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ON SIMULTANEOUS APPROXIMATION FOR CERTAIN BASKAKOV DURRMEYER TYPE OPERATORS

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ABSTRACT. In the present paper, we study a certain integral modification of the well known Baskakov operators with the weight function of Beta basis function. We establish pointwise convergence, an asymptotic formula an error estimation and an inverse result in simultaneous approximation for these new operators.

Key words and phrases: Baskakov operators, Simultaneous approximation, Asymptotic formula, Pointwise convergence, Error estimation, Inverse theorem.

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1. INTRODUCTION

For

$$f \in C_\gamma[0,\infty) \equiv \{f \in C[0,\infty) : |f(t)| \leq Mt^\gamma$$

for some $M > 0, \gamma > 0\}$ we consider a certain type of Baskakov-Durrmeyer operator as

$$(1.1) \qquad B_n(f(t),x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t) f(t) dt + (1+x)^{-n} f(0)$$
$$= \int_0^\infty W_n(x,t) f(t) dt$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$
$$b_{n,k}(t) = \frac{1}{B(n+1,k)} \cdot \frac{t^{k-1}}{(1+t)^{n+k+1}}$$

and

$$W_n(x,t) = \sum_{k=1}^{\infty} p_{n,k}(x) b_{n,k}(t) + (1+x)^{-n}\delta(t),$$

$\delta(t)$ being the Dirac delta function. The norm- $||\cdot||_\gamma$ on the class $C_\gamma[0,\infty)$ is defined as $||f||_\gamma = \sup_{0\leq t<\infty} |f(t)|t^{-\gamma}$.

The operators defined by (1.1) are the integral modification of the well known Baskakov operators with weight functions of some Beta basis functions. Very recently Finta [2] also studied some other approximation properties of these operators. The behavior of these operators is very similar to the operators recently introduced in [6], [9] and also studied in [8]. These operators reproduce not only the constant functions but also the linear functions, which is the interesting property of such operators. The other usual Durrmeyer type integral modification of the Baskakov operators [5] reproduce only the constant functions, so one can not apply the iterative combinations easily to improve the order of approximation for the usual Baskakov Durrmeyer operators. For recent work in this area we refer to [7]. In the present paper we study some direct results which include pointwise convergence, asymptotic formula, error estimation and inverse theorem in the simultaneous approximation for the unbounded functions of growth of order t^γ.

2. BASIC RESULTS

In this section we mention certain lemmas which will be used in the sequel.

Lemma 2.1 ([3]). *For $m \in N \cup \{0\}$, if the m^{th} order moment be defined as*

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \left(\frac{k}{n} - x\right)^m,$$

then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ and

$$nU_{n,m+1}(x) = x(1+x)(U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)).$$

Consequently we have $U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 2.2. *Let the function* $T_{n,m}(x), m \in N \cup \{0\}$*, be defined as*

$$\begin{aligned} T_{n,m}(x) &= B_n\big((t-x)^m x\big) \\ &= \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(t-x)^m dt + (1+x)^{-n}(-x)^m. \end{aligned}$$

Then $T_{n,0}(x) = 1$, $T_{n,1} = 0$, $T_{n,2}(x) = \frac{2x(1+x)}{n-1}$ *and also there holds the recurrence relation*

$$(n-m)T_{n,m+1}(x) = x(1+x)\left[T_{n,m}^{(1)}(x) + 2mT_{n,m-1}(x)\right] + m(1+2x)T_{n,m}(x).$$

Proof. By definition, we have

$$\begin{aligned} T_{n,m}^{(1)}(x) = \sum_{k=1}^{\infty} p_{n,k}^{(1)}(x) \int_0^{\infty} b_{n,k}(t)(t-x)^m dt \\ - m \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(t-x)^{m-1} dt \\ - n(1+x)^{-n-1}(-x)^m - m(1+x)^{-n}(-x)^{m-1}. \end{aligned}$$

Using the identities

$$x(1+x)p_{n,k}^{(1)}(x) = (k-nx)p_{n,k}(x)$$

and

$$t(1+t)b_{n,k}^{(1)}(t) = [(k-1)-(n+2)t]b_{n,k}(t),$$

we have

$$\begin{aligned} & x(1+x)\left[T_{n,m}^{(1)}(x) + mT_{n,m-1}(x)\right] \\ &= \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} (k-nx)b_{n,k}(t)(t-x)^m dt + n(1+x)^{-n}(-x)^{m+1} \\ &= \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} \big[(k-1)-(n+2)t+(n+2)(t-x) \\ &\qquad + (1+2x)\big] b_{n,k}(t)(t-x)^m dt + n(1+x)^{-n}(-x)^{m+1} \\ &= \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} t(1+t)b_{n,k}^{(1)}(t)(t-x)^m dt \\ &\qquad + (n+2)[T_{n,m+1}(x) - (1+x)^{-n}(-x)^{m+1}] \\ &\qquad + (1+2x)[T_{n,m}(x) - (1+x)^{-n}(-x)^m] + n(1+x)^{-n}(-x)^{m+1} \\ &= -(m+1)(1+2x)[T_{n,m}(x) - (1+x)^{-n}(-x)^m] \\ &\qquad - (m+2)[T_{n,m+1} - (1+x)^{-n}(-x)^{m+1}] \\ &\qquad - mx(1+x)[T_{n,m-1}(x) - (1+x)^{-n}(-x)^{m-1}] \\ &\qquad + (n+2)[T_{n,m+1} - (1+x)^{-n}(-x)^{m+1}] \\ &\qquad + (1+2x)[T_{n,m}(x) - (1+x)^{-n}(-x)^m] + n(1+x)^{-n}(-x)^{m+1}. \end{aligned}$$

