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A POTPOURRI OF SCHWARZ RELATED INEQUALITIES IN INNER PRODUCT SPACES (I)

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ABSTRACT. In this paper we obtain some new Schwarz related inequalities in inner product spaces over the real or complex number field. Applications for the generalized triangle inequality are also given.

Key words and phrases: Schwarz's inequality, Triangle inequality, Inner product spaces.

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1. INTRODUCTION

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications, is the Schwarz inequality; that may be written in two forms:

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2, \quad x,y\in H \qquad \text{(quadratic form)} \tag{1.1}$$

or, equivalently,

$$|\langle x,y\rangle| \le \|x\|\|y\|, \quad x,y\in H \qquad \text{(simple form).} \tag{1.2}$$

The case of equality holds in (1.1) (or (1.2)) if and only if the vectors x and y are linearly dependent.

In 1966, S. Kurepa [14], gave the following refinement of the quadratic form for the complexification of a real inner product space:

Theorem 1.1. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real Hilbert space and $(H_{\mathbb{C}},\langle\cdot,\cdot\rangle_{\mathbb{C}})$ the complexification of H. Then for any pair of vectors $a\in H$, $z\in H_{\mathbb{C}}$*

$$|\langle z,a\rangle_{\mathbb{C}}|^2 \le \frac{1}{2}\|a\|^2\left(\|z\|_{\mathbb{C}}^2+|\langle z,\bar{z}\rangle_{\mathbb{C}}|\right)\le \|a\|^2\|z\|_{\mathbb{C}}^2. \tag{1.3}$$

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In 1985, S.S. Dragomir [2, Theorem 2] obtained a refinement of the simple form of the Schwarz inequality as follows:

Theorem 1.2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $x,y,e\in H$ *with* $\|e\|=1$. *Then we have the inequality*

$$\|x\|\,\|y\|\geq|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\geq|\langle x,y\rangle|\,. \tag{1.4}$$

For other similar results, see [8] and [9].

A refinement of the *weaker version* of the Schwarz inequality, i.e.,

$$\operatorname{Re}\langle x,y\rangle\leq\|x\|\,\|y\|\,,\quad x,y\in H \tag{1.5}$$

has been established in [5]:

Theorem 1.3. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $x,y,e\in H$ *with* $\|e\|=1$. *If* $r_1,r_2>0$ *and* $x,y\in H$ *are such that*

$$\|x-y\|\geq r_2\geq r_1\geq|\|x\|-\|y\||\,, \tag{1.6}$$

then we have the following refinement of the weak Schwarz inequality

$$\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\geq\frac{1}{2}\left(r_2^2-r_1^2\right)\qquad(\geq 0)\,. \tag{1.7}$$

The constant $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a larger quantity.*

For other recent results see the paper mentioned above, [5].

In practice, one may need reverses of the Schwarz inequality, namely, upper bounds for the quantities

$$\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\,,\qquad\|x\|^2\,\|y\|^2-(\operatorname{Re}\langle x,y\rangle)^2$$

and

$$\frac{\|x\|\,\|y\|}{\operatorname{Re}\langle x,y\rangle}$$

or the corresponding expressions where $\operatorname{Re}\langle x,y\rangle$ is replaced by either $|\operatorname{Re}\langle x,y\rangle|$ or $|\langle x,y\rangle|\,,$ under suitable assumptions for the vectors x,y in an inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$.

In this class of results, we mention the following recent reverses of the Schwarz inequality due to the present author, that can be found, for instance, in the book [6], where more specific references are provided:

Theorem 1.4. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$. *If* $a,A\in\mathbb{K}$ *and* $x,y\in H$ *are such that either*

$$\operatorname{Re}\langle Ay-x,x-ay\rangle\geq 0, \tag{1.8}$$

or, equivalently,

$$\left\|x-\frac{A+a}{2}y\right\|\leq\frac{1}{2}|A-a|\,\|y\|\,, \tag{1.9}$$

then the following reverse for the quadratic form of the Schwarz inequality

$$\begin{aligned}(0\leq)\,\|x\|^2\,\|y\|^2-|\langle x,y\rangle|^2&\leq\begin{cases}\frac{1}{4}|A-a|^2\,\|y\|^4-\left|\frac{A+a}{2}\|y\|^2-\langle x,y\rangle\right|^2\\[2ex]\frac{1}{4}|A-a|^2\,\|y\|^4-\|y\|^2\operatorname{Re}\langle Ay-x,x-ay\rangle\end{cases}\\&\leq\frac{1}{4}|A-a|^2\,\|y\|^4\end{aligned} \tag{1.10}$$

holds.

If in addition, we have $\operatorname{Re}(A\bar{a}) > 0$, *then*

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right)\langle x, y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \leq \frac{1}{2} \cdot \frac{|A+a|}{\sqrt{\operatorname{Re}(A\bar{a})}} |\langle x, y\rangle|, \tag{1.11}$$

and

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{|A-a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y\rangle|^2 . \tag{1.12}$$

Also, if (1.8) or (1.9) are valid and $A \neq -a$, *then we have the reverse of the simple form of the Schwarz inequality*

$$\begin{aligned}(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| &\leq \|x\| \|y\| - \left|\operatorname{Re}\left[\frac{\bar{A}+\bar{a}}{|A+a|} \langle x, y\rangle\right]\right| \\ &\leq \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{A}+\bar{a}}{|A+a|} \langle x, y\rangle\right] \leq \frac{1}{4} \cdot \frac{|A-a|^2}{|A+a|} \|y\|^2 .\end{aligned} \tag{1.13}$$

The multiplicative constants $\frac{1}{4}$ *and* $\frac{1}{2}$ *above are best possible.*

For some classical results related to the Schwarz inequality, see [1], [11], [15], [16], [17] and the references therein.

