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MONOTONICITY PROPERTIES OF THE RELATIVE ERROR OF A PADÉ APPROXIMATION FOR MILLS' RATIO

IOSIF PINELIS

DEPARTMENT OF MATHEMATICAL SCIENCES
MICHIGAN TECHNOLOGICAL UNIVERSITY
HOUGHTON, MI 49931, USA
ipinelis@mtu.edu

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ABSTRACT. Based on a "monotonicity" analogue of the l'Hospital Rule, monotonicity properties of the relative error are established for a Padé approximation of Mills' ratio.

Key words and phrases: Mills' ratio, Complementary error function, Failure rate, Padé approximation, Continuous fractions, Relative error, L'Hospital's Rule, Monotonicity.

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1. INTRODUCTION, STATEMENT OF RESULTS, AND DISCUSSION

Let

$$\varphi(t) := \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2} \quad \text{and} \quad \overline{\Phi}(t) := \int_t^\infty \varphi(u)\, du = \frac{1}{2}\,\mathrm{erfc}\left(\frac{t}{\sqrt{2}}\right)$$

denote, as usual, the density and tail function of the standard normal law; erfc denotes the complementary error function. The ratio

$$r(t) := \frac{\overline{\Phi}(t)}{\varphi(t)}$$

is known as Mills' ratio; see, e.g., [6, Sect. 2.26]; its reciprocal, $\frac{\varphi(t)}{\overline{\Phi}(t)}$, is the so-called failure rate for the standard normal law.

The most well-known results on Mills' ratio include the inequality

$$r(t) < \frac{1}{t} \quad \forall t > 0 \tag{1.1}$$

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and the asymptotic relation

$$r(t) \sim \frac{1}{t} \quad \text{as} \quad t \to \infty; \tag{1.2}$$

as usual, $a \sim b$ means $\frac{a}{b} \to 1$; cf., e.g., inequalities [6, Sect. 2.26, Eq. (1)] due to Gordon [3] and the special case with $p = 2$ of inequalities [6, Section 2.26, Eq. (7)] due to Gautschi [2].

There are many different kinds of inequalities, as well as asymptotic and numerical results, for Mills' ratio in the literature. See e.g. [7, Chapter 3] and references therein; in particular, a wealth of such information is given in Shenton [12].

In this paper, we shall consider monotonicity properties of the relative error

$$\delta_k(t) := \frac{r(t) - r_k(t)}{r(t)} \tag{1.3}$$

for a sequence $(r_k(t))$ of certain rational approximations of Mills' ratio $r(t)$; as far as we know, such properties have not yet been considered. Such monotonicity properties may be used in an obvious manner: if it is known that $|\delta_k(t)|$ is monotonically decreasing in $t > 0$ and $|\delta_k(t_0)| < \delta$ for some $t_0 > 0$ and $\delta > 0$, then $|\delta_k(t)| < \delta$ for all $t \in [t_0, \infty)$ – cf. Remark 1.6 below; also, if it is known that, say, $t^{2k}|\delta_k(t)|$ is monotonically increasing in $t \in (0, \infty)$ from 0 to $k!$, then $|\delta_k(t)| < \frac{k!}{t^{2k}}$ for all $t \in (0, \infty)$ – cf. part (ii) of Theorem 1.5.

Our main results here are based on the following "monotonicity" analogue of the l'Hospital Rule, stated and proved in [8].

Theorem 1.1. *Let f and g be differentiable functions on an interval (a, b) such that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$, g' is nonzero and does not change sign, and $\frac{f'}{g'}$ is increasing (decreasing) on (a, b). Then $\frac{f}{g}$ is increasing (respectively, decreasing) on (a, b). (Note that the conditions here imply that g is nonzero and does not change sign on (a, b).)*

Further developments of Theorem 1.1 and other applications were given: in [8], applications to certain information inequalities; in [9], extensions to non-monotonic ratios of functions, with applications to certain probability inequalities arising in bioequivalence studies and to problems of convexity; in [10], applications to probability inequalities for sums of bounded random variables.