Thus, we get

$$(n-m)T_{n,m+1}(x) = x(1+x)[T_{n,m}^{(1)}(x) + 2mT_{n,m-1}(x)] + m(1+2x)T_{n,m}(x).$$

This completes the proof of recurrence relation. From the above recurrence relation, it is easily verified for all $x \in [0,\infty)$ that

$$T_{n,m}(x) = O\big(n^{-[(m+1)/2]}\big).$$

□

Remark 2.3. It is easily verified from Lemma 2.1 that for each $x \in (0,\infty)$

$$B_n(t^i,x) = \frac{(n+i-1)!(n-i)!}{n!(n-1)!}x^i + i(i-1)\frac{(n+i-2)!(n-i)!}{n!(n-1)!}x^{i-1} + O(n^{-2}).$$

Corollary 2.4. *Let δ be a positive number. Then for every $\gamma > 0, x \in (0,\infty)$, there exists a constant $M(s,x)$ independent of n and depending on s and x such that*

$$\left\| \int_{|t-x|>\delta} W_n(x,t)t^\gamma dt \right\|_{C[a,b]} \le M(s,x)n^{-s}, \qquad s = 1,2,3,\dots$$

Lemma 2.5. *There exist the polynomials $Q_{i,j,r}(x)$ independent of n and k such that*

$$\{x(1+x)\}^r D^r\left[p_{n,k}(x)\right] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j Q_{i,j,r}(x)p_{n,k}(x),$$

where $D \equiv \frac{d}{dx}$.

By C_0, we denote the class of continuous functions on the interval $(0,\infty)$ having a compact support and C_0^r is the class of r times continuously differentiable functions with $C_0^r \subset C_0$. The function f is said to belong to the generalized Zygmund class $Liz(\alpha,1,a,b)$, if there exists a constant M such that $\omega_2(f,\delta) \le M\delta^\alpha$, $\delta > 0$, where $\omega_2(f,\delta)$ denotes the modulus of continuity of 2nd order on the interval $[a,b]$. The class $Liz(\alpha,1,a,b)$ is more commonly denoted by $Lip^*(\alpha,a,b)$. Suppose $G^{(r)} = \{g : g \in C_0^{r+2}, \operatorname{supp} g \subset [a',b'] \text{ where } [a',b'] \subset (a,b)\}$. For r times continuously differentiable functions f with $\operatorname{supp} f \subset [a',b']$ the Peetre's K-functionals are defined as

$$K_r(\xi,f) = \inf_{g\in G^{(r)}} \left[\left\|f^{(r)} - g^{(r)}\right\|_{C[a',b']} + \xi\left\{\left\|g^{(r)}\right\|_{C[a',b']} + \left\|g^{(r+2)}\right\|_{C[a',b']}\right\}\right], \qquad 0 < \xi \le 1.$$

For $0 < \alpha < 2$, $C_0^r(\alpha,1,a,b)$ denotes the set of functions for which

$$\sup_{0<\xi\le 1} \xi^{-\alpha/2}K_r(\xi,f,a,b) < C.$$

Lemma 2.6. *Let $0 < a' < a'' < b'' < b' < b < \infty$ and $f^{(r)} \in C_0$ with $\operatorname{supp} f \subset [a'',b'']$ and if $f \in C_0^r(\alpha,1,a',b')$, we have $f^{(r)} \in Liz(\alpha,1,a',b')$ i.e. $f^{(r)} \in Lip^*(\alpha,a',b')$ where $Lip^*(\alpha,a',b')$ denotes the Zygmund class satisfying $K_r(\delta,f) \le C\delta^{\alpha/2}$.*

Proof. Let $g \in G^{(r)}$, then for $f \in C_0^r(\alpha,1,a',b')$, we have

$$\begin{aligned}
\left|\triangle_\delta^2 f^{(r)}(x)\right| &\le \left|\triangle_\delta^2(f^{(r)} - g^{(r)})(x)\right| + \left|\triangle_\delta^2 g^{(r)}(x)\right| \\
&\le \left\|\triangle_\delta^2(f^{(r)} - g^{(r)})\right\|_{C[a',b']} + \delta^2\left\|g^{(r+2)}\right\|_{C[a',b']} \\
&\le 4M_1K_r(\delta^2,f) \le M_2\delta^\alpha.
\end{aligned}$$

□

Lemma 2.7. *If f is r times differentiable on $[0,\infty)$, such that $f^{(r-1)} = O(t^\alpha)$, $\alpha > 0$ as $t \to \infty$, then for $r = 1,2,3,\dots$ and $n > \alpha + r$ we have*

$$B_n^{(r)}(f,x) = \frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)f^{(r)}(t)dt.$$

Proof. First

$$B_n^{(1)}(f,x) = \sum_{k=1}^{\infty} p_{n,k}^{(1)}(x) \int_0^\infty b_{n,k}(t)f(t)dt - n(1+x)^{-n-1}f(0).$$

