition (3.1). Namely, he proved that $\{a_n\} \in \mathcal{F}_p$, $p > 1$ iff $\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} < \infty$, where

$$\Delta_s^{(p)} = \left\{ \frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p \right\}^{1/p}.$$

Let \mathcal{BV} denote the class of null sequence $\{a_n\}$ of bounded variation, i.e. $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$.



On a Bojanić–Stanojević Type Inequality and its Applications

Živorad Tomovski

Title Page

Contents



Go Back

Close

Quit

Page 6 of 10

The class \mathcal{C} was defined by Garrett and Stanojević [4] as follows: a null sequence of real numbers satisfy the condition \mathcal{C} if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ independent of n , such that

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon, \quad \text{for every } n.$$

On the other hand, Stanojević [6] proved the following inclusion between the classes \mathcal{F}_p , \mathcal{C} and \mathcal{BV} .

Lemma 3.1. [6]. *For all $1 < p \leq 2$ the following inclusion holds: $\mathcal{F}_p \subset \mathcal{BV} \cap \mathcal{C}$.*

In [8] we defined an extension \mathcal{C}_r , $r = 0, 1, 2, \dots$, of the Garrett-Stanojević class. Namely, a null sequence $\{a_k\}$ belongs to the class \mathcal{C}_r , $r = 0, 1, 2, \dots$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

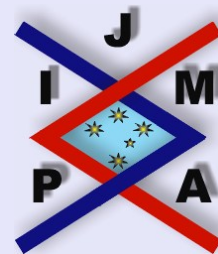
$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| < \varepsilon, \quad \text{for all } n.$$

When $r = 0$, we denote $\mathcal{C}_r = \mathcal{C}$.

Denote by I_m the dyadic interval $[2^{m-1}, 2^m)$, for $m \geq 1$. A null sequence $\{a_n\}$ belongs to the class F_{pr} , $p > 1$, $r = 0, 1, 2, \dots$ if

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} < \infty, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

It is obvious that $F_{pr} \subset F_p$. For $r = 0$, we obtain the Fomin's class F_p .



Title Page

Contents



Go Back

Close

Quit

Page 7 of 10

Theorem 3.2. For all $1 < p \leq 2$ and $r = 0, 1, 2, \dots$ the following inclusion holds $F_{pr} \subset BV \cap C_r$.

Proof. By Lemma 3.1, it is clear that $F_{pr} \subset BV$. It suffices to show that

$$\left\| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right\| = o(1), \quad n \rightarrow \infty.$$

Since

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} = 2 \sum_{m=1}^{\infty} \left\{ 2^{(m-1)[(r+1)p-1]} \sum_{k \in I_m} |\Delta a_k|^p \right\}^{1/p},$$

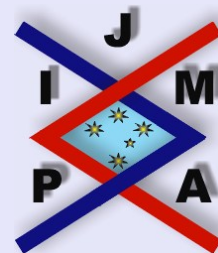
we have

$$\sum_{k=1}^{\infty} k^{(r+1)p-1} |\Delta a_k|^p < \infty.$$

Applying the Theorem 2.1, we obtain

$$\left\| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right\| \leq M_p \left(\sum_{k=n}^{\infty} k^{(r+1)p-1} |\Delta a_k|^p \right)^{1/p} = o(1), \quad n \rightarrow \infty.$$

□



On a Bojanić–Stanojević Type Inequality and its Applications

Živorad Tomovski

Title Page

Contents



Go Back

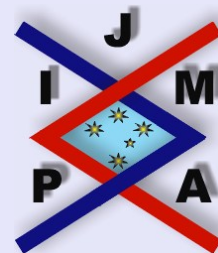
Close

Quit

Page 8 of 10

References

- [1] R. BOJANIĆ AND Č.V. STANOJEVIĆ, A class of L^1 -convergence, *Trans. Amer. Math. Soc.*, **269** (1982), 677–683.
- [2] G.A. FOMIN, On linear method for summing Fourier series, *Mat. Sb* (Russian), **66** (107), (1964), 144–152.
- [3] G.A. FOMIN, A class of trigonometric series, *Math. Zametki* (Russian), **23** (1978), 117–124.
- [4] J.W. GARRETT AND Č.V. STANOJEVIĆ, Necessary and sufficient conditions for L^1 convergence of trigonometric series, *Proc. Amer. Math. Soc.*, **60** (1976), 68–72.
- [5] S. SIDON, Hinreichende Bedingungen fur den Fourier charakter einer Trigonometrischen Reihe, *J. London, Math. Soc.*, **14** (1939), 158.
- [6] Č.V. STANOJEVIĆ, Classes of L^1 convergence of Fourier series and Fourier Stiltjes series, *Proc. Amer. Math. Soc.*, **82** (1981), 209–215.
- [7] S.A. TELYAKOVSKII, On a sufficient condition of Sidon for the integrability of trigonometric series, *Math. Zametki* (Russian), (1973), 742–748.
- [8] Ž. TOMOVSKI, An extension of the Garrett- Stanojević class, *Approx. Theory Appl.*, **16**(1) (2000), 46–51. [ONLINE] A corrected version is available in the *RGMA Research Report Collection*, **3**(4), Article 3, 2000. URL: <http://rgmia.vu.edu.au/v3n4.html>
- [9] Ž. TOMOVSKI, An extension of the Sidon-Fomin inequality and applications, *Math. Inequal. Appl.*, (to appear).



On a Bojanić–Stanojević Type
Inequality and its Applications

Živorad Tomovski

Title Page

Contents



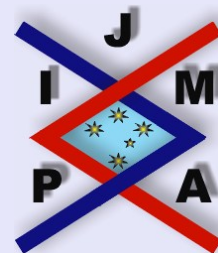
Go Back

Close

Quit

Page 9 of 10

[10] A. ZYGMUND, *Trigonometric Series*, Cambridge Univ. Press, 1959.



**On a Bojanić–Stanojević Type
Inequality and its Applications**

Živorad Tomovski

Title Page

Contents



Go Back

Close

Quit

Page 10 of 10