Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 5, Article 73, 2002

A MONOTONICITY PROPERTY OF THE Γ-FUNCTION

HENDRIK VOGT AND JÜRGEN VOIGT

FACHRICHTUNG MATHEMATIK,
TECHNISCHE UNIVERSITÄT DRESDEN,
D-01062 DRESDEN, GERMANY.
vogt@math.tu-dresden.de

voigt@math.tu-dresden.de

Received 11 January, 2001; accepted 29 July, 2002
Communicated by A. Laforgia

ABSTRACT. It is shown that the function $x \mapsto 1 + \frac{1}{x}\ln\Gamma(x+1) - \ln(x+1)$ is strictly completely monotone on $(-1, \infty)$ and tends to one as $x \to -1$, to zero as $x \to \infty$. This property is derived from a suitable integral representation of $\ln\Gamma(x+1)$.

Key words and phrases: Gamma function, Strictly completely monotone.

2000 *Mathematics Subject Classification.* 33B15.

The starting point of this note was an inequality,

$$1 \le \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{n-d}{n}}}{\Gamma\left(\frac{n-d}{2}+1\right)} \le e^{\frac{d}{2}}, \tag{1}$$

for all pairs of integers $0 \le d \le n$, in [5, Lemma 2.1]. Note that the left hand side of this inequality is an immediate consequence of the logarithmic convexity of the Γ-function; see [5]. Looking for a stream-lined proof of inequality (1), we first found a proof of the more general inequality

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \le e^{\frac{p}{q}-1}, \tag{2}$$

valid for all $0 < q \le p$, and finally showed

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \le \frac{p+1}{q+1}, \tag{3}$$

for all $-1 < q \le p$. These inequalities will be immediate consequences of the following result.

ISSN (electronic): 1443-5756

The authors were partly supported by the DFG..
007-01

Theorem 1. *The function* $f(x) := 1 + \frac{1}{x}\ln\Gamma(x+1) - \ln(x+1)$ *is strictly completely monotone on* $(-1,\infty)$*,*

$$\lim_{x\to -1} f(x) = 1, \quad \lim_{x\to\infty} f(x) = 0,$$
$$f(0) = \lim_{x\to 0} f(x) = 1 - \gamma.$$

(Here, γ *is the Euler-Mascheroni constant, and* strictly completely monotone *means* $(-1)^n f^{(n)}(x) > 0$ *for all* $x \in (-1,\infty)$*,* $n \in \mathbb{N}_0$*).*

Proof. The main ingredient of the proof is the integral representation

$$\ln\Gamma(x+1) = x\ln(x+1) - x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-t}\frac{1}{t}(1-e^{-xt})\,dt,$$

which is an immediate consequence of [6, formula 1.9 (2) (p. 21)] and [6, formula 1.7.2 (18) (p. 17)]. We obtain

$$f(x) = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-t}\frac{1}{xt}(1-e^{-xt})\,dt.$$

The function

$$g(y) := \frac{1}{y}(1-e^{-y}) = \int_0^1 e^{-sy}\,ds$$

is strictly completely monotone on $\mathbb{R}$. Since $\frac{1}{t} - \frac{1}{e^t-1} > 0$ for all $t > 0$, we conclude that f is strictly completely monotone. As $y \to \infty$, $g(y)$ tends to zero, and hence $\lim_{x\to\infty} f(x) = 0$. The definition of f shows $\lim_{x\to 0} f(x) = 1 + \psi(1) = 1 - \gamma$; cf. [6, formula 1.7 (4) (p. 15)]. Finally,

$$\lim_{x\to -1} f(x) = 1 + \lim_{x\to -1}\left(\frac{1}{x}\big(\ln\Gamma(x+2) - \ln(x+1)\big) - \ln(x+1)\right) = 1.$$

□

Corollary 2. *Inequalities* (3), (2) *and* (1) *are valid for the indicated ranges.*

Proof. Inequality (3) is just a reformulation of the monotonicity of the function f from Theorem 1. Continuing (3) to the right,

$$\frac{p+1}{q+1} \le \frac{p}{q} \le e^{\frac{p}{q}-1} \quad (0 < q \le p),$$

we obtain (2). Setting $q = \frac{n-d}{2}$, $p = \frac{n}{2}$ we get (1). □

Remark 3.

(a) In [4] it was shown that the function $\xi \mapsto \xi\left(\Gamma\left(1+\frac{1}{\xi}\right)\right)^\xi$ is increasing on $(0,\infty)$. This fact follows immediately from our Theorem 1, because of

$$\ln\left(\frac{1}{x}\Gamma(x+1)^{\frac{1}{x}}\right) + 1 = -\ln x + \frac{1}{x}\Gamma(x+1) + 1 = \ln(x+1) - \ln x + f(x).$$

(In fact, the latter function even is strictly completely monotone as well.)

(b) For other recent results on (complete) monotonicity properties of the Γ-function we refer to [1, 2, 3].

REFERENCES

[1] H. ALZER, On some inequalities for the gamma and psi functions, *Math. Comp.*, **66**(217) (1997), 373–389.

[2] G. D. ANDERSON AND S.-L. QIU, A monotoneity property of the gamma function, *Proc. Amer. Math. Soc.*, **125**(11) (1997), 3355–3362.

[3] Á. ELBERT AND A. LAFORGIA, On some properties of the gamma function, *Proc. Amer. Math. Soc.*, **128**(9) (2000), 2667–2673.

[4] D. KERSHAW AND A. LAFORGIA, Monotonicity results for the gamma function, *Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Natur.*, **119**(3-4) (1985), 127–133.

[5] A. KOLDOBSKY AND M. LIFSHITS, Average volume of sections of star bodies, In: *Geometric Aspects of Functional Analysis*, V. D. Milman and G. Schechtmann (eds.), *Lect. Notes Math.*, **1745**, Springer, Berlin, 2000, 119–146.

[6] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. TRICOMI, *Higher Trancscendental Functions*, McGraw-Hill Book Company, New York-Toronto-London, 1953.