The main aim of the present paper is to point out other results in connection with both the quadratic and simple forms of the Schwarz inequality. As applications, some reverse results for the generalised triangle inequality, i.e., upper bounds for the quantity

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\|$$

under various assumptions for the vectors $x_i \in H$, $i \in \{1, \ldots, n\}$, are established.

2. INEQUALITIES RELATED TO SCHWARZ'S

The following result holds.

Proposition 2.1. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$. *The subsequent statements are equivalent.*

(i) *The following inequality holds*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq (\geq)\, r; \tag{2.1}$$

(ii) *The following reverse (improvement) of Schwarz's inequality holds*

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq (\geq) \frac{1}{2} r^2 \|x\| \|y\| . \tag{2.2}$$

The constant $\frac{1}{2}$ *is best possible in (2.2).*

Proof. It is obvious by taking the square in (2.2) and performing the required calculations. □

Remark 2.2. Since

$$\begin{aligned}\| \|y\| x - \|x\| y\| &= \| \|y\| (x-y) + (\|y\| - \|x\|) y\| \\ &\leq \|y\| \|x-y\| + |\|y\| - \|x\|| \|y\| \\ &\leq 2 \|y\| \|x-y\|\end{aligned}$$

hence a sufficient condition for (2.1) to hold is

$$\|x-y\| \leq \frac{r}{2}\|x\|. \tag{2.3}$$

Remark 2.3. Utilising the Dunkl-Williams inequality [10]

$$\|a-b\| \geq \frac{1}{2}(\|a\|+\|b\|)\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|, \quad a,b \in H\backslash\{0\} \tag{2.4}$$

with equality if and only if either $\|a\| = \|b\|$ or $\|a\|+\|b\| = \|a-b\|$, we can state the following inequality

$$\frac{\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle}{\|x\|\|y\|} \leq 2\left(\frac{\|x-y\|}{\|x\|+\|y\|}\right)^2, \quad x,y \in H\backslash\{0\}. \tag{2.5}$$

Obviously, if $x,y \in H\backslash\{0\}$ are such that

$$\|x-y\| \leq \eta(\|x\|+\|y\|), \tag{2.6}$$

with $\eta \in (0,1]$, then one has the following reverse of the Schwarz inequality

$$\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle \leq 2\eta^2\|x\|\|y\| \tag{2.7}$$

that is similar to (2.2).

The following result may be stated as well.

Proposition 2.4. *If $x,y \in H\backslash\{0\}$ and $\rho > 0$ are such that*

$$\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\| \leq \rho, \tag{2.8}$$

then we have the following reverse of Schwarz's inequality

$$\begin{aligned}(0 \leq)\ \|x\|\|y\| - |\langle x,y\rangle| &\leq \|x\|\|y\| - \operatorname{Re}\langle x,y\rangle \\ &\leq \frac{1}{2}\rho^2\|x\|\|y\|.\end{aligned} \tag{2.9}$$

The constant $\frac{1}{2}$ in (2.9) cannot be replaced by a smaller quantity.

Proof. Taking the square in (2.8), we get

$$\frac{\|x\|^2}{\|y\|^2} - \frac{2\operatorname{Re}\langle x,y\rangle}{\|x\|\|y\|} + \frac{\|y\|^2}{\|x\|^2} \leq \rho^2. \tag{2.10}$$

Since, obviously

$$2 \leq \frac{\|x\|^2}{\|y\|^2} + \frac{\|y\|^2}{\|x\|^2} \tag{2.11}$$

with equality iff $\|x\| = \|y\|$, hence by (2.10) we deduce the second inequality in (2.9). □

Remark 2.5. In [13], Hile obtained the following inequality

$$\left\|\|x\|^v x - \|y\|^v y\right\| \leq \frac{\|x\|^{v+1} - \|y\|^{v+1}}{\|x\|-\|y\|}\|x-y\|, \tag{2.12}$$

provided $v > 0$ and $\|x\| \neq \|y\|$.

If in (2.12) we choose $v = 1$ and take the square, then we get

$$\|x\|^4 - 2\|x\|\|y\|\operatorname{Re}\langle x,y\rangle + \|y\|^4 \leq (\|x\|+\|y\|)^2\|x-y\|^2. \tag{2.13}$$

Since,

$$\|x\|^4 + \|y\|^4 \geq 2\|x\|^2\|y\|^2,$$

hence, by (2.13) we deduce

$$(0 \le) \, \|x\| \, \|y\| - \operatorname{Re} \langle x, y \rangle \le \frac{1}{2} \cdot \frac{(\|x\| + \|y\|)^2 \, \|x - y\|^2}{\|x\| \, \|y\|}, \tag{2.14}$$

provided $x, y \in H \backslash \{0\}$.

