Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 1, Issue 1, Article 3, 2000

A STEFFENSEN TYPE INEQUALITY

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Received 26 October, 1999; accepted 7 December, 1999
Communicated by D.B. Hinton

ABSTRACT. Steffensen's inequality deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$. In this paper we prove an inequality which is similar to Steffensen's inequality. The most general form of this inequality deals with integrals over a measure space. We also consider the discrete case.

Key words and phrases: Steffensen inequality, upper-separating subsets.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

The most basic inequality which deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$ is the following inequality, which was established by J.F. Steffensen in 1919, [3].

Theorem 1.1. (STEFFENSEN'S INEQUALITY) *Let a and b be real numbers such that $a < b$, f and g be integrable functions from $[a, b]$ into $\mathbb{R}$ such that f is nonincreasing and for every $x \in [a, b]$, $0 \leq g(x) \leq 1$. Then*

$$\int_{b-\lambda}^{b} f(x)dx \leq \int_{a}^{b} f(x)g(x)dx \leq \int_{a}^{a+\lambda} f(x)dx,$$

where $\lambda = \int_a^b g(x)dx$.

The following is a discrete analogue of Steffensen's inequality, [1]:

Theorem 1.2. (DISCRETE STEFFENSEN'S INEQUALITY). *Let $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of nonnegative real numbers, and let $(y_i)_{i=1}^n$ be a finite sequence of real numbers such*

ISSN (electronic): 1443-5756

009-99

that for every i, $0 \le y_i \le 1$. *Let* k_1 $k_2 \in \{1, \dots, n\}$ *be such that* $k_2 \le \sum_{i=1}^{n} y_i \le k_1$. *Then*

$$\sum_{i=n-k_2+1}^{n} x_i \le \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k_1} x_i.$$

In section 2 we consider the discrete case. Our first result is the following.

Theorem 1.3. *Let* $\ell \ge 0$ *be a real number,* $(x_i)_{i=1}^n$ *be a nonincreasing finite sequence of real numbers in* $[\ell, \infty)$*, and* $(y_i)_{i=1}^n$ *be a finite sequence of nonnegative real numbers. Let* $\Phi : [\ell, \infty) \to [0, \infty)$ *be strictly increasing, convex, and such that* $\Phi(xy) \ge \Phi(x)\Phi(y)$ *for all* $x, y, xy \ge \ell$. *Let* $k \in \{1, \dots, n\}$ *be such that* $k \ge \ell$ *and* $\Phi(k) \ge \sum_{i=1}^n y_i$. *Then either*

$$\sum_{i=1}^{n} \Phi(x_i) y_i \le \Phi\left(\sum_{i=1}^{k} x_i\right) \quad \text{or} \quad \sum_{i=1}^{k} y_i \ge 1.$$

Theorem 1.3 takes an especially simple form if $\Phi(x) = x^\alpha$, where $\alpha \ge 1$.

Theorem 1.4. *Let* $(x_i)_{i=1}^n$ *be a nonincreasing finite sequence of nonnegative real numbers, and let* $(y_i)_{i=1}^n$ *be a finite sequence of nonnegative real numbers. Assume that* $\alpha \ge 1$. *Let* $k \in \{1, \dots, n\}$ *be such that*

$$k \ge \left(\sum_{i=1}^{n} y_i\right)^{\frac{1}{\alpha}}.$$

Then either

$$\sum_{i=1}^{n} x_i^\alpha y_i \le \left(\sum_{i=1}^{k} x_i\right)^\alpha \quad \textit{or} \quad \sum_{i=1}^{k} y_i \ge 1.$$

As an example of an application of Theorem 1.4 we obtain the following result:

Theorem 1.5. *Let* α *and* β *be real numbers such that* $\alpha \ge 1 + \beta$, $0 \le \beta \le 1$. *Let* $(x_i)_{i=1}^n$ *be a nonincreasing sequence of nonnegative real numbers. Assume that*

$$\sum_{i=1}^{n} x_i \le A, \qquad \sum_{i=1}^{n} x_i^\alpha \ge B^\alpha,$$

where A *and* B *are positive real numbers. Let* $k \in \{1, 2 \dots, n\}$ *be such that*

$$k \ge \left(\frac{A}{B}\right)^{\frac{\beta}{\alpha - 1}}.$$

Then

$$\sum_{i=1}^{k} x_i^\beta \ge B^\beta.$$

For $\beta = 1$ *this is a result from* [1].

