A GENERAL NOTE ON INCREASING SEQUENCES

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Received: 23 December, 2006

Accepted: 08 August, 2007

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 40D15, 40F05, 40G99.

Key words: Absolute summability, Summability factors, Almost and power increasing sequences, Infinite series.

Abstract: In the present paper, a general theorem on $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series has been proved under more weaker conditions. Also we have obtained a new result concerning the $\mid C, 1 \mid_k$ summability factors.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We denote by $\mathcal{BV}_{\mathcal{O}}$ the expression $\mathcal{BV} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}}$ and $\mathcal{BV}$ are the set of all null sequences and the set of all sequences with bounded variation, respectively. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [6, 8])

$$\sum_{n=1}^{\infty} n^{k-1} \left|u_n^\alpha - u_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty} \frac{|t_n^\alpha|^k}{n} < \infty. \tag{1.4}$$

If we take $\alpha = 1$, then we get $|C,1|_k$ summability.

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.5}$$

vol. 8, iss. 3, art. 82, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.6}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2, 3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \left|\Delta \sigma_{n-1}\right|^k < \infty, \tag{1.7}$$

where

$$\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.8}$$

In the special case $p_n = 1$ for all values of n, $\left|\bar{N}, p_n\right|_k$ summability is the same as $\left|C, 1\right|_k$ summability.

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issn: 1443-5756

2. Known Results

Mishra and Srivastava [10] have proved the following theorem concerning the $\left|\bar{N}, p_n\right|$ summability factors.

Theorem A. *Let* (X_n) *be a positive non-decreasing sequence and let there be sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \quad \textit{as} \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty, \tag{2.3}$$

$$|\lambda_n|\,X_n = O(1). \tag{2.4}$$

If

$$\sum_{v=1}^{n} \frac{|s_v|}{v} = O(X_n) \quad \textit{as} \quad n \to \infty \tag{2.5}$$

and (p_n) *is a sequence such that*

$$P_n = O(np_n), \tag{2.6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{2.7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $\left|\bar{N}, p_n\right|$.

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

Later on Bor [4] generalized Theorem A for $\left|\bar{N}, p_n\right|_k$ summability in the following form.

Theorem B. *Let* (X_n) *be a positive non-decreasing sequence and the sequences* (β_n) *and* (λ_n) *are such that conditions (2.1) – (2.7) of Theorem A are satisfied with the condition (2.5) replaced by:*

$$\sum_{v=1}^{n} \frac{|s_v|^k}{v} = O(X_n) \quad \text{as} \quad n \to \infty. \tag{2.8}$$

Then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $\left|\bar{N}, p_n\right|_k, k \geq 1.$

It may be noticed that if we take $k = 1$, then we get Theorem A.

Quite recently Bor [5] has proved Theorem B under weaker conditions by taking an almost increasing sequence instead of a positive non-decreasing sequence.

Theorem C. *Let* (X_n) *be an almost increasing sequence. If the conditions (2.1) – (2.4) and (2.6) – (2.8) are satisfied, then the series* $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Remark 1. It should be noted that, under the conditions of Theorem B, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [4]).

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issn: 1443-5756

3. Main Result

The aim of this paper is to prove Theorem C under weaker conditions. For this we need the concept of quasi β-power increasing sequences. A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{3.1}$$

holds for all $n \geq m \geq 1$. It should be noted that almost every increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$.

Now we shall prove the following theorem.

Theorem 3.1. *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If the conditions (2.1) – (2.4), (2.6) – (2.8) and*

$$(\lambda_n) \in \mathcal{BV}_{\mathcal{O}} \tag{3.2}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

It should be noted that if we take (X_n) as an almost increasing sequence, then we get Theorem C. In this case, condition (3.2) is not needed.

We require the following lemma for the proof of Theorem 3.1.