The following inequality is due to Goldstein, Ryff and Clarke [12, p. 309]:

$$\|x\|^{2r} + \|y\|^{2r} - 2 \|x\|^r \|y\|^r \cdot \frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \, \|y\|} \le \begin{cases} r^2 \|x\|^{2r-2} \|x - y\|^2 & \text{if } r \ge 1 \\ \\ \|y\|^{2r-2} \|x - y\|^2 & \text{if } r < 1 \end{cases} \tag{2.15}$$

provided $r \in \mathbb{R}$ and $x, y \in H$ with $\|x\| \ge \|y\|$.

Utilising (2.15) we may state the following proposition containing a different reverse of the Schwarz inequality in inner product spaces.

Proposition 2.6. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x, y \in H \backslash \{0\}$ and $\|x\| \ge \|y\|$, then we have*

$$\begin{aligned} 0 &\le \|x\| \, \|y\| - |\langle x, y \rangle| \le \|x\| \, \|y\| - \operatorname{Re} \langle x, y \rangle \\ &\le \begin{cases} \frac{1}{2} r^2 \left(\frac{\|x\|}{\|y\|} \right)^{r-1} \|x - y\|^2 & \textit{if } r \ge 1, \\ \\ \frac{1}{2} \left(\frac{\|x\|}{\|y\|} \right)^{1-r} \|x - y\|^2 & \textit{if } r < 1. \end{cases} \end{aligned} \tag{2.16}$$

Proof. It follows from (2.15), on dividing by $\|x\|^r \|y\|^r$, that

$$\left(\frac{\|x\|}{\|y\|} \right)^r + \left(\frac{\|y\|}{\|x\|} \right)^r - 2 \cdot \frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \, \|y\|} \le \begin{cases} r^2 \cdot \frac{\|x\|^{r-2}}{\|y\|^r} \|x - y\|^2 & \text{if } r \ge 1, \\ \\ \frac{\|y\|^{r-2}}{\|x\|^r} \|x - y\|^2 & \text{if } r < 1. \end{cases} \tag{2.17}$$

Since

$$\left(\frac{\|x\|}{\|y\|} \right)^r + \left(\frac{\|y\|}{\|x\|} \right)^r \ge 2,$$

hence, by (2.17) one has

$$2 - 2 \cdot \frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \, \|y\|} \le \begin{cases} r^2 \frac{\|x\|^{r-2}}{\|y\|^r} \|x - y\|^2 & \text{if } r \ge 1, \\ \\ \frac{\|y\|^{r-2}}{\|x\|^r} \|x - y\|^2 & \text{if } r < 1. \end{cases}$$

Dividing this inequality by 2 and multiplying with $\|x\| \, \|y\|$, we deduce the desired result in (2.16). □

Another result providing a different additive reverse (refinement) of the Schwarz inequality may be stated.

Proposition 2.7. *Let $x, y \in H$ with $y \ne 0$ and $r > 0$. The subsequent statements are equivalent:*

(i) *The following inequality holds:*

$$\left\| x - \frac{\langle x, y \rangle}{\|y\|^2} \cdot y \right\| \le (\ge) \, r; \tag{2.18}$$

(ii) *The following reverse (refinement) of the quadratic Schwarz inequality holds:*

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \le (\ge) \, r^2 \|y\|^2. \tag{2.19}$$

The proof is obvious on taking the square in (2.18) and performing the calculation.

Remark 2.8. Since

$$\left\| \|y\|^2 x - \langle x, y\rangle y \right\| = \left\| \|y\|^2 (x-y) - \langle x-y, y\rangle y \right\| \leq \|y\|^2 \|x-y\| + |\langle x-y, y\rangle| \|y\| \leq 2\|x-y\| \|y\|^2,$$

hence a sufficient condition for the inequality (2.18) to hold is that

$$\|x-y\| \leq \frac{r}{2}. \tag{2.20}$$

The following proposition may give a complementary approach:

Proposition 2.9. *Let $x, y \in H$ with $\langle x, y\rangle \neq 0$ and $\rho > 0$. If*

$$\left\| x - \frac{\langle x, y\rangle}{|\langle x, y\rangle|} \cdot y \right\| \leq \rho, \tag{2.21}$$

then

$$(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| \leq \frac{1}{2}\rho^2. \tag{2.22}$$

The multiplicative constant $\frac{1}{2}$ is best possible in (2.22).

The proof is similar to the ones outlined above and we omit it.

For the case of complex inner product spaces, we may state the following result.

Proposition 2.10. *Let $(H; \langle \cdot, \cdot \rangle)$ be a complex inner product space and $\alpha \in \mathbb{C}$ a given complex number with $\operatorname{Re}\alpha, \operatorname{Im}\alpha > 0$. If $x, y \in H$ are such that*

$$\left\| x - \frac{\operatorname{Im}\alpha}{\operatorname{Re}\alpha} \cdot y \right\| \leq r, \tag{2.23}$$

then we have the inequality

$$(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| \leq \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha} \cdot r^2. \tag{2.24}$$

The equality holds in the second inequality in (2.24) if and only if the case of equality holds in (2.23) and $\operatorname{Re}\alpha \cdot \|x\| = \operatorname{Im}\alpha \cdot \|y\|$.