Now using the identities

$$p_{n,k}^{(1)}(x) = n[p_{n+1,k-1}(x) - p_{n+1,k}(x)], \tag{2.1}$$

$$b_{n,k}^{(1)}(t) = (n+1)[b_{n+1,k-1}(t) - b_{n+1,k}(t)]. \tag{2.2}$$

for $k \geq 1$, we have

$$\begin{aligned} B_n^{(1)}(f,x) &= \sum_{k=1}^{\infty} n[p_{n+1,k-1}(x) - p_{n+1,k}(x)] \int_0^\infty b_{n,k}(t)f(t)dt - n(1+x)^{-n-1}f(0) \\ &= np_{n+1,0}(x)\int_0^\infty b_{n,1}(t)f(t)dt - n(1+x)^{-n-1}f(0) \\ &\qquad + n\sum_{k=1}^{\infty} p_{n+1,k}(x)\int_0^\infty [b_{n,k+1}(t) - b_{n,k}(t)]f(t)dt \\ &= n(1+x)^{-n-1}\int_0^\infty (n+1)(1+t)^{-n-2}f(t)dt - n(1+x)^{-n-1}f(0) \\ &\qquad + n\sum_{k=1}^{\infty} p_{n+1,k}(x)\int_0^\infty -\frac{1}{n}b_{n-1,k+1}^{(1)}(t)f(t)dt. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} B_n^{(1)}(f,x) &= n(1+x)^{-n-1}f(0) + n(1+x)^{-n-1}\int_0^\infty (1+t)^{-n-1}f^{(1)}(t)dt \\ &\qquad - n(1+x)^{-n-1}f(0) + \sum_{k=1}^{\infty} p_{n+1,k}(x)\int_0^\infty b_{n-1,k+1}(t)f^{(1)}(t)dt \\ &= \sum_{k=0}^{\infty} p_{n+1,k}(x)\int_0^\infty b_{n-1,k+1}(t)f^{(1)}(t)dt. \end{aligned}$$

Thus the result is true for $r = 1$. We prove the result by induction method. Suppose that the result is true for $r = i$, then

$$B_n^{(i)}(f,x) = \frac{(n+i-1)!(n-i)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+i,k}(x)\int_0^\infty b_{n-i,k+i}(t)f^{(i)}(t)dt.$$

Thus using the identities (2.1) and (2.2), we have

$$\begin{aligned} &B_n^{(i+1)}(f,x) \\ &= \frac{(n+i-1)!(n-i)!}{n!(n-1)!}\sum_{k=1}^{\infty}(n+i)[p_{n+i+1,k-1}(x) - p_{n+i+1,k}(x)]\int_0^\infty b_{n-i,k+i}(t)f^{(i)}(t)dt \\ &\qquad + \frac{(n+i-1)!(n-i)!}{n!(n-1)!}(-(n+i)(1+x)^{-n-i-1})\int_0^\infty b_{n-i,i}(t)f^{(i)}(t)dt \end{aligned}$$

$$\begin{aligned}
&= \frac{(n+i)!(n-i)!}{n!(n-1)!} p_{n+i+1,0}(x) \int_0^\infty b_{n-i,i+1}(t) f^{(i)}(t)dt \\
&\qquad - \frac{(n+i)!(n-i)!}{n!(n-1)!} p_{n+i+1,0}(x) \int_0^\infty b_{n-i,i}(t) f^{(i)}(t)dt \\
&\qquad + \frac{(n+i)!(n-i)!}{n!(n-1)!} \sum_{k=1}^{\infty} p_{n+i+1,k}(x) \int_0^\infty [b_{n-i,k+i+1}(t) - b_{n-i,k+i}(t)] f^{(i)}(t)dt \\
&= \frac{(n+i)!(n-i)!}{n!(n-1)!} p_{n+i+1,0}(x) \int_0^\infty -\frac{1}{(n-i)} b^{(1)}_{n-i-1,i+1}(t) f^{(i)}(t)dt \\
&\qquad + \frac{(n+i)!(n-i)!}{n!(n-1)!} \sum_{k=1}^{\infty} p_{n+i+1,k}(x) \int_0^\infty -\frac{1}{(n-i)} b^{(1)}_{n-i-1,k+i+1}(t) f^{(i)}(t)dt.
\end{aligned}$$

Integrating by parts, we obtain

$$B_n^{(i+1)}(f,x) = \frac{(n+i)!(n-i-1)!}{n!(n-1)!} \sum_{k=0}^{\infty} p_{n+i+1,k}(x) \int_0^\infty b_{n-i-1,k+i+1}(t) f^{(i+1)}(t)dt.$$

This completes the proof of the lemma. □

3. DIRECT THEOREMS

In this section we present the following results.

Theorem 3.1. *Let* $f \in C_\gamma[0,\infty)$ *and* $f^{(r)}$ *exists at a point* $x \in (0,\infty)$. *Then we have*

$$B_n^{(r)}(f,x) = f^{(r)}(x)$$

as $n \to \infty$.

Proof. By Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Hence

$$\begin{aligned}
B_n^{(r)}(f,x) &= \int_0^\infty W_n^{(r)}(t,x) f(t)dt \\
&= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt + \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt \\
&=: R_1 + R_2.
\end{aligned}$$

First to estimate R_1, using the binomial expansion of $(t-x)^i$ and Remark 2.3, we have

$$\begin{aligned}
R_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{v=0}^{i} \binom{i}{v} (-x)^{i-v} \frac{\partial^r}{\partial x^r} \int_0^\infty W_n(t,x) t^v dt \\
&= \frac{f^{(r)}(x)}{r!} \frac{d^r}{dx^r} \left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!} x^r + \text{terms containing lower powers of } x \right] \\
&= f^{(r)}(x) \left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!} \right] \to f^{(r)}(x)
\end{aligned}$$

as $n \to \infty$. Next applying Lemma 2.5, we obtain

$$R_2 = \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt,$$

$$|R_2| \leq \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r} \sum_{k=1}^\infty |k-nx|^j p_{n,k}(x)$$

$$\int_0^\infty b_{n,k}(t)|\varepsilon(t,x)||t-x|^r dt + \frac{(n+r+1)!}{(n-1)!}(1+x)^{-n-r}|\varepsilon(0,x)|x^r.$$