The main result of section 3 is Theorem 3.2. This theorem is similar to Theorem 1.3, but it involves integrals over a measure space instead of finite sums. The key tool that we use to state and to prove Theorem 3.2 is the concept of separating subsets introduced and studied in [1]. If we take a measure space to be just a closed interval of the real line $\mathbb{R}$, we obtain the following simplest case of Theorem 3.2:

Theorem 1.6. *Let $\ell \geq 0$ be a real number, a and b be real numbers such that $a < b$, f and g be integrable functions from $[a, b]$ into $[\ell, \infty)$ and $[0, \infty)$ respectively, such that f is nonincreasing. Let $\Phi : [\ell, \infty) \to [0, \infty)$ be strictly increasing, convex, and such that $\Phi(xy) \geq \Phi(x)\Phi(y)$ for all x, y, $xy \geq \ell$. Let λ be a real number such that $\Phi(\lambda) = \int_a^b g(x)dx$. Assume that $\lambda \leq b - a$ and*

$$f(a) - f(a - \lambda) \leq \int_a^{a+\lambda} [f(x) - f(a + \lambda)]\, dx.$$

Then either

$$\int_a^b (\Phi \circ f)g\, dx \leq \Phi\left(\int_a^{a+\lambda} f\, dx\right) \quad \textit{or} \quad \int_a^{a+\lambda} g\, dx \geq 1.$$

Remark 1.1. In Theorems 1.3, 1.4, 1.6 and 3.2 the assumption that Φ is convex can be weakened: it is enough to assume that Φ is Wright-convex, where Wright-convexity means [4] that $\Phi(t_2) - \Phi(t_1) \leq \Phi(t_2 + \delta) - \Phi(t_1 + \delta)$ for all $t_1, t_2, \delta \in [0, \infty)$ such that $t_1 \leq t_2$. It is known that each convex function is Wright-convex, but the converse is not true.

2. THE DISCRETE CASE

Proof. of Theorem 1.3

$$\begin{aligned}
\sum_{i=1}^n \Phi(x_i)y_i &= \sum_{i=1}^k \Phi(x_i)y_i + \sum_{i=k+1}^n \Phi(x_i)y_i \\
&\leq \sum_{i=1}^k \Phi(x_i)y_i + \Phi(x_k) \sum_{i=k+1}^n y_i \\
&= \sum_{i=1}^k \Phi(x_i)y_i + \Phi(x_k) \left(\sum_{i=1}^n y_i - \sum_{i=1}^k y_i\right) \\
&= \sum_{i=1}^k y_i \left[\Phi(x_i) - \Phi(x_k)\right] + \Phi(x_k) \sum_{i=1}^n y_i.
\end{aligned}$$

Since $\Phi(k) \geq \sum_{i=1}^n y_i$ and $\Phi(kx_k) \geq \Phi(k)\Phi(x_k)$, we obtain

$$\sum_{i=1}^n \Phi(x_i)y_i \leq \sum_{i=1}^k y_i \left[\Phi(x_i) - \Phi(x_k)\right] + \Phi(kx_k).$$

Since Φ is Wright-convex,

$$\begin{aligned}
\Phi(x_i) - \Phi(x_k) &\leq \Phi(x_i + (k-1)x_k) - \Phi(x_k + (k-1)x_k) \\
&= \Phi(x_i + (k-1)x_k) - \Phi(kx_k) \\
&\leq \Phi\left(\sum_{i=1}^k x_i\right) - \Phi(kx_k).
\end{aligned}$$

Therefore

$$\sum_{i=1}^n \Phi(x_i)y_i \leq \left[\Phi\left(\sum_{i=1}^k x_i\right) - \Phi(kx_k)\right] \sum_{i=1}^k y_i + \Phi(kx_k).$$