Proof. Observe that the condition (2.23) is equivalent to

$$[\operatorname{Re}\alpha]^2 \|x\|^2 + [\operatorname{Im}\alpha]^2 \|y\|^2 \leq 2 \operatorname{Re}\alpha \operatorname{Im}\alpha \operatorname{Re}\langle x, y\rangle + [\operatorname{Re}\alpha]^2 r^2. \tag{2.25}$$

On the other hand, on utilising the elementary inequality

$$2 \operatorname{Re}\alpha \operatorname{Im}\alpha \|x\| \|y\| \leq [\operatorname{Re}\alpha]^2 \|x\|^2 + [\operatorname{Im}\alpha]^2 \|y\|^2, \tag{2.26}$$

with equality if and only if $\operatorname{Re}\alpha \cdot \|x\| = \operatorname{Im}\alpha \cdot \|y\|$, we deduce from (2.25) that

$$2 \operatorname{Re}\alpha \operatorname{Im}\alpha \|x\| \|y\| \leq 2 \operatorname{Re}\alpha \operatorname{Im}\alpha \operatorname{Re}\langle x, y\rangle + r^2 [\operatorname{Re}\alpha]^2 \tag{2.27}$$

giving the desired inequality (2.24).

The case of equality follows from the above and we omit the details. □

The following different reverse for the Schwarz inequality that holds for both real and complex inner product spaces may be stated as well.

Theorem 2.11. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbb{K}=\mathbb{C},\mathbb{R}$. If $\alpha\in\mathbb{K}\backslash\{0\}$, then*

$$\begin{aligned}0 &\leq \|x\|\,\|y\|-|\langle x,y\rangle| \\ &\leq \|x\|\,\|y\|-\operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle x,y\rangle\right] \\ &\leq \frac{1}{2}\cdot\frac{\left[|\operatorname{Re}\alpha|\,\|x-y\|+|\operatorname{Im}\alpha|\,\|x+y\|\right]^2}{|\alpha|^2} \\ &\leq \frac{1}{2}\cdot I^2,\end{aligned} \tag{2.28}$$

where

$$I:=\begin{cases}\max\{|\operatorname{Re}\alpha|,|\operatorname{Im}\alpha|\}\left(\|x-y\|+\|x+y\|\right); \\ \left(|\operatorname{Re}\alpha|^p+|\operatorname{Im}\alpha|^p\right)^{\frac{1}{p}}\left(\|x-y\|^q+\|x+y\|^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max\{\|x-y\|,\|x+y\|\}\left(|\operatorname{Re}\alpha|+|\operatorname{Im}\alpha|\right).\end{cases} \tag{2.29}$$

Proof. Observe, for $\alpha\in\mathbb{K}\backslash\{0\}$, that

$$\begin{aligned}\|\alpha x-\bar{\alpha}y\|^2 &= |\alpha|^2\|x\|^2-2\operatorname{Re}\langle\alpha x,\bar{\alpha}y\rangle+|\alpha|^2\|y\|^2 \\ &= |\alpha|^2\left(\|x\|^2+\|y\|^2\right)-2\operatorname{Re}\left[\alpha^2\langle x,y\rangle\right].\end{aligned}$$

Since $\|x\|^2+\|y\|^2\geq 2\|x\|\,\|y\|$, hence

$$\|\alpha x-\bar{\alpha}y\|^2\geq 2|\alpha|^2\left\{\|x\|\,\|y\|-\operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle x,y\rangle\right]\right\}. \tag{2.30}$$

On the other hand, we have

$$\begin{aligned}\|\alpha x-\bar{\alpha}y\| &= \|(\operatorname{Re}\alpha+i\operatorname{Im}\alpha)x-(\operatorname{Re}\alpha-i\operatorname{Im}\alpha)y\| \\ &= \|\operatorname{Re}\alpha\,(x-y)+i\operatorname{Im}\alpha\,(x+y)\| \\ &\leq |\operatorname{Re}\alpha|\,\|x-y\|+|\operatorname{Im}\alpha|\,\|x+y\|.\end{aligned} \tag{2.31}$$

Utilising (2.30) and (2.31) we deduce the third inequality in (2.28).

For the last inequality we use the following elementary inequality

$$\alpha a+\beta b\leq\begin{cases}\max\{\alpha,\beta\}(a+b) \\ \left(\alpha^p+\beta^p\right)^{\frac{1}{p}}\left(a^q+b^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases} \tag{2.32}$$

provided $\alpha,\beta,a,b\geq 0$. □

The following result may be stated.