The second term in the above expression tends to zero as $n \to \infty$. Since $\varepsilon(t,x) \to 0$ as $t \to x$ for a given $\varepsilon > 0$ there exists a δ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. If $\alpha \geq \max\{\gamma, r\}$, where α is any integer, then we can find a constant $M_3 > 0$, $|\varepsilon(t,x)(t-x)^r| \leq M_3|t-x|^\alpha$, for $|t-x| \geq \delta$. Therefore

$$\begin{aligned} |R_2| &\leq M_3 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=0}^\infty p_{n,k}(x)|k-nx|^j \\ &\qquad \times \left\{ \varepsilon \int_{|t-x|<\delta} b_{n,k}(t)|t-x|^r dt + \int_{|t-x|\geq\delta} b_{n,k}(t)|t-x|^\alpha dt \right\} \\ &=: R_3 + R_4. \end{aligned}$$

Applying the Cauchy-Schwarz inequality for integration and summation respectively, we obtain

$$R_3 \leq \varepsilon M_3 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left\{ \sum_{k=1}^\infty p_{n,k}(x)(k-nx)^{2j} \right\}^{\frac{1}{2}} \left\{ \sum_{k=1}^\infty p_{n,k}(x) \int_0^\infty b_{n,k}(t)(t-x)^{2r} dt \right\}^{\frac{1}{2}}.$$

Using Lemma 2.1 and Lemma 2.2, we get

$$R_3 = \varepsilon \cdot O(n^{r/2})O(n^{-r/2}) = \varepsilon \cdot o(1).$$

Again using the Cauchy-Schwarz inequality, Lemma 2.1 and Corollary 2.4, we get

$$\begin{aligned} R_4 &\leq M_4 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=1}^\infty p_{n,k}(x)|k-nx|^j \int_{|t-x|\geq\delta} b_{n,k}(t)|t-x|^\alpha dt \\ &\leq M_4 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=1}^\infty p_{n,k}(x)|k-nx|^j \left\{ \int_{|t-x|\geq\delta} b_{n,k}(t) dt \right\}^{\frac{1}{2}} \left\{ \int_{|t-x|\geq\delta} b_{n,k}(t)(t-x)^{2\alpha} dt \right\}^{\frac{1}{2}} \\ &\leq M_4 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left\{ \sum_{k=1}^\infty p_{n,k}(x)(k-nx)^{2j} \right\}^{\frac{1}{2}} \left\{ \sum_{k=1}^\infty p_{n,k}(x) \int_0^\infty b_{n,k}(t)(t-x)^{2\alpha} dt \right\}^{\frac{1}{2}} \\ &= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{j/2})O(n^{-\alpha/2}) = O(n^{(r-\alpha)/2}) = o(1). \end{aligned}$$

Collecting the estimates of $R_1 - R_4$, we obtain the required result. □

Theorem 3.2. *Let $f \in C_\gamma[0,\infty)$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$. Then*

$$\begin{aligned} &\lim_{n\to\infty} n\left[B_n^{(r)}(f,x) - f^{(r)}(x)\right] \\ &\qquad = r(r-1)f^{(r)}(x) + r(1+2x)f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x). \end{aligned}$$

Proof. Using Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and $\varepsilon(t,x)=O((t-x)^\gamma)$, $t\to\infty$ for $\gamma>0$. Applying Lemma 2.2, we have

$$\begin{aligned}
&n\left[B_n^{(r)}(f(t),x)-f^{(r)}(x)\right]\\
&=n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt-f^{(r)}(x)\right]\\
&\qquad+n\int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}dt\\
&=: E_1+E_2.
\end{aligned}$$

First, we have

$$\begin{aligned}
E_1&=n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty W_n^{(r)}(t,x)t^j dt-nf^{(r)}(x)\\
&=\frac{f^{(r)}(x)}{r!}n\left[B_n^{(r)}(t^r,x)-(r!)\right]+\frac{f^{(r+1)}(x)}{(r+1)!}n\left[(r+1)(-x)B_n^{(r)}(t^r,x)+B_n^{(r)}(t^{r+1},x)\right]\\
&\quad+\frac{f^{(r+2)}(x)}{(r+2)!}n\left[\frac{(r+2)(r+1)}{2}x^2B_n^{(r)}(t^r,x)+(r+2)(-x)B_n^{(r)}(t^{r+1},x)+B_n^{(r)}(t^{r+2},x)\right].
\end{aligned}$$

Therefore, by Remark 2.3, we have

$$\begin{aligned}
E_1&=nf^{(r)}(x)\left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!}-1\right]\\
&\quad+\frac{nf^{(r+1)}(x)}{(r+1)!}\left[(x-1)(-x)\left\{\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\right\}\right.\\
&\quad+\left\{\frac{(n+r)!(n-r-1)!}{n!(n-1)!}(r+1)!x+(r+1)r\frac{(n+r-1)!(n-r-1)!}{n!(n-1)!}r!\right\}\\
&\quad+\frac{nf^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)}{2}x^2\frac{(n+r-1)!(n-r)!}{n!(n-1)!}r!\right]\\
&\quad+(r+2)(-x)\left\{\frac{(n+r)!(n-r-1)!}{2}x(r+1)!+(r+1)r\frac{(n-r-1)!(n-r-1)!}{n!(n-1)!}r!\right\}\\
&\quad+\left\{\frac{(n+r+1)!(n-r-2)!}{n!(n-1)!}\frac{(r+2)!}{2}x^2\right.\\
&\quad\left.\left.+(r+2)(r+1)\ \frac{(n+r)!(n-r-2)!}{n!(n-1)!}(r+1)!x\right\}\right]+O(n^{-2}).
\end{aligned}$$

In order to complete the proof of the theorem it is sufficient to show that $E_2\to 0$ as $n\to\infty$ which easily follows proceeding along the lines of the proof of Theorem 3.1 and by using Lemma 2.1, Lemma 2.2 and Lemma 2.5. □

Lemma 3.3. *Let* $0 < \alpha < 2$, $0 < a < a' < a'' < b'' < b' < b < \infty$. *If* $f \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$ *and* $\left\| B_n^{(r)}(f, \cdot) - f^{(r)} \right\|_{C[a,b]} = O(n^{-\alpha/2})$, *then*

$$K_r(\xi, f) = M_5 \left\{ n^{-\alpha/2} + n\xi K_r(n^{-1}, f) \right\}.$$

Consequently $K_r(\xi, f) \leq M_6 \xi^{\alpha/2}, M_6 > 0$.