It follows that

$$\sum_{i=1}^{n}\Phi(x_i)y_i-\Phi\left(\sum_{i=1}^{k}x_i\right)\le\left[\Phi\left(\sum_{i=1}^{k}x_i\right)-\Phi(kx_k)\right]\left(\sum_{i=1}^{k}y_i-1\right), \tag{2.1}$$

since

$$\sum_{i=1}^{k}x_i\ge kx_k,\qquad \Phi\left(\sum_{i=1}^{k}x_i\right)-\Phi(kx_k)\ge 0.$$

Assume first that

$$\Phi\left(\sum_{i=1}^{k}x_i\right)-\Phi(kx_k)=0.$$

Since Φ is strictly increasing we obtain that

$$\sum_{i=1}^{k}x_i=kx_k\quad\text{and therefore}\quad x_1=\cdots=x_k.$$

Then

$$\begin{aligned}\Phi\left(\sum_{i=1}^{k}x_i\right)-\sum_{i=1}^{n}\Phi(x_i)y_i&\ge\Phi(kx_k)-\Phi(x_k)\sum_{i=1}^{n}y_i\\&\ge\Phi(k)\Phi(x_k)-\Phi(x_k)\sum_{i=1}^{n}y_i\\&=\Phi(x_k)\left(\Phi(k)-\sum_{i=1}^{n}y_i\right)\ge 0.\end{aligned}$$

Thus, in the case $\Phi\left(\sum_{i=1}^{k}x_i\right)-\Phi(kx_k)=0$ we obtain that

$$\sum_{i=1}^{n}\Phi(x_i)y_i\le\Phi\left(\sum_{i=1}^{k}x_i\right),$$

and we are done.

Assume now that $\Phi\left(\sum_{i=1}^{k}x_i\right)-\Phi(kx_k)>0$. Then equation (2.1) implies that either

$$\sum_{i=1}^{n}\Phi(x_i)y_i\le\Phi\left(\sum_{i=1}^{k}x_i\right)\quad\text{or}\quad\sum_{i=1}^{k}y_i\ge 1.$$

□

Proof. of Theorem 1.5 Take x_i^{β} instead of x_i and $\frac{\alpha-1}{\beta}$ instead of α in Theorem 1.4. Then we get that

$$k\ge\left(\sum_{i=1}^{n}y_i\right)^{\frac{\beta}{\alpha-1}}$$

implies that either

$$\sum_{i=1}^{n}x_i^{\alpha-1}y_i\le\left(\sum_{k=1}^{k}x_i^{\beta}\right)^{\frac{\alpha-1}{\beta}}\quad\text{or}\quad\sum_{i=1}^{k}y_i\ge 1.$$

Take $y_i = \frac{x_i}{B}$ for $i = 1, \dots, n$, then

$$\sum_{i=1}^{n} y_i = \frac{1}{B} \sum_{i=1}^{n} x_i \le \frac{A}{B}.$$

Since $k \ge \left(\frac{A}{B}\right)^{\frac{\beta}{\alpha-1}}$, we obtain that

$$k \ge \left(\sum_{i=1}^{n} y_i\right)^{\frac{\beta}{\alpha-1}}.$$

This implies that either

$$\begin{aligned} \sum_{i=1}^{k} x_i^{\beta} &\ge \left(\sum_{i=1}^{n} x_i^{\alpha-1} y_i\right)^{\frac{\beta}{\alpha-1}} \\ &= \left(\frac{1}{B} \sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{\beta}{\alpha-1}} \\ &\ge \left(\frac{B^{\alpha}}{B}\right)^{\frac{\beta}{\alpha-1}} = B^{\beta}, \end{aligned}$$

or

$$\sum_{i=1}^{k} x_i = B \sum_{i=1}^{k} y_i \ge B.$$

However, if

$$\sum_{i=1}^{k} x_i \ge B,$$

then, since $0 \le \beta \le 1$,

$$\sum_{i=1}^{k} x_i^{\beta} \ge \left(\sum_{i=1}^{k} x_i\right)^{\beta} \ge B^{\beta}.$$