Proposition 2.12. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product over $\mathbb{K}$ and $e\in H$, $\|e\|=1$. If $\lambda\in(0,1)$, then*

$$\begin{aligned}&\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right] \\ &\qquad\leq\frac{1}{4}\cdot\frac{1}{\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)y\|^2-|\langle\lambda x+(1-\lambda)y,e\rangle|^2\right].\end{aligned} \tag{2.33}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Firstly, note that the following equality holds true

$$\langle x - \langle x, e\rangle e, y - \langle y, e\rangle e\rangle = \langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle .$$

Utilising the elementary inequality

$$\operatorname{Re}\langle z, w\rangle \le \frac{1}{4}\|z + w\|^2, \qquad z, w \in H$$

we have

$$\begin{aligned}
&\operatorname{Re}\langle x - \langle x, e\rangle e, y - \langle y, e\rangle e\rangle \\
&\quad = \frac{1}{\lambda(1-\lambda)} \operatorname{Re}\langle \lambda x - \langle \lambda x, e\rangle e, (1-\lambda) y - \langle (1-\lambda) y, e\rangle e\rangle \\
&\quad \le \frac{1}{4} \cdot \frac{1}{\lambda(1-\lambda)} \left[\|\lambda x + (1-\lambda) y\|^2 - |\langle \lambda x + (1-\lambda) y, e\rangle|^2\right],
\end{aligned}$$

proving the desired inequality (2.33). □

Remark 2.13. For $\lambda = \frac{1}{2}$, we get the simpler inequality:

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right] \le \left\|\frac{x+y}{2}\right\|^2 - \left|\left\langle \frac{x+y}{2}, e\right\rangle\right|^2, \tag{2.34}$$

that has been obtained in [6, p. 46], for which the sharpness of the inequality was established.

The following result may be stated as well.

Theorem 2.14. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $p \ge 1$. *Then for any* $x, y \in H$ *we have*

$$\begin{aligned}
0 &\le \|x\|\,\|y\| - |\langle x, y\rangle| \\
&\le \|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle \\
&\le \frac{1}{2} \times \begin{cases} \left[(\|x\| + \|y\|)^{2p} - \|x+y\|^{2p}\right]^{\frac{1}{p}}, \\[2ex] \left[\|x-y\|^{2p} - |\|x\| - \|y\||^{2p}\right]^{\frac{1}{p}}. \end{cases}
\end{aligned} \tag{2.35}$$

Proof. Firstly, observe that

$$2\left(\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle\right) = (\|x\| + \|y\|)^2 - \|x+y\|^2 .$$

Denoting $D := \|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle$, then we have

$$2D + \|x+y\|^2 = (\|x\| + \|y\|)^2 . \tag{2.36}$$

Taking in (2.36) the power $p \ge 1$ and using the elementary inequality

$$(a+b)^p \ge a^p + b^p; \quad a, b \ge 0, \tag{2.37}$$

we have

$$(\|x\| + \|y\|)^{2p} = \left(2D + \|x+y\|^2\right)^p \ge 2^p D^p + \|x+y\|^{2p},$$

giving

$$D^p \le \frac{1}{2^p}\left[(\|x\| + \|y\|)^{2p} - \|x+y\|^{2p}\right],$$

which is clearly equivalent to the first branch of the third inequality in (2.35).

With the above notation, we also have

$$2D + (\|x\| - \|y\|)^2 = \|x-y\|^2 . \tag{2.38}$$

Taking the power $p \geq 1$ in (2.38) and using the inequality (2.37) we deduce

$$\|x-y\|^{2p} \geq 2^p D^p + |\|x\| - \|y\||^{2p},$$

from where we get the last part of (2.35). □

3. MORE SCHWARZ RELATED INEQUALITIES

Before we point out other inequalities related to the Schwarz inequality, we need the following identity that is interesting in itself.

Lemma 3.1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $e \in H$, $\|e\| = 1$, $\alpha \in H$ and $\gamma, \Gamma \in \mathbb{K}$. Then we have the identity:*

$$\begin{aligned} \|x\|^2 - |\langle x,e\rangle|^2 &= (\operatorname{Re}\Gamma - \operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle - \operatorname{Re}\gamma) \\ &\quad + (\operatorname{Im}\Gamma - \operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle - \operatorname{Im}\gamma) \\ &\quad + \left\|x - \frac{\gamma+\Gamma}{2}e\right\|^2 - \frac{1}{4}|\Gamma-\gamma|^2. \end{aligned} \tag{3.1}$$

Proof. We start with the following known equality (see for instance [3, eq. (2.6)])

$$\|x\|^2 - |\langle x,e\rangle|^2 = \operatorname{Re}\left[(\Gamma - \langle x,e\rangle)\left(\overline{\langle x,e\rangle} - \bar{\gamma}\right)\right] - \operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \tag{3.2}$$

holding for $x \in H$, $e \in H$, $\|e\| = 1$ and $\gamma, \Gamma \in \mathbb{K}$.

We also know that (see for instance [4])

$$-\operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle = \left\|x - \frac{\gamma+\Gamma}{2}e\right\|^2 - \frac{1}{4}|\Gamma-\gamma|^2. \tag{3.3}$$

Since

$$\begin{aligned} &\operatorname{Re}\left[(\Gamma - \langle x,e\rangle)\left(\overline{\langle x,e\rangle} - \bar{\gamma}\right)\right] \\ &\quad = (\operatorname{Re}\Gamma - \operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle - \operatorname{Re}\gamma) \\ &\quad\quad + (\operatorname{Im}\Gamma - \operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle - \operatorname{Im}\gamma), \end{aligned} \tag{3.4}$$

hence, by (3.2) – (3.4), we deduce the desired identity (3.1). □

The following general result providing a reverse of the Schwarz inequality may be stated.