Proof. It is sufficient to prove

$$K_r(\xi, f) = M_7 \{ n^{-\alpha/2} + n\xi K_r(n^{-1}, f) \},$$

for sufficiently large n. Because $\operatorname{supp} f \subset [a'', b'']$ therefore by Theorem 3.2 there exists a function $h^{(i)} \in G^{(r)}$, $i = r, r+2$, such that

$$\left\| B_n^{(r)}(f, \bullet) - h^{(i)} \right\|_{C[a',b']} \leq M_8 n^{-1}.$$

Therefore,

$$K_r(\xi, f) \leq 3M_9 n^{-1} + \left\| B_n^{(r)}(f, \bullet) - f^{(r)} \right\|_{C[a',b']} + \xi \left\{ \left\| B_n^{(r)}(f, \bullet) \right\|_{C[a',b']} + \left\| B_n^{(r+2)}(f, \bullet) \right\|_{C[a',b']} \right\}.$$

Next, it is sufficient to show that there exists a constant M_{10} such that for each $g \in G^{(r)}$

$$\left\| B_n^{(r+2)}(f, \bullet) \right\|_{C[a',b']} \leq M_{10} n \{ \left\| f^{(r)} - g^{(r)} \right\|_{C[a',b']} + n^{-1} \left\| g^{(r+2)} \right\|_{C[a',b']}. \tag{3.1}$$

Also using the linearity property, we have

$$\left\| B_n^{(r+2)}(f, \bullet) \right\|_{C[a',b']} \leq \left\| B_n^{(r+2)}(f - g, \bullet) \right\|_{C[a',b']} + \left\| B_n^{(r+2)}(g, \bullet) \right\|_{C[a',b']}. \tag{3.2}$$

Applying Lemma 2.5, we get

$$\int_0^\infty \left| \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t) \right| dt \leq \sum_{\substack{2i+j \leq r+2 \\ i,j \geq 0}} \sum_{k=1}^{\infty} n^i |k - nx|^j \frac{|Q_{i,j,r+2}(x)|}{\{x(1+x)\}^{r+2}} \times p_{n,k}(x) \int_0^\infty b_{n,k}(t) dt + \frac{d^{r+2}}{dx^{r+2}} [(1+x)^{-n}].$$

Therefore by the Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$\left\| B_n^{(r)}(f - g, \bullet) \right\|_{C[a',b']} \leq M_{11} n \left\| f^{(r)} - g^{(r)} \right\|_{C[a',b']}, \tag{3.3}$$

where the constant M_{11} is independent of f and g. Next by Taylor's expansion, we have

$$g(t) = \sum_{i=0}^{r+1} \frac{g^{(i)}(x)}{i!} (t-x)^i + \frac{g^{(r+2)}(\xi)}{(r+2)!} (t-x)^{r+2},$$

where ξ lies between t and x. Using the above expansion and the fact that $\int_0^\infty \frac{\partial^m}{\partial x^m} W_n(x,t)(t-x)^i dt = 0$ for $m > i$, we get

$$\left\| B_n^{(r+2)}(g, \bullet) \right\|_{C[a',b']} \leq M_{12} \left\| g^{(r+2)} \right\|_{C[a',b']} \cdot \left\| \int_0^\infty \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t)(t-x)^{r+2} dt \right\|_{C[a',b']}. \tag{3.4}$$

Also by Lemma 2.5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
E &\equiv \int_0^\infty \left| \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t) \right| (t-x)^{r+2} dt \\
&\leq \sum_{\substack{2i+j\leq r+2\\ i,j\geq 0}} \sum_{k=1}^{\infty} n^i p_{n,k}(x) |k-nx|^j \frac{|Q_{i,j,r+2}(x)|}{\{x(1+x)\}^{r+2}} \int_0^\infty b_{n,k}(t)(t-x)^{r+2} dt \\
&\quad + \frac{d^{r+2}}{dx^{r+2}}[(-x)^{r+2}(1+x)^{-n}] \\
&\leq \sum_{\substack{2i+j\leq r+2\\ i,j\geq 0}} \frac{|Q_{i,j,r+2}(x)|}{\{x(1+x)\}^{r+2}} \left(\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j} \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{k=1}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t)(t-x)^{2r+4} dt \right)^{\frac{1}{2}} \left(\int_0^\infty b_{n,k}(t) dt \right)^{\frac{1}{2}} \\
&\quad + \frac{d^{r+2}}{dx^{r+2}}[(-x)^{r+2}(1+x)^{-n}] \\
&= \sum_{\substack{2i+j\leq r+2\\ i,j\geq 0}} n^i \frac{|Q_{i,j,r+2}(x)|}{\{x(1+x)\}^{r+2}} O(n^{j/2}) O\left(n^{-\left(1+\frac{r}{2}\right)} \right).
\end{aligned}$$

Hence

$$||B_n^{(r+2)}(g,\bullet)||_{C[a',b']} \leq M_{13}||g^{(r+2)}||_{C[a',b']}. \tag{3.5}$$

Combining the estimates of (3.2)-(3.5), we get (3.1). The other consequence follows form [1]. This completes the proof of the lemma. □

Theorem 3.4. *Let $f \in C_\gamma[0,\infty)$ and suppose $0 < a < a_1 < b_1 < b < \infty$. Then for all n sufficiently large, we have*

$$\left\| B_n^{(r)}(f,\bullet) - f^{(r)} \right\|_{C[a_1,b_1]} \leq \max\left\{ M_{14}\omega_2\left(f^{(r)}, n^{-\frac{1}{2}}, a, b \right) + M_{15} n^{-1} \|f\|_\gamma \right\},$$

where $M_{14} = M_{14}(r), M_{15} = M_{15}(r,f)$.