To begin the discussion here, let us illustrate possible applications of Theorem 1.1 with the following simple refinement of (1.1) and (1.2):

Proposition 1.2. *The ratio $\frac{r(t)}{1/t}$ is increasing in $t \in (0, \infty)$ from 0 to 1, and so, the relative absolute error $\frac{1/t - r(t)}{r(t)}$ is decreasing in $t \in (0, \infty)$ from ∞ to 0.*

This proposition is immediate from Theorem 1.1 and the usual l'Hospital rule for limits, because for $t > 0$ one has $\frac{r(t)}{1/t} = \frac{\overline{\Phi}(t)}{\varphi(t)/t}$ and

$$\frac{\overline{\Phi}(t)'}{(\varphi(t)/t)'} = \frac{1}{1 + 1/t^2},$$

which is increasing in $t \in (0, \infty)$ from 0 to 1.

It is not hard to obtain the following generalization of (1.1) and (1.2).

Proposition 1.3. *For all* $k \in \{0,1,\dots\}$ *and all* $t>0$,

$$r(t) = \frac{1}{t} - \frac{1}{t^3} + \frac{1\cdot 3}{t^5} - \frac{1\cdot 3\cdot 5}{t^7} + \cdots + (-1)^k \frac{(2k-1)!!}{t^{2k+1}} + R_k(t), \tag{1.4}$$

where

$$R_k(t) := (-1)^{k+1} \frac{(2k+1)!!}{\varphi(t)} \int_t^\infty \frac{\varphi(u)}{u^{2k+2}}\, du, \quad \text{so that}$$

$$|R_k(t)| = (-1)^{k+1} R_k(t) < \frac{(2k+1)!!}{t^{2k+2}}\, r(t) < \frac{(2k+1)!!}{t^{2k+3}}. \tag{1.5}$$

As usual,

$$(2k-1)!! := 1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1) \ \text{ and } \quad (2k)!! := 2\cdot 4\cdot 6\cdot \cdots \cdot 2k$$

for all $k \in \{1,2,\dots\}$. Also, $(-1)!! := 1$ and $0!! := 1$.

Inequality $|R_k(t)| < \dfrac{(2k+1)!!}{t^{2k+3}}$ for all $t>0$, which follows from (1.5), is well known; see e.g. [12, page 180].

Identity (1.4) is immediate from the recurrence relation

$$\int_t^\infty \frac{\varphi(u)}{u^m}\, du = \frac{\varphi(t)}{t^{m+1}} - (m+1)\int_t^\infty \frac{\varphi(u)}{u^{m+2}}\, du \quad \forall t>0\ \forall m \in \{0,1,\dots\},$$

which in turn is obtained by integration by parts. Inequalities (1.5) follow from the Chebyshev type inequality $\displaystyle\int_t^\infty \frac{\varphi(u)}{u^{2k+2}}\, du < \frac{1}{t^{2k+2}} \int_t^\infty \varphi(u)\, du$ and inequality (1.1), respectively.

Proposition 1.2 can be generalized and strengthened as follows.

Proposition 1.4. *For every* $k \in \{0,1,\dots\}$

- **(i)** *the relative absolute error* $\Delta_k(t) := \dfrac{|R_k(t)|}{r(t)}$ *is decreasing in* $t \in (0,\infty)$ *from* ∞ *to* 0*;*
- **(ii)** *on the other hand, the ratio* $\dfrac{\Delta_k(t)}{t^{-(2k+2)}} = t^{2k+2}\dfrac{|R_k(t)|}{r(t)}$ *is increasing in* $t \in (0,\infty)$ *from* 0 *to* $(2k+1)!!$.

In particular, Proposition 1.2 is a special case (with $k=0$) of part (i) of Proposition 1.4. On the other hand, part (ii) of Proposition 1.4 may be considered as a refinement of the first inequality in (1.5).

The proof of Proposition 1.4 is given below in Section 3.

As the first inequality in (1.5) shows, relation (1.4) provides an asymptotic estimate of $r(t)$, which has a relative error $O(1/t^{2k+2})$, which is small as $t\to\infty$. However, because $\int_t^\infty \frac{\varphi(u)}{u^{2k+2}}\, du > \int_t^{t+1} \frac{\varphi(u)}{u^{2k+2}}\, du > \frac{\varphi(t+1)}{(t+1)^{2k+2}}\ \forall t>0$, the remainder $|R_k(t)|$ rapidly tends to ∞ as $k\to\infty$, for each fixed $t>0$. This is a serious disadvantage of the asymptotic expansion given by (1.4). The disadvantage disappears if one uses, instead of (1.4), the corresponding rational Padé approximation, as follows.