Proposition 3.2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $e \in H$, $\|e\| = 1$, $x \in H$ and $\gamma, \Gamma \in \mathbb{K}$. Then we have the inequality:*

$$(0 \leq)\ \|x\|^2 - |\langle x,e\rangle|^2 \leq \left\|x - \frac{\gamma+\Gamma}{2}\cdot e\right\|^2. \tag{3.5}$$

The case of equality holds in (3.5) if and only if

$$\operatorname{Re}\langle x,e\rangle = \operatorname{Re}\left(\frac{\gamma+\Gamma}{2}\right), \qquad \operatorname{Im}\langle x,e\rangle = \operatorname{Im}\left(\frac{\gamma+\Gamma}{2}\right). \tag{3.6}$$

Proof. Utilising the elementary inequality for real numbers

$$\alpha\beta \leq \frac{1}{4}(\alpha+\beta)^2, \qquad \alpha,\beta \in \mathbb{R};$$

with equality iff $\alpha = \beta$, we have

$$(\operatorname{Re}\Gamma - \operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle - \operatorname{Re}\gamma) \leq \frac{1}{4}(\operatorname{Re}\Gamma - \operatorname{Re}\gamma)^2 \tag{3.7}$$

and

$$(3.8)\qquad (\operatorname{Im}\Gamma - \operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle - \operatorname{Im}\gamma) \leq \frac{1}{4}(\operatorname{Im}\Gamma - \operatorname{Im}\gamma)^2$$

with equality if and only if

$$\operatorname{Re}\langle x,e\rangle = \frac{\operatorname{Re}\Gamma + \operatorname{Re}\gamma}{2} \quad \text{and} \quad \operatorname{Im}\langle x,e\rangle = \frac{\operatorname{Im}\Gamma + \operatorname{Im}\gamma}{2}.$$

Finally, on making use of (3.7), (3.8) and the identity (3.1), we deduce the desired result (3.5). □

The following result may be stated as well.

Proposition 3.3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $e \in H$, $\|e\| = 1$, $x \in H$ and $\gamma, \Gamma \in \mathbb{K}$. If $x \in H$ is such that*

$$(3.9)\qquad \operatorname{Re}\gamma \leq \operatorname{Re}\langle x,e\rangle \leq \operatorname{Re}\Gamma \quad \textit{and} \quad \operatorname{Im}\gamma \leq \operatorname{Im}\langle x,e\rangle \leq \operatorname{Im}\Gamma,$$

then we have the inequality

$$(3.10)\qquad \|x\|^2 - |\langle x,e\rangle|^2 \geq \left\| x - \frac{\gamma+\Gamma}{2}e \right\|^2 - \frac{1}{4}|\Gamma - \gamma|^2.$$

The case of equality holds in (3.10) if and only if

$$\operatorname{Re}\langle x,e\rangle = \operatorname{Re}\Gamma \textit{ or } \operatorname{Re}\langle x,e\rangle = \operatorname{Re}\gamma$$

and

$$\operatorname{Im}\langle x,e\rangle = \operatorname{Im}\Gamma \textit{ or } \operatorname{Im}\langle x,e\rangle = \operatorname{Im}\gamma.$$

Proof. From the hypothesis we obviously have

$$(\operatorname{Re}\Gamma - \operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle - \operatorname{Re}\gamma) \geq 0$$

and

$$(\operatorname{Im}\Gamma - \operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle - \operatorname{Im}\gamma) \geq 0.$$

Utilising the identity (3.1) we deduce the desired result (3.10). The case of equality is obvious. □

Further on, we can state the following reverse of the quadratic Schwarz inequality:

Proposition 3.4. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $e \in H$, $\|e\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x \in H$ are such that either*

$$(3.11)\qquad \operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \geq 0$$

or, equivalently,

$$(3.12)\qquad \left\| x - \frac{\gamma+\Gamma}{2}e \right\| \leq \frac{1}{2}|\Gamma - \gamma|,$$

then

$$\begin{aligned}(3.13)\qquad (0 \leq)\ \|x\|^2 - |\langle x,e\rangle|^2 &\leq (\operatorname{Re}\Gamma - \operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle - \operatorname{Re}\gamma) \\ &\qquad + (\operatorname{Im}\Gamma - \operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle - \operatorname{Im}\gamma) \\ &\leq \frac{1}{4}|\Gamma - \gamma|^2.\end{aligned}$$

The case of equality holds in (3.13) if it holds either in (3.11) or (3.12).

The proof is obvious by Lemma 3.1 and we omit the details.

Remark 3.5. We remark that the inequality (3.13) may also be used to get, for instance, the following result

$$\|x\|^2 - |\langle x, e\rangle|^2 \le \left[(\operatorname{Re}\Gamma - \operatorname{Re}\langle x, e\rangle)^2 + (\operatorname{Im}\Gamma - \operatorname{Im}\langle x, e\rangle)^2\right]^{\frac{1}{2}} \times \left[(\operatorname{Re}\langle x, e\rangle - \operatorname{Re}\gamma)^2 + (\operatorname{Im}\langle x, e\rangle - \operatorname{Im}\gamma)^2\right]^{\frac{1}{2}}, \tag{3.14}$$

that provides a different bound than $\frac{1}{4}|\Gamma - \gamma|^2$ for the quantity $\|x\|^2 - |\langle x, e\rangle|^2$.