Proof. For sufficiently small $\delta > 0$, we define a function $f_{2,\delta}(t)$ corresponding to $f \in C_\gamma[0,\infty)$ by

$$f_{2,\delta}(t) = \delta^{-2} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \left(f(t) - \Delta_\eta^2 f(t) \right) dt_1 dt_2,$$

where $\eta = \frac{t_1+t_2}{2}$, $t \in [a,b]$ and $\Delta_\eta^2 f(t)$ is the second forward difference of f with step length η. Following [4] it is easily checked that:

(i) $f_{2,\delta}$ has continuous derivatives up to order $2k$ on $[a,b]$,
(ii) $\|f_{2,\delta}^{(r)}\|_{C[a_1,b_1]} \leq \widehat{M}_1 \delta^{-r} \omega_2(f,\delta,a,b)$,
(iii) $\|f - f_{2,\delta}\|_{C[a_1,b_1]} \leq \widehat{M}_2 \omega_2(f,\delta,a,b)$,
(iv) $\|f_{2,\delta}\|_{C[a_1,b_1]} \leq \widehat{M}_3 \|f\|_\gamma$,

where $\widehat{M}_i$, $i = 1,2,3$ are certain constants that depend on $[a,b]$ but are independent of f and n [4].

We can write

$$\begin{aligned}&\left\|B_n^{(r)}(f,\bullet)-f^{(r)}\right\|_{C[a_1,b_1]}\\&\leq\left\|B_n^{(r)}(f-f_{2,\delta},\bullet)\right\|_{C[a_1,b_1]}+\left\|B_n^{(r)}(f_{2,\delta},\bullet)-f_{2,\delta}^{(r)}\right\|_{C[a_1,b_1]}+\left\|f^{(r)}-f_{2,\delta}^{(r)}\right\|_{C[a_1,b_1]}\\&=:H_1+H_2+H_3.\end{aligned}$$

Since $f_{2,\delta}^{(r)}=\left(f^{(r)}\right)_{2,\delta}(t)$, by property (iii) of the function $f_{2,\delta}$, we get

$$H_3\leq\widehat{M}_4\omega_2(f^{(r)},\delta,a,b).$$

Next on an application of Theorem 3.2, it follows that

$$H_2\leq\widehat{M}_5n^{-1}\sum_{j=r}^{r+2}\left\|f_{2,\delta}^{(j)}\right\|_{C[a,b]}.$$

Using the interpolation property due to Goldberg and Meir [4], for each $j=r,r+1,r+2$, it follows that

$$\left\|f_{2,\delta}^{(j)}\right\|_{C[a_1,b_1]}\leq\widehat{M}_6\left\{||f_{2,\delta}||_{C[a,b]}+\left\|f_{2,\delta}^{(r+2)}\right\|_{C[a,b]}\right\}.$$

Therefore by applying properties (iii) and (iv) of the of the function $f_{2,\delta}$, we obtain

$$H_2\leq\widehat{M}_74\cdot n^{-1}\left\{||f||_\gamma+\delta^{-2}\omega_2(f^{(r)},\delta)\right\}.$$

Finally we shall estimate H_1, choosing a^*,b^* satisfying the conditions $0<a<a^*<a_1<b_1<b^*<b<\infty$. Suppose $\psi(t)$ denotes the characteristic function of the interval $[a^*,b^*]$, then

$$\begin{aligned}H_1\leq&\left\|B_n^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),\bullet)\right\|_{C[a_1,b_1]}\\&+\left\|B_n^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t)),\bullet)\right\|_{C[a_1,b_1]}\\=:&\,H_4+H_5.\end{aligned}$$

Using Lemma 2.7, it is clear that

$$\begin{aligned}&B_n^{(r)}\big(\psi(t)(f(t)-f_{2,\delta}(t)),x\big)\\&\quad=\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)\psi(t)(f^{(r)}(t)-f_{2,\delta}^{(r)}(t))dt.\end{aligned}$$

Hence

$$\left\|B_n^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),\bullet)\right\|_{C[a_1,b_1]}\leq\widehat{M}_8\left\|f^{(r)}-f_{2,\delta}^{(r)}\right\|_{C[a^*,b^*]}.$$

Next for $x\in[a_1,b_1]$ and $t\in[0,\infty)\setminus[a^*,b^*]$, we choose a $\delta_1>0$ satisfying $|t-x|\geq\delta_1$.