For all $k \in \{1,2,\dots\}$ and $t>0$, consider the ratio

$$r_k(t) := \cfrac{1}{t + \cfrac{1}{t + \cfrac{2}{t + \cfrac{3}{\ddots\, t + \cfrac{k-1}{t}}}}}, \tag{1.6}$$

which is the kth initial segment of the Laplace [5] continuous fraction for Mills' ratio $r(t)$. Let also $r_0(t) := 0$ for all $t > 0$.

Take any $p \in \{0, 1, \ldots\}$. Then, for $k = 2p$, the expression $tr_k(t)$ is the $[(p-1)/p]$ Padé approximant [1] (or, rather, a Padé-Laurent one) of the (divergent) asymptotic expansion (cf. (1.4))

$$tr(t) \simeq 1 - \frac{1}{t^2} + \frac{1 \cdot 3}{t^4} - \frac{1 \cdot 3 \cdot 5}{t^6} + \cdots = 1 - s + 1 \cdot 3 \cdot s^2 - 1 \cdot 3 \cdot 5 \cdot s^3 + \cdots$$

in the powers of $s := \frac{1}{t^2}$ as $t \to \infty$; for $k = 2p + 1$, the expression $tr_k(t)$ is the $[p/p]$ Padé approximant of the above asymptotic expansion in the powers of s. Indeed, it is easy to see that $tr_k(t)$ may be rewritten as the ratio of two polynomials in $s = \frac{1}{t^2}$ of degrees $p-1$ and p when $k = 2p$ and of degrees p and p when $k = 2p + 1$; moreover, it follows from (1.9) below, (1.3), and (1.2) that

$$|tr(t) - tr_k(t)| = O\left(\frac{1}{t^{2k}}\right) = O\left(s^k\right) \tag{1.7}$$

as $t \to \infty$ or, equivalently, $s \to 0$. (Alternatively, see Section 4.6 in [1], especially formula (6.9) therein.)

Moreover, this Padé approximation is wrapping in the sense that for all $m \in \{1, 2, \ldots\}$ and all $t > 0$

$$r_{2m-2}(t) < r_{2m}(t) < r(t) < r_{2m+1}(t) < r_{2m-1}(t); \tag{1.8}$$

this follows from general properties of continuous fractions; cf., e.g., [12, Eq. (19)].

This Padé approximation is convergent: for every $t > 0$,

$$r_k(t) \longrightarrow r(t) \quad \text{as} \quad k \to \infty;$$

this follows, e.g., from the more general results in [1]; see, in particular, the partial proof of (6.15) therein. On the rate of this convergence, see e.g. [12, page 181].

Other rational approximations and inequalities comparing such approximations with Mills' ratio are known as well; see e.g. [12, page 188, inequalities (i)–(vi)] and [4, Eq. (3.7)]; these results may also be found in [7, Subsections 3.7.6(b) and 3.6.3, respectively]. However, because of the uniqueness of Padé approximants, the important property (1.7) is not shared by any rational approximations of Mills' ratio other than the Padé ones.

In what follows, let the relative error $\delta_k(t)$ be still defined by (1.3), where $r_k(t)$ is specifically defined by (1.6).

The following theorem, which is the main result of this paper, may be compared with Proposition 1.4.

Theorem 1.5.

(i) *For all $k \in \{0, 1, \ldots\}$ and all $t > 0$,*

$$(-1)^k \delta_k(t) > 0.$$

(ii) *For all $k \in \{1, 2, \ldots\}$, the ratio $\frac{|\delta_k(t)|}{t^{-2k}} = t^{2k} |\delta_k(t)|$ is increasing in $t \in (0, \infty)$ from 0 to $k!$. In particular, for all $t > 0$,*

$$|\delta_k(t)| < \frac{k!}{t^{2k}}. \tag{1.9}$$

(iii) *For all $k \in \{1, 2, \ldots\}$, the absolute relative error $|\delta_k(t)|$ is decreasing in $t \in (0, \infty)$ from ∞ to 0 if k is odd, and from 1 to 0 if k is even.*

The proof of Theorem 1.5 is given below in Section 3.