The following result may be stated as well.

Theorem 3.6. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $\alpha, \gamma > 0$, $\beta \in \mathbb{K}$ with $|\beta|^2 \ge \alpha\gamma$. If $x, a \in H$ are such that $a \neq 0$ and*

$$\left\| x - \frac{\beta}{\alpha} a \right\| \le \frac{\left(|\beta|^2 - \alpha\gamma\right)^{\frac{1}{2}}}{\alpha} \|a\|, \tag{3.15}$$

then we have the following reverses of Schwarz's inequality

$$\|x\| \|a\| \le \frac{\operatorname{Re}\beta \cdot \operatorname{Re}\langle x, a\rangle + \operatorname{Im}\beta \cdot \operatorname{Im}\langle x, a\rangle}{\sqrt{\alpha\gamma}} \le \frac{|\beta| |\langle x, a\rangle|}{\sqrt{\alpha\gamma}} \tag{3.16}$$

and

$$(0 \le) \|x\|^2 \|a\|^2 - |\langle x, a\rangle|^2 \le \frac{|\beta|^2 - \alpha\gamma}{\alpha\gamma} |\langle x, a\rangle|^2. \tag{3.17}$$

Proof. Taking the square in (3.15), it becomes equivalent to

$$\|x\|^2 - \frac{2}{\alpha}\operatorname{Re}\left[\bar{\beta}\langle x, a\rangle\right] + \frac{|\beta|^2}{\alpha^2}\|a\|^2 \le \frac{|\beta|^2 - \alpha\gamma}{\alpha^2}\|a\|^2,$$

which is clearly equivalent to

$$\alpha\|x\|^2 + \gamma\|a\|^2 \le 2\operatorname{Re}\left[\bar{\beta}\langle x, a\rangle\right] = 2\left[\operatorname{Re}\beta \cdot \operatorname{Re}\langle x, a\rangle + \operatorname{Im}\beta \cdot \operatorname{Im}\langle x, a\rangle\right]. \tag{3.18}$$

On the other hand, since

$$2\sqrt{\alpha\gamma}\|x\| \|a\| \le \alpha\|x\|^2 + \gamma\|a\|^2, \tag{3.19}$$

hence by (3.18) and (3.19) we deduce the first inequality in (3.16).

The other inequalities are obvious. □

Remark 3.7. The above inequality (3.16) contains in particular the reverse (1.11) of the Schwarz inequality. Indeed, if we assume that $\alpha = 1$, $\beta = \frac{\delta+\Delta}{2}$, $\delta, \Delta \in \mathbb{K}$, with $\gamma = \operatorname{Re}(\Delta\bar{\gamma}) > 0$, then the condition $|\beta|^2 \ge \alpha\gamma$ is equivalent to $|\delta + \Delta|^2 \ge 4\operatorname{Re}(\Delta\bar{\gamma})$ which is actually $|\Delta - \delta|^2 \ge 0$. With this assumption, (3.15) becomes

$$\left\| x - \frac{\delta + \Delta}{2} \cdot a \right\| \le \frac{1}{2}|\Delta - \delta| \|a\|,$$

which implies the reverse of the Schwarz inequality

$$\|x\| \|a\| \leq \frac{\operatorname{Re}\left[\left(\bar{\Delta}+\bar{\delta}\right)\langle x, a\rangle\right]}{2\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}} \leq \frac{|\Delta+\delta|}{2\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}}|\langle x, a\rangle|,$$

which is (1.11).

The following particular case of Theorem 3.6 may be stated:

Corollary 3.8. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\varphi \in [0, 2\pi)$, $\theta \in \left(0, \frac{\pi}{2}\right)$. If $x, a \in H$ are such that $a \neq 0$ and*

$$\|x - (\cos\varphi + i\sin\varphi)\, a\| \leq \cos\theta \|a\|, \tag{3.20}$$

then we have the reverses of the Schwarz inequality

$$\|x\| \|a\| \leq \frac{\cos\varphi \operatorname{Re}\langle x, a\rangle + \sin\varphi \operatorname{Im}\langle x, a\rangle}{\sin\theta}. \tag{3.21}$$

In particular, if

$$\|x - a\| \leq \cos\theta \|a\|,$$

then

$$\|x\| \|a\| \leq \frac{1}{\sin\theta}\operatorname{Re}\langle x, a\rangle;$$

and if

$$\|x - ia\| \leq \cos\theta \|a\|,$$

then

$$\|x\| \|a\| \leq \frac{1}{\sin\theta}\operatorname{Im}\langle x, a\rangle.$$

4. Reverses of the Generalised Triangle Inequality

In [7], the author obtained the following reverse result for the generalised triangle inequality

$$\sum_{i=1}^{n}\|x_i\| \geq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{4.1}$$

provided $x_i \in H$, $i \in \{1, \dots, n\}$ are vectors in a real or complex inner product $(H;\langle\cdot,\cdot\rangle)$:

Theorem 4.1. *Let $e, x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $k_i \geq 0$, $i \in \{1, \dots, n\}$ are such that*

$$(0 \leq)\, \|x_i\| - \operatorname{Re}\langle e, x_i\rangle \leq k_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.2}$$

then we have the inequality

$$(0 \leq) \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} k_i. \tag{4.3}$$

The equality holds in (4.3) if and only if

$$\sum_{i=1}^{n}\|x_i\| \geq \sum_{i=1}^{n} k_i \tag{4.4}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i \right) e. \tag{4.5}$$

By utilising some of the results obtained in Section 2, we point out several reverses of the generalised triangle inequality (4.1) that are corollaries of Theorem 4.1.