Therefore by Lemma 2.5 and the Cauchy-Schwarz inequality, we have

$$I\equiv B_n^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t),x)$$

and

$$\begin{aligned}|I|\leq&\sum_{\substack{2i+j\leq r\\i,j\geq0}}n^i\frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r}\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^\infty b_{n,k}(t)(1-\psi(t))|f(t)-f_{2,\delta}(t)|dt\\&+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}(1-\psi(0))|f(0)-f_{2,\delta}(0)|.\end{aligned}$$

For sufficiently large n, the second term tends to zero. Thus

$$\begin{aligned}|I| &\leq \widehat{M}_9||f||_\gamma \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_{|t-x|\geq\delta_1} b_{n,k}(t)dt \\ &\leq \widehat{M}_9||f||_\gamma \delta_1^{-2m} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \left(\int_0^\infty b_{n,k}(t)dt\right)^{\frac{1}{2}} \left(\int_0^\infty b_{n,k}(t)(t-x)^{4m}dt\right)^{\frac{1}{2}} \\ &\leq \widehat{M}_9||f||_\gamma \delta_1^{-2m} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left\{\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right\}^{\frac{1}{2}} \left\{\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty b_{n,k}(t)(t-x)^{4m}dt\right\}^{\frac{1}{2}}.\end{aligned}$$

Hence by using Lemma 2.1 and Lemma 2.2, we have

$$I \leq \widehat{M}_{10}||f||_\gamma \delta_1^{-2m} O\left(n^{(i+\frac{j}{2}-m)}\right) \leq \widehat{M}_{11} n^{-q}||f||_\gamma,$$

where $q = m - \frac{r}{2}$. Now choosing $m > 0$ satisfying $q \geq 1$, we obtain $I \leq \widehat{M}_{11}n^{-1}\|f\|_\gamma$. Therefore by property (iii) of the function $f_{2,\delta}(t)$, we get

$$\begin{aligned}H_1 &\leq \widehat{M}_8 \left\|f^{(r)} - f_{2,\delta}^{(r)}\right\|_{C[a^*,b^*]} + \widehat{M}_{11}n^{-1}||f||_\gamma \\ &\leq \widehat{M}_{12}\omega_2(f^{(r)},\delta,a,b) + \widehat{M}_{11}n^{-1}||f|||_\gamma.\end{aligned}$$

Choosing $\delta = n^{-\frac{1}{2}}$, the theorem follows. □

4. INVERSE THEOREM

This section is devoted to the following inverse theorem in simultaneous approximation:

Theorem 4.1. *Let* $0 < \alpha < 2$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$ *and suppose* $f \in C_\gamma[0,\infty)$. *Then in the following statements* $(i) \Rightarrow (ii)$

(i) $||B_n^{(r)}(f,\bullet)||_{C[a_1,b_1]} = O(n^{-\alpha/2})$,
(ii) $f^{(r)} \in Lip^*(\alpha,a_2,b_2)$,

where $Lip^*(\alpha,a_2,b_2)$ *denotes the Zygmund class satisfying* $\omega_2(f,\delta,a_2,b_2) \leq M\delta^\alpha$.

Proof. Let us choose a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Also suppose $g \in C_0^\infty$ with $\text{supp}g \in [a'',b'']$ and $g(x) = 1$ on the interval $[a_2,b_2]$. For $x \in [a',b']$ with $D \equiv \frac{d}{dx}$, we have

$$\begin{aligned}B_n^{(r)}(fg,x) &- (fg)^{(r)}(x) \\ &= D^r(B_n((fg)(t)-(fg)(x)),x) \\ &= D^r(B_n(f(t)(g(t)-g(x)),x)) + D^r(B_n(g(x)(f(t)-f(x)),x)) \\ &=: J_1 + J_2.\end{aligned}$$

Using the Leibniz formula, we have

$$\begin{aligned}
J_1 &= \frac{\partial^r}{\partial x^r}\int_0^\infty W_n(x,t)f(t)[g(t)-g(x)]dt \\
&= \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_n^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}[f(t)(g(t)-g(x))]dt \\
&= -\sum_{i=0}^{r-1}\binom{r}{i} g^{(r-i)}(x)B_n^{(i)}(f,x) + \int_0^\infty W_n^{(r)}(x,t)f(t)(g(t)-g(x))dt \\
&=: J_3 + J_4.
\end{aligned}$$

Applying Theorem 3.4, we have

$$J_3 = -\sum_{i=0}^{r-1}\binom{r}{i} g^{(r-i)}(x)f^{(i)}(x) + O\left(n^{-\frac{\alpha}{2}}\right),$$

uniformly in $x \in [a', b']$. Applying Theorem 3.2, the Cauchy-Schwarz inequality, Taylor's expansions of f and g and Lemma 2.2, we are led to

$$\begin{aligned}
J_4 &= \sum_{i=0}^{r}\frac{g^{(i)}(x)f^{(r-i)}(x)}{i!(r-i)!}r! + o\left(n^{-\frac{1}{2}}\right) \\
&= \sum_{i=0}^{r}\binom{r}{i} g^{(i)}(x)f^{(r-i)}(x) + o\left(n^{-\frac{\alpha}{2}}\right),
\end{aligned}$$

uniformly in $x \in [a', b']$. Again using the Leibniz formula, we have

$$\begin{aligned}
J_2 &= \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_n^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}[g(t)(f(t)-f(x))]dt \\
&= \sum_{i=0}^{r}\binom{r}{i} g^{(r-i)}(x)B_n^{(i)}(f,x) - (fg)^{(r)}(x) \\
&= \sum_{i=0}^{r}\binom{r}{i} g^{(r-i)}(x)f^{(i)}(x) - (fg)^{(r)}(x) + o(n^{-\alpha/2}) \\
&= O\left(n^{-\frac{\alpha}{2}}\right),
\end{aligned}$$

uniformly in $x \in [a', b']$. Combining the above estimates, we get

$$\left\|B_n^{(r)}(fg,\bullet) - (fg)^{(r)}\right\|_{C[a',b']} = O\left(n^{-\frac{\alpha}{2}}\right).$$