Remark 1.6. Part (iii) of Theorem 1.5 is useful in the following way. Suppose that the absolute relative error $|\delta_k(t_0)|$ is no greater than some positive number δ, for some $t_0 > 0$; then $|\delta_k(t)| < \delta$ for *all* $t > t_0$. For example, it is easy to see (for instance, using (1.8)) that $|\delta_8(4)| < 0.5 \times 10^{-6}$; it follows then from part (iii) of Theorem 1.5 that the absolute relative error $|\delta_8(t)|$ of the approximation

$$r(t) \approx r_8(t) = \frac{t(t^6 + 27t^4 + 185t^2 + 279)}{t^8 + 28t^6 + 210t^4 + 420t^2 + 105}$$

is less than 0.5×10^{-6} for *all* $t > 4$.

2. AUXILIARY RESULTS

For all real t, define polynomial expressions $g_k(t)$ and $f_k(t)$ recursively by the following formulas: for all $k \in \{2, 3, \ldots\}$

$$g_k(t) = tg_{k-1}(t) + (k-1)g_{k-2}(t) \quad \text{and} \quad f_k(t) = tf_{k-1}(t) + (k-1)f_{k-2}(t);$$
$$g_0(t) = 1; \quad f_0(t) = 0; \quad g_1(t) = t; \quad f_1(t) = 1.$$

Then for all $t > 0$ and $k \in \{0, 1, \ldots\}$

$$r_k(t) = \frac{f_k(t)}{g_k(t)}; \tag{2.1}$$

see, e.g., [12, Eq. (17)]. Also, for all $k \in \{1, 2, \ldots\}$,

$$f_k g_{k-1} - g_k f_{k-1} = (-1)^{k-1}(k-1)!\,; \tag{2.2}$$

see, e.g., [12, Eq. (18)]. Further, let

$$\gamma_k(t) := \int_t^\infty (t-u)^k\, \varphi(u)\, du. \tag{2.3}$$

Then it is obvious that for all $k \in \{0, 1, \ldots\}$

$$(-1)^k \gamma_k > 0; \tag{2.4}$$

also, one can show that

$$\gamma_k = g_k \overline{\Phi} - f_k \varphi; \tag{2.5}$$

see, e.g., [12, Eq. (21)]. In addition, it is immediate from (2.3) that for all $k \in \{1, 2, \ldots\}$

$$\gamma_k' = k\gamma_{k-1}. \tag{2.6}$$

Here and in what follows, f' denotes the derivative of f.

Remark 2.1.

- One has

$$\begin{aligned} &g_2(t) = t^2 + 1, \quad &&g_3(t) = t^3 + 3t, \quad &&g_4(t) = t^4 + 6t^2 + 3, \\ &f_2(t) = t, &&f_3(t) = t^2 + 2, &&f_4(t) = t^3 + 5t. \end{aligned}$$

- By induction, for all $k \in \{0, 1, \ldots\}$,
  - all the coefficients of f_k and g_k are nonnegative integers;
  - $\deg(g_k) = k$ and $\deg(f_k) = k - 1$ (the degree of the zero polynomial is defined to be -1); the leading coefficients of all f_k and g_k equal 1, except for f_0;
  - polynomial g_k is even if k is even and odd if k is odd, while f_k is odd if k is even and even if k is odd;

- moreover, if f_k or g_k is even, then all the coefficients of its even powers, up through the leading one, are strictly positive; if f_k or g_k is odd, then all the coefficients of its odd powers, up through the leading one, are strictly positive.

Lemma 2.2. *For all $k \in \{1, 2, \ldots\}$ and all real t,*

$$g_k'(t) = kg_{k-1}(t) \quad \text{and} \quad f_k'(t) = tf_k(t) + kf_{k-1}(t) - g_k(t).$$

Proof. From the definition of $g_k(t)$, it is easy to see that the identity $g_k'(t) = kg_{k-1}(t)$ is true for $k \in \{1, 2\}$; also, for $k \geq 3$ one has by induction

$$\begin{aligned} g_k'(t) &= (tg_{k-1}(t) + (k-1)g_{k-2}(t))' \\ &= g_{k-1}(t) + tg_{k-1}'(t) + (k-1)g_{k-2}'(t) \\ &= g_{k-1}(t) + t(k-1)g_{k-2}(t) + (k-1)(k-2)g_{k-3}(t) \\ &= g_{k-1}(t) + (k-1)g_{k-1}(t) = kg_{k-1}(t). \end{aligned}$$

The proof of the identity $f_k'(t) = tf_k(t) + kf_{k-1}(t) - g_k(t)$ is similar. □

3. Proofs of Main Results

Proof of Proposition 1.4.