Lemma 3.2 ([9]). *Except for the condition (3.2), under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of Theorem 3.1, the following conditions hold, when (2.3) is satisfied:*

$$nX_n\beta_n = O(1), \tag{3.3}$$

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issn: 1443-5756

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.4}$$

Proof of Theorem 3.1. Let (T_n) denote the $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v}, \tag{3.5}$$

and thus

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v}, \qquad n \geq 1. \tag{3.6}$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_v \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v p_v} \right) + \frac{\lambda_n s_n}{n} \\ &= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta \lambda_v}{(v+1) p_{v+1}} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say}. \end{aligned}$$

To prove Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{3.7}$$

Increasing Sequences
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journal of inequalities in pure and applied mathematics
issn: 1443-5756

Firstly by using Abel's transformation, we have

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= \sum_{n=1}^{m}\left(\frac{P_n}{np_n}\right)^{k-1}|\lambda_n|^{k-1}|\lambda_n|\frac{|s_n|^k}{n}\\
&= O(1)\sum_{n=1}^{m}|\lambda_n|\frac{|\,|s_n|\,|^k}{n}\\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{|s_v|^k}{v}+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{|s_n|^k}{n}\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)|\lambda_m|X_m=O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

Now, using the fact that $P_{v+1}=O((v+1)p_{v+1})$, by (2.6), and then applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n-1}P_vs_v\Delta\lambda_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}|s_v|p_v|\Delta\lambda_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k|s_v|^kp_v|\Delta\lambda_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}
\end{aligned}$$

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issn: 1443-5756

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k |s_v|^k p_v |\Delta\lambda_v|^k \sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v|\Delta\lambda_v|}{p_v}\right)^{k-1} |s_v|^k |\Delta\lambda_v|$$

$$= O(1)\sum_{v=1}^{m} |s_v|^k |\Delta\lambda_v| \left(\frac{P_v}{vp_v}\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m} v\beta_v \frac{|s_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{|s_r|^k}{r} + O(1)m\beta_m\sum_{v=1}^{m}\frac{|s_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m$$

$$= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}|\beta_v|X_v + O(1)m\beta_m X_m$$

$$= O(1)$$

as $m \to \infty$, in view of the hypotheses of Theorem 3.1 and Lemma 3.2.

Again, since $\Delta(\frac{P_v}{vp_v}) = O(\frac{1}{v})$, by (2.6) and (2.7) (see [10]), as in $T_{n,1}$ we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|s_v||\lambda_v|\frac{1}{v}\right\}^k$$

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issn: 1443-5756

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)p_v\left|s_v\right|\left|\lambda_v\right|\frac{1}{v}\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k p_v\left|s_v\right|^k\left|\lambda_v\right|^k\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k\left|s_v\right|^k p_v\left|\lambda_v\right|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k p_v\left|s_v\right|^k\left|\lambda_v\right|^k\frac{1}{P_v}\cdot\frac{v}{v}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1}\left|\lambda_v\right|^{k-1}\left|\lambda_v\right|\frac{\left|s_v\right|^k}{v}$$
$$= O(1)\sum_{v=1}^{m}\left|\lambda_v\right|\frac{\left|s_v\right|^k}{v}$$
$$= O(1)\sum_{v=1}^{m-1}X_v\beta_v + O(1)X_m\left|\lambda_m\right| = O(1)\quad as\quad m\to\infty.$$

Finally, using Hölder's inequality, as in $T_{n,3}$ we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,4}\mid^k = \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n-1}s_v\frac{P_v}{v}\lambda_v\right|^k$$
$$= \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n-1}s_v\frac{P_v}{vp_v}p_v\lambda_v\right|^k$$

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issn: 1443-5756

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k \left(\frac{P_v}{vp_v}\right)^k p_v |\lambda_v|^k \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right)^{k-1}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{vp_v}\right)^k |s_v|^k p_v |\lambda_v|^k \frac{1}{P_v} \cdot \frac{v}{v}$$
$$= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{|s_v|^k}{v}$$
$$= O(1) \sum_{v=1}^{m-1} X_v \beta_v + O(1) X_m |\lambda_m| = O(1) \quad \text{as} \quad m \to \infty.$$

Therefore we get

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 3.1.

Finally if we take $p_n = 1$ for all values of n in the theorem, then we obtain a new result concerning the $|C, 1|_k$ summability factors. □

vol. 8, iss. 3, art. 82, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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issn: 1443-5756