Therefore in both cases we have that

$$\sum_{i=1}^{k} x_i^{\beta} \ge B^{\beta}.$$

□

Example 2.1. Let $(x_i)_{i=1}^{n}$ be a nonincreasing sequence in $[0, \infty)$ such that $\sum_{i=1}^{k} x_i \le 400$ and $\sum_{i=1}^{k} x_i^2 \ge 10,000$. Then $\sqrt{x_1} + \sqrt{x_2} \ge 10$. For a proof take $\alpha = 2$, $\beta = \frac{1}{2}$, $A = 400$, and $B = 100$ in Theorem 1.5. The result is the best possible since if $n \ge 16$ and $x_1 = \dots = x_{16} = 25$, $x_{17} = \dots = x_n = 0$, we have that $\sum_{i=1}^{n} x_i = 400$, $\sum_{i=1}^{n} x_i^2 = 10,000$, and $\sqrt{x_1} + \sqrt{x_2} = 10$.

3. THE CASE OF INTEGRALS OVER A MEASURE SPACE.

Let $X = (X, \mathcal{A}, \mu)$ be a measure space. From now on we will assume that $0 < \mu(X) < \infty$.

Definition 3.1. [1]. Let $f \in L^{\circ}(X)$, where $L^{\circ}(X)$ means the set of all measurable functions on X. Let $(U, c) \in \mathcal{A} \times \mathbb{R}$. We say that the pair (U, c) is *upper-separating* for f iff

$$\{x \in X : f(x) > c\} \overset{a}{\subseteq} U \overset{a}{\subseteq} \{x \in X : f(x) \geq c\}$$

where $A \overset{a}{\subseteq} B$ means that A is almost contained in B, i.e. $\mu(A \setminus B) = 0$. We say that a subset U of X is *upper-separating* for f if there exists $c \in \mathbb{R}$ such that (U, c) is an upper-separating pair for f.

It is possible to prove, [1], that if μ is continuous (for a definition of a continuous measure see, for example, [2]), then, given $f \in L^{\circ}(X)$, for any real number λ such that $0 \leq \lambda \leq \mu(X)$, there exists an upper-separating subset U for f such that $\mu(U) = \lambda$.

Lemma 3.1. [1]. *Let $\Phi : [0, \infty) \to \mathbb{R}$ be convex and increasing. Let $c \in [0, \infty)$ and let $f \in L^1(X)$ have nonnegative values and satisfy the condition*

$$0 \leq f - c \leq \int_X (f - c) d\mu \quad a.e. \tag{3.1}$$

Then

$$\Phi \circ f - \Phi(c) \leq \Phi \left(\int_X f d\mu \right) - \Phi \left(c\mu(X) \right) \quad a.e.$$

Proof. The conclusion is trivial if $f = c$ a.e. Suppose that $\mu \left(\{x \in X : f(x) > c\} \right) > 0$. Then the left inequality (3.1) implies that

$$\int_X (f - c) d\mu > 0.$$

On the other hand, by integrating the right inequality (3.1), we obtain

$$\int_X (f - c) d\mu \leq \left(\int_X (f - c) d\mu \right) \mu(X),$$

which implies $\mu(X) \geq 1$. Since Φ is Wright-convex, we obtain that

$$\begin{aligned} \Phi \circ f - \Phi(c) &\leq \Phi \left(f + c(\mu(X) - 1) \right) - \Phi \left(c + c(\mu(X) - 1) \right) \\ &= \Phi \left(f - c + c\mu(X) \right) - \Phi \left(c\mu(X) \right) \quad \text{a.e.} \end{aligned}$$

Because Φ is increasing it follows by (3.1) that

$$\begin{aligned} \Phi \circ f - \Phi(c) &\leq \Phi \left(\int_X (f - c) d\mu + \int_X c \, d\mu \right) - \Phi \left(c\mu(X) \right) \\ &= \Phi(\int_X f \, d\mu) - \Phi \left(c\mu(X) \right). \end{aligned}$$