Corollary 4.2. *Let* $e, x_i \in H \backslash \{0\}$, $i \in \{1, \dots, n\}$ *with* $\|e\| = 1$. *If*

$$\left\| \frac{x_i}{\|x_i\|} - e \right\| \le r_i \qquad \text{for each} \qquad i \in \{1, \dots, n\}, \tag{4.6}$$

then

$$\begin{aligned} (0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| &\le \frac{1}{2} \sum_{i=1}^{n} r_i^2 \|x_i\| \\ &\le \frac{1}{2} \times \begin{cases} \left(\max\limits_{1 \le i \le n} r_i \right)^2 \sum\limits_{i=1}^{n} \|x_i\|; \\ \\ \left(\sum\limits_{i=1}^{n} r_i^{2p} \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \max\limits_{1 \le i \le n} \|x_i\| \sum\limits_{i=1}^{n} r_i^2. \end{cases} \end{aligned} \tag{4.7}$$

Proof. The first part follows from Proposition 2.1 on choosing $x = x_i$, $y = e$ and applying Theorem 4.1. The last part is obvious by Hölder's inequality. ☐

Remark 4.3. One would obtain the same reverse inequality (4.7) if one were to use Theorem 2.4. In this case, the assumption (4.6) should be replaced by

$$\| \|x_i\| x_i - e\| \le r_i \|x_i\| \qquad \text{for each} \qquad i \in \{1, \dots, n\}. \tag{4.8}$$

On utilising the inequalities (2.5) and (2.15) one may state the following corollary of Theorem 4.1.

Corollary 4.4. *Let* $e, x_i \in H \backslash \{0\}$, $i \in \{1, \dots, n\}$ *with* $\|e\| = 1$. *Then we have the inequality*

$$(0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \min \{A, B\}, \tag{4.9}$$

where

$$A := 2 \sum_{i=1}^{n} \|x_i\| \left(\frac{\|x_i - e\|}{\|x_i\| + 1} \right)^2,$$

and

$$B := \frac{1}{2} \sum_{i=1}^{n} \frac{(\|x_i\| + 1)^2 \|x_i - e\|^2}{\|x_i\|}.$$

For vectors located outside the closed unit ball $\bar{B}(0,1) := \{z \in H | \|z\| \le 1\}$, we may state the following result.

Corollary 4.5. *Assume that* $x_i \notin \bar{B}(0,1)$, $i \in \{1,\dots,n\}$ *and* $e \in H$, $\|e\| = 1$. *Then we have the inequality:*

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \begin{cases} \frac{1}{2} p^2 \sum_{i=1}^{n} \|x_i\|^{p-1} \|x_i - e\|^2, & \text{if } p \geq 1 \\ \\ \frac{1}{2} \sum_{i=1}^{n} \|x_i\|^{1-p} \|x_i - e\|^2, & \text{if } p < 1. \end{cases} \tag{4.10}$$

The proof follows by Proposition 2.6 and Theorem 4.1.

For complex spaces one may state the following result as well.

Corollary 4.6. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be a complex inner product space and* $\alpha_i \in \mathbb{C}$ *with* $\operatorname{Re}\alpha_i, \operatorname{Im}\alpha_i > 0$, $i \in \{1,\dots,n\}$. *If* $x_i, e \in H$, $i \in \{1,\dots,n\}$ *with* $\|e\| = 1$ *and*

$$\left\| x_i - \frac{\operatorname{Im}\alpha_i}{\operatorname{Re}\alpha_i} \cdot e \right\| \leq d_i, \qquad i \in \{1,\dots,n\}, \tag{4.11}$$

then

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{2} \sum_{i=1}^{n} \frac{\operatorname{Re}\alpha_i}{\operatorname{Im}\alpha_i} \cdot d_i^2. \tag{4.12}$$

The proof follows by Theorems 2.10 and 4.1 and the details are omitted.

Finally, by the use of Theorem 2.14, we can state:

Corollary 4.7. *If* $x_i, e \in H$, $i \in \{1,\dots,n\}$ *with* $\|e\| = 1$ *and* $p \geq 1$, *then we have the inequalities:*

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{2} \times \begin{cases} \sum_{i=1}^{n} \left[(\|x_i\| + 1)^{2p} - \|x_i + e\|^{2p} \right]^{\frac{1}{p}}, \\ \\ \sum_{i=1}^{n} \left[\|x_i - e\|^{2p} - \left| \|x_i\| - 1 \right|^{2p} \right]^{\frac{1}{p}}. \end{cases} \tag{4.13}$$

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