Thus by Lemma 2.5 and Lemma 2.6, we have $(fg)^{(r)} \in Lip^*(\alpha, a', b')$ also $g(x) = 1$ on the interval $[a_2, b_2]$, it proves that $f^{(r)} \in Lip^*(\alpha, a_2, b_2)$. This completes the validity of the implication $(i) \Rightarrow (ii)$ for the case $0 < \alpha \leq 1$. To prove the result for $1 < \alpha < 2$ for any interval $[a^*, b^*] \subset (a_1, b_1)$, let a_2^*, b_2^* be such that $(a_2, b_2) \subset (a_2^*, b_2^*)$ and $(a_2^*, b_2^*) \subset (a_1^*, b_1^*)$. Letting $\delta > 0$ we shall prove the assertion $\alpha < 2$. From the previous case it implies that $f^{(r)}$ exists and belongs to $Lip(1-\delta, a_1^*, b_1^*)$. Let $g \in C_0^\infty$ be such that $g(x) = 1$ on $[a_2, b_2]$ and $\operatorname{supp} g \subset (a_2^*, b_2^*)$. Then with $\chi_2(t)$ denoting the characteristic function of the interval $[a_1^*, b_1^*]$, we have

$$\begin{aligned}&\left\|B_n^{(r)}(fg,\bullet)-(fg)^{(r)}\right\|_{C[a_2^*,b_2^*]}\\&\leq ||D^r[B_n(g(\cdot)(f(t)-f(\cdot)),\bullet)]||_{C[a_2^*,b_2^*]}+||D^r[B_n(f(t)(g(t)-g(\cdot)),\bullet)]||_{C[a_2^*,b_2^*]}\\&=: P_1+P_2.\end{aligned}$$

To estimate P_1, by Theorem 3.4, we have

$$\begin{aligned}P_1&=\left\|D^r[B_n(g(\cdot)(f(t),\bullet)]-(fg)^{(r)}\right\|_{C[a_2^*,b_2^*]}\\&=\left\|\sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(\cdot)B_n^{(i)}(f,\bullet)-(fg)^{(r)}\right\|_{C[a_2^*,b_2^*]}\\&=\left\|\sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(\cdot)f^{(i)}-(fg)^{(r)}\right\|_{C[a_2^*,b_2^*]}+O(n^{-\alpha/2})\\&=O\left(n^{-\frac{\alpha}{2}}\right).\end{aligned}$$

Also by the Leibniz formula and Theorem 3.2, have

$$\begin{aligned}P_2&\leq\left\|\sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(\cdot)B_n(f,\bullet)+B_n^{(r)}(f(t)(g(t)-g(\cdot))\chi_2(t),\bullet)\right\|_{C[a_2^*,b_2^*]}+O(n^{-1})\\&=: ||P_3+P_4||_{C[a_2^*,b_2^*]}+O(n^{-1}).\end{aligned}$$

Then by Theorem 3.4, we have

$$P_3=\sum_{i=0}^{r-1}\binom{r}{i}g^{(r-i)}(x)f^{(i)}(x)+O\left(n^{-\frac{\alpha}{2}}\right),$$

uniformly in $x\in[a_2^*,b_2^*]$. Applying Taylor's expansion of f, we have

$$\begin{aligned}P_4&=\int_{i=0}^{\infty}W_n^{(r)}(x,t)[f(t)(g(t)-g(x))\chi_2(t)dt\\&=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^{\infty}W_n^{(r)}(x,t)(t-x)^i(g(t)-g(x))dt\\&\quad+\int_0^{\infty}W_n^{(r)}(x,t)\frac{(f^{(r)}(\xi)-f^{(r)}(x))}{r!}(t-x)^r(g(t)-g(x))\chi_2(t)dt,\end{aligned}$$

where ξ lying between t and x. Next by Theorem 3.4, the first term in the above expression is given by

$$\sum_{m=0}^{r}\binom{r}{m}g^{(m)}f^{(r-m)}(x)+O\left(n^{-\frac{\alpha}{2}}\right),$$

uniformly in $x \in [a_2^*, b_2^*]$. Also by mean value theorem and using Lemma 2.5, we can obtain the second term as follows:

$$\left\| \int_0^\infty W_n^{(r)}(x,t) \frac{(f^{(r)}(\xi) - f^{(r)}(x))}{r!} (t-x)^r (g(t) - g(x)) \chi_2(t) dt \right\|_{C[a_2^*, b_2^*]}$$

$$\leq \sum_{\substack{2m+s \leq r \\ m,s \geq 0}} n^{m+s} \left\| \frac{|Q_{m,s,r}(x)|^r}{x(1+x)} \int_0^\infty W_n(x,t) |t-x|^{\delta+r+1} \frac{|f^{(r)}(\xi) - f^{(r)}(x)|}{r!} |g'(\eta)| \chi_2(t) dt \right\|_{C[a_2^*, b_2^*]}$$

$$= O\left(n^{-\frac{\delta}{2}}\right),$$

choosing δ such that $0 \leq \delta \leq 2 - \alpha$. Combining the above estimates we get

$$\left\| B_n^{(r)}(fg, \bullet) - (fg)^{(r)} \right\|_{C[a_2^*, b_2^*]} = O\left(n^{-\frac{\alpha}{2}}\right).$$

Since $\mathrm{supp} fg \subset (a_2^*, b_2^*)$, it follows from Lemma 2.5 and Lemma 2.6 that $(fg)^{(r)} \in Liz(\alpha, 1, a_2^*, b_2^*)$. Since $g(x) = 1$ on $[a_2, b_2]$, we have $f^{(r)} \in Liz(\alpha, 1, a_2^*, b_2^*)$. This completes the proof of the theorem. □

Remark 4.2. As noted in the first section, these operators also reproduce the linear functions so we can easily apply the iterative combinations to the operators (1.1) to improve the order of approximation.

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