(i) One has for $t > 0$

$$\frac{(\varphi(t)|R_k(t)|)'}{\overline{\Phi}(t)'} = \frac{(2k+1)!!}{t^{2k+2}}, \tag{3.1}$$

which is decreasing in $t \in (0, \infty)$. Also, obviously $\varphi(t)|R_k(t)| \to 0$ and $\overline{\Phi}(t) \to 0$ as $t \to \infty$. In addition, $\Delta_k(t) \to \infty$ as $t \downarrow 0$. Hence, by Theorem 1.1, relation (3.1), and the usual l'Hospital Rule for limits, $\Delta_k(t) = \dfrac{\varphi(t)|R_k(t)|}{\overline{\Phi}(t)}$ is decreasing in $t \in (0, \infty)$ from ∞ to 0. Thus, part (i) of the proposition is proved.

(ii) One has for $t > 0$

$$\frac{1}{(2k+1)!!} \frac{(\varphi(t)|R_k(t)|)'}{\left(t^{-(2k+2)}\,\overline{\Phi}(t)\right)'} = \left[1 + (2k+2)\frac{\overline{\Phi}(t)}{t\varphi(t)}\right]^{-1}. \tag{3.2}$$

Using Theorem 1.1 and the usual l'Hospital Rule for limits, it is easy to see that $\dfrac{\overline{\Phi}(t)}{\varphi(t)}$ is decreasing in $t \in (0, \infty)$, from $\dfrac{\overline{\Phi}(0)}{\varphi(0)}$ to 0. Hence, $\dfrac{\overline{\Phi}(t)}{t\varphi(t)}$ is decreasing in $t \in (0, \infty)$, from ∞ to 0, and so, by (3.2), $\dfrac{(\varphi(t)|R_k(t)|)'}{\left(t^{-(2k+2)}\,\overline{\Phi}(t)\right)'}$ is increasing in $t \in (0, \infty)$ from 0 to $(2k+1)!!$. Therefore, again by Theorem 1.1 and the usual l'Hospital Rule, $\dfrac{\Delta_k(t)}{t^{-(2k+2)}} = \dfrac{\varphi(t)|R_k(t)|}{t^{-(2k+2)}\,\overline{\Phi}(t)}$ is increasing in $t \in (0, \infty)$ from 0 to $(2k+1)!!$.

□

Proof of Theorem 1.5.

(i) In view of (1.3) and (2.5),

$$\delta_k = \frac{\gamma_k}{g_k \overline{\Phi}}. \tag{3.3}$$

This, (2.4), and Remark 2.1 yield part (i) of the theorem.

(ii) It follows from Remark 2.1 that for all $t > 0$,

$$\frac{g_k(t)}{t^k} = 1 + \frac{c_{k-1}}{t} + \cdots + \frac{c_0}{t^k},$$

where $c_0, \ldots, c_{k-1}$ are nonnegative constants. Hence, $\frac{g_k(t)}{t^k}$ is non-increasing in $t \in (0, \infty)$ and $\frac{g_k(t)}{t^k} \geq 1$ for all $t > 0$. Therefore, the ratio $\frac{t^{2k} |\delta_k(t)|}{t^k g_k(t) |\delta_k(t)|} = \frac{t^k}{g_k(t)}$ is non-decreasing in $t \in (0, \infty)$. It is also obvious that $\frac{t^k}{g_k(t)} \to 1$ as $t \to \infty$. Thus, to complete the proof of part (ii) of the theorem, it remains to show that $t^k g_k(t) |\delta_k(t)|$ is increasing in $t \in (0, \infty)$ from 0 to $k!$. It follows from (3.3) that

$$t^k g_k(t) |\delta_k(t)| = \prod_{j=1}^{k} \frac{t |\gamma_j(t)|}{|\gamma_{j-1}(t)|}$$

