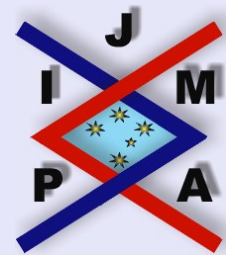


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Abstract

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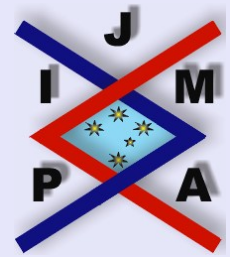


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## Abstract

The well-known *second moment Heisenberg-Weyl inequality (or uncertainty relation)* states: Assume that  $f : \mathbb{R} \rightarrow \mathbb{C}$  is a complex valued function of a random real variable  $x$  such that  $f \in L^2(\mathbb{R})$ , where  $\mathbb{R} = (-\infty, \infty)$ . Then the product of the second moment of the random real  $x$  for  $|f|^2$  and the second moment of the random real  $\xi$  for  $|\hat{f}|^2$  is at least  $E_{\mathbb{R},|f|^2} / 4\pi$ , where  $\hat{f}$  is the Fourier transform of  $f$ ,  $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx$  and  $f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi) d\xi$ , and  $E_{\mathbb{R},|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx$ . This uncertainty relation is well-known in classical quantum mechanics. In 2004, the author generalized the afore-mentioned result to *the higher order moments for  $L^2(\mathbb{R})$  functions  $f$* . In this paper, a refined form of the generalized Heisenberg-Weyl type inequality is established.

*2000 Mathematics Subject Classification:* 26, 33, 42, 60, 62.

*Key words:* Heisenberg-Weyl Type Inequality, Uncertainty Principle, Gram determinant.

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# 1. Introduction

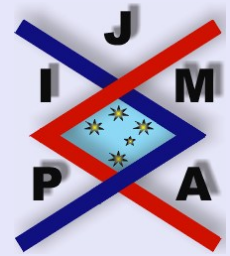
The serious question of certainty in science was high-lighted by Heisenberg, in 1927, via his “uncertainty principle” [2]. He demonstrated, for instance, the impossibility of specifying simultaneously the position and the speed (or the momentum) of an electron within an atom. In 1933, according to Wiener [7] “*a pair of transforms cannot both be very small.*” This uncertainty principle was stated in 1925 by Wiener, according to Wiener’s autobiography [8, p. 105–107], in a lecture in Göttingen. The following result of the *Heisenberg-Weyl Inequality* is credited to Pauli according to Weyl [6, p. 77, p. 393–394]. In 1928, according to Pauli [6] “*the less the uncertainty in  $|f|^2$ , the greater the uncertainty in  $|\widehat{f}|^2$ , and conversely.*” This result does not actually appear in Heisenberg’s seminal paper [2] (in 1927).

In 1998, Burke Hubbard [1] wrote a remarkable book on wavelets. According to her, most people first learn the Heisenberg uncertainty principle in connection with quantum mechanics, but it is also a central statement of information processing. The following second order moment Heisenberg-Weyl inequality provides a precise quantitative formulation of the above-mentioned uncertainty principle.

## 1.1. Second Moment Heisenberg-Weyl Inequality ([1], [4], [5])

For any  $f \in L^2(\mathbb{R})$ ,  $f : \mathbb{R} \rightarrow \mathbb{C}$ , such that

$$\|f\|_{2,\mathbb{R}}^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{\mathbb{R},|f|^2},$$



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any fixed but arbitrary constants  $x_m, \xi_m \in \mathbb{R}$ , and for the second order moments

$$(\mu_2)_{\mathbb{R},|f|^2} = \sigma_{\mathbb{R},|f|^2}^2 = \int_{\mathbb{R}} (x - x_m)^2 |f(x)|^2 dx$$

and

$$(\mu_2)_{\mathbb{R},|\hat{f}|^2} = \sigma_{\mathbb{R},|\hat{f}|^2}^2 = \int_{\mathbb{R}} (\xi - \xi_m)^2 |\hat{f}(\xi)|^2 d\xi,$$

the second order moment Heisenberg-Weyl inequality

$$(H_1) \quad \sigma_{\mathbb{R},|f|^2}^2 \cdot \sigma_{\mathbb{R},|\hat{f}|^2}^2 \geq \frac{\|f\|_{2,\mathbb{R}}^4}{16\pi^2},$$

holds. Equality holds in  $(H_1)$  if and only if the generalized Gaussians

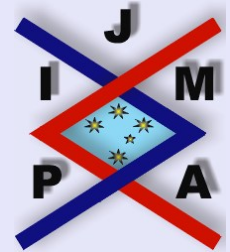
$$f(x) = c_o \exp(2\pi i x \xi_m) \exp(-c(x - x_m)^2)$$

hold for some constants  $c_o \in \mathbb{C}$  and  $c > 0$ .

The Heisenberg-Weyl inequality in *spectral analysis* says that the product of the effective duration  $\Delta x$  and the effective bandwidth  $\Delta \xi$  of a signal cannot be less than the value  $1/4\pi$ , where  $\Delta x^2 = \sigma_{\mathbb{R},|f|^2}^2 / E_{\mathbb{R},|f|^2}$  and  $\Delta \xi^2 = \sigma_{\mathbb{R},|\hat{f}|^2}^2 / E_{\mathbb{R},|\hat{f}|^2}$  with  $f : \mathbb{R} \rightarrow \mathbb{C}$ ,  $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$  defined as in  $(H_1)$ , and

$$(PPR) \quad E_{\mathbb{R},|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = E_{\mathbb{R},|\hat{f}|^2}$$

according to the Plancherel-Parseval-Rayleigh identity.