□

Theorem 3.2. *Let $\ell \geq 0$ be a real number. Let $\Phi : [\ell, \infty) \to \mathbb{R}$ be convex strictly increasing, and such that $\Phi(xy) \geq \Phi(x)\Phi(y)$ for all x, y, $xy \geq \ell$. Let $f, g \in L'(X)$ be such that $f \geq \ell$ and $g \geq 0$ a.e.. Let λ be a real number and such that $\Phi(\lambda) = \int_X g \, d\mu$. Assume that $0 \leq \lambda \leq \mu(X)$,* *and let (U,c) be an upper-separating pair for f such that $\mu(U)=\lambda$. Assume that $f-c \le \int_U (f-c)\,d\mu$ a.e. on U. Then either*

$$\int_X (\Phi\circ f)g\,d\mu \le \Phi\left(\int_U f\,d\mu\right) \quad \textit{or} \quad \int_U g\,d\mu \ge 1.$$

Proof.

$$\begin{aligned}\int_X (\Phi\circ f)g\,d\mu &= \int_U (\Phi\circ f)g\,d\mu + \int_{X\setminus U}(\Phi\circ f)g\,d\mu\\ &\le \int_U (\Phi\circ f)g\,d\mu + \Phi(c)\int_{X\setminus U} g\,d\mu\\ &= \int_U (\Phi\circ f)g\,d\mu + \Phi(c)\left(\int_X g\,d\mu - \int_U g\,d\mu\right)\\ &= \int_U g\left(\Phi\circ f - \Phi(c)\right)d\mu + \Phi(c)\Phi(\lambda).\end{aligned}$$

By Lemma 3.1

$$\begin{aligned}\int_X (\Phi\circ f)g\,d\mu &\le \left[\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda)\right]\int_U g\,d\mu + \Phi(c)\Phi(\lambda)\\ &\le \left[\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda)\right]\int_U g\,d\mu + \Phi(c\lambda).\end{aligned}$$

It follows that

$$\int_X (\Phi\circ f)g\,d\mu - \Phi\left(\int_U f\,d\mu\right) \le \left[\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda)\right]\left(\int_U g d\mu - 1\right). \tag{3.2}$$

Since (U,c) is upper-separating for f, $f\ge c$ on U. Hence

$$\int_U f\,d\mu \ge c\lambda \quad \text{and therefore} \quad \Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda) \ge 0.$$

Assume first that

$$\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda) = 0, \quad \text{then} \quad \Phi\left(\int_U f\,d\mu\right) = \Phi\left(\int_U c\,d\mu\right).$$

Since Φ is strictly increasing,

$$\int_U f\,d\mu = \int_U c\,d\mu, \quad \text{hence} \quad \int_U (f-c)d\mu = 0.$$

Since $f \geq c$ on U, we obtain that $f = c$ a.e. on U. Then

$$\begin{aligned}\Phi\left(\int\limits_U f\,d\mu\right) - \int\limits_X (\Phi\circ f)g\,d\mu &= \Phi\left(\int\limits_U c\,d\mu\right) - \int\limits_X (\Phi\circ f)g\,d\mu \\ &= \Phi(c\lambda) - \int\limits_X (\Phi\circ f)g\,d\mu \\ &\geq \Phi(c)\Phi(\lambda) - \int\limits_X (\Phi\circ f)g\,d\mu.\end{aligned}$$

Since (U,c) is upper-separating for f, we obtain that $f = c$ a.e. on U and $f \leq c$ a.e. on $X\backslash U$. Hence $f \leq c$ a.e. on X. It follows that

$$\begin{aligned}\Phi\left(\int\limits_U f\,d\mu\right) - \int\limits_X (\Phi\circ f)g\,d\mu &\geq \Phi(c)\Phi(\lambda) - \int\limits_X \Phi(c)g\,d\mu \\ &= \Phi(c)\left[\Phi(\lambda) - \int\limits_X g\,d\mu\right] = 0.\end{aligned}$$

This proves Theorem 1.6 in the case $\Phi\left(\int\limits_U f\,d\mu\right) - \Phi(c\lambda) = 0$.

Assume now that $\Phi\left(\int\limits_U f\,d\mu\right) - \Phi(c\lambda) > 0$, then equation 3.2 implies that either

$$\int\limits_X (\Phi\circ f)g\,d\mu - \Phi\left(\int\limits_U f\,d\mu\right) \leq 0 \quad \text{or} \quad \int\limits_U g\,d\mu \geq 1.$$

□

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