for all $k \in \{1, 2, \ldots\}$ and all $t > 0$. Hence, to complete the proof of part (ii), it suffices to show that, for each $j \in \{1, 2, \ldots\}$, the ratio $\frac{t |\gamma_j(t)|}{|\gamma_{j-1}(t)|}$ is increasing in $t \in (0, \infty)$ from 0 to j or, equivalently, that the ratio $\frac{|\gamma_{j-1}(t)| / t}{|\gamma_j(t)|}$ is decreasing in $t \in (0, \infty)$ from ∞ to $\frac{1}{j}$. But the latter claim follows by induction in j using the identity

$$\frac{(|\gamma_{j-1}(t)| / t)'}{|\gamma_j(t)|'} = \frac{1}{jt^2} + \frac{j-1}{j} \frac{|\gamma_{j-2}(t)| / t}{|\gamma_{j-1}(t)|} \quad \forall j \in \{2, 3, \ldots\} \; \forall t > 0, \tag{3.4}$$

Theorem 1.1, and the usual l'Hospital Rule for limits; in turn, identity (3.4) follows from (2.4) and (2.6). The basis of the induction is provided here by the identity

$$\frac{(|\gamma_0(t)| / t)'}{|\gamma_1(t)|'} = \frac{1}{t^2} + \frac{\varphi(t)/t}{\overline{\Phi}(t)} \quad \forall t > 0$$

and Proposition 1.2.

(iii) Using part (i) of the theorem and recalling (1.3) and (2.1), one has

$$|\delta_k| + (-1)^{k+1} = (-1)^k \delta_k + (-1)^{k+1} = \frac{(-1)^k \frac{f_k \varphi}{g_k}}{-\overline{\Phi}}. \tag{3.5}$$

Next, using Lemma 2.2 and identity (2.2), one obtains

$$\frac{\left((-1)^k \frac{f_k \varphi}{g_k}\right)'}{\left(-\overline{\Phi}\right)'} = \frac{k!}{g_k^2} + (-1)^{k+1}.$$

The latter expression is decreasing to $(-1)^{k+1}$ in $t \in (0, \infty)$, in view of Remark 2.1. Now identity (3.5), Theorem 1.1, and the l'Hospital Rule for limits imply that $|\delta_k(t)|$ is decreasing to 0 in $t \in (0, \infty)$. It remains to notice that, in view of (3.3), one has the following:

- $|\delta_k(0+)| = \infty$ if k is odd, because then $g_k(0) = 0$;
- $|\delta_k(0)| = \dfrac{\gamma_k(0)}{g_k(0)\overline{\Phi}(0)} = 1$ if k is even; the first equality here follows from (3.3) and (2.4); the second equality follows from (2.5) (because f_k is odd for even k, and so, $f_k(0) = 0$).

□

References

[1] G.A. BAKER, JR., *Padé Approximants*, Second Ed., Cambridge Univ. Press, 1996.

[2] W. GAUTSCHI, Some elementary inequalities relating to the gamma and incomplete gamma functions, *J. Math. and Phys.*, **38** (1959), 77–81.

[3] R.D. GORDON, Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument, *Ann. Math. Statist.*, **12** (1941), 364–366.

[4] H.L. GRAY AND W.R. Schucany, On the evaluation of distribution functions, *JASA*, **63** (1968), 715–720.

[5] P.S. LAPLACE, *Traité de Mécanique Celeste*, Vol. 4, Paris, 1805.

[6] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer, 1970.

[7] J.K. PATEL AND C.R. READ, *Handbook of the Normal Distribution*, Marcel Dekker, 1982.

[8] I. PINELIS, L'Hospital's rules for monotonicity, with applications, *J. Ineq. Pure & Appl. Math.*, **3**(1) (2002), Article 5. (`http://jipam.vu.edu.au/v3n1/010_01.html`).

[9] I. PINELIS, L'Hospital type rules for oscillation, with applications, *J. Ineq. Pure & Appl. Math.*, **2**(3) (2001), Article 33. (`http://jipam.vu.edu.au/v2n3/011_01.html`).

[10] I. PINELIS, L'Hospital type rules for monotonicity: an application to probability inequalities for sums of bounded random variables, *J. Ineq. Pure & Appl. Math.*, **3**(1) (2002), Article 7. (`http://jipam.vu.edu.au/v3n1/013_01.html`).

[11] W.R. SCHUCANY AND H.L. GRAY, A new approximation related to the error function, *Math. Comp.*, (1968), 201–202.

[12] L.R. SHENTON, Inequalities for the normal integral including a new continued fraction. *Biometrika*, **41** (1954), 177–189.