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## 1.2. Fourth Moment Heisenberg-Weyl Inequality ([4, pp. 26–27])

For any  $f \in L^2(\mathbb{R})$ ,  $f : \mathbb{R} \rightarrow \mathbb{C}$ , such that

$$\|f\|_{2,\mathbb{R}}^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{\mathbb{R},|f|^2},$$

any fixed but arbitrary constants  $x_m, \xi_m \in \mathbb{R}$ , and for the fourth order moments

$$(\mu_4)_{\mathbb{R},|f|^2} = \int_{\mathbb{R}} (x - x_m)^4 |f(x)|^2 dx$$

and

$$(\mu_4)_{\mathbb{R},|\widehat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^4 |\widehat{f}(\xi)|^2 d\xi,$$

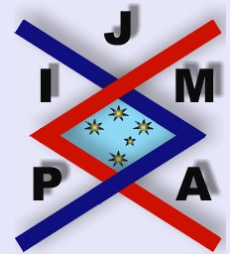
the fourth order moment Heisenberg-Weyl inequality

$$(H_2) \quad (\mu_4)_{\mathbb{R},|f|^2} \cdot (\mu_4)_{\mathbb{R},|\widehat{f}|^2} \geq \frac{1}{64\pi^4} E_{2,\mathbb{R},f}^2,$$

holds, where

$$E_{2,\mathbb{R},f} = 2 \int_{\mathbb{R}} \left[ (1 - 4\pi^2 \xi_m^2 x_\delta^2) |f(x)|^2 - x_\delta^2 |f'(x)|^2 - 4\pi \xi_m x_\delta^2 \operatorname{Im} \left( f(x) \overline{f'(x)} \right) \right] dx,$$

with  $x_\delta = x - x_m$ ,  $\xi_\delta = \xi - \xi_m$ ,  $\operatorname{Im}(\cdot)$  is the imaginary part of  $(\cdot)$ , and  $|E_{2,\mathbb{R},f}| < \infty$ .



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The “inequality” ( $H_2$ ) holds, unless  $f(x) = 0$ .

We note that if the ordinary differential equation of second order

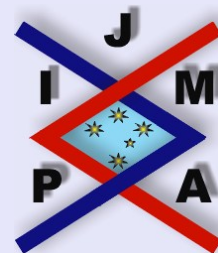
$$(ODE) \quad f''_\alpha(x) = -2c_2x_\delta^2 f_\alpha(x)$$

holds, with  $\alpha = -2\pi\xi_m i$ ,  $f_\alpha(x) = e^{\alpha x} f(x)$ , and a constant  $c_2 = \frac{1}{2}k_2^2 > 0$ ,  $k_2 \in \mathbb{R}$  and  $k_2 \neq 0$ , then “equality” in ( $H_2$ ) seems to occur. However, the solution of this differential equation (**ODE**), given by the function

$$f(x) = \sqrt{|x_\delta|} e^{2\pi i x \xi_m} \left[ c_{20} J_{-1/4} \left( \frac{1}{2} |k_2| x_\delta^2 \right) + c_{21} J_{1/4} \left( \frac{1}{2} |k_2| x_\delta^2 \right) \right],$$

in terms of the Bessel functions  $J_{\pm 1/4}$  of the first kind of orders  $\pm 1/4$ , leads to a contradiction, because this  $f \notin L^2(\mathbb{R})$ . Furthermore, a limiting argument is required for this problem. For the proof of this inequality see [4]. It is *open* to investigate cases, where the integrand on the right-hand side of the integral of  $E_{2,\mathbb{R},f}$  will be nonnegative. For instance, for  $x_m = \xi_m = 0$ , this integrand is:  $|f(x)|^2 - x^2 |f'(x)|^2 (\geq 0)$ .

In 2004, we [4] generalized the Heisenberg-Pauli-Weyl inequality in  $\mathbb{R} = (-\infty, \infty)$ . In this paper, a refined form of this generalized *Heisenberg-Weyl type inequality* is established in  $I = [0, \infty)$ . Afterwards, an open problem is proposed on some pertinent extremum principle. However, the above-mentioned Fourier transform is considered in  $\mathbb{R}$ , while our results in this paper are restricted to  $I = [0, \infty)$ . Furthermore, the corresponding inequality is investigated in  $\mathbb{R}$ , as well. Our second moment Heisenberg-Weyl type inequality and the fourth moment Heisenberg-Weyl type inequality are of the following forms ( $R_i$ ), ( $i = 1, 2$ ).



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### 1.3. Second Moment Heisenberg-Weyl Type Inequality ([4])

For any  $f \in L^2(I)$ ,  $I = [0, \infty)$ ,  $f : I \rightarrow \mathbb{C}$ , such that  $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$ , any fixed but arbitrary constant  $x_m \in \mathbb{R}$ , and for the second order moment

$$(\mu_2)_{I,|f|^2} = \sigma_{I,|f|^2}^2 = \int_I (x - x_m)^2 |f(x)|^2 dx,$$

the second order moment Heisenberg-Weyl type inequality

$$(R_1) \quad (\mu_2)_{I,|f|^2} \cdot \|f'\|_{2,I}^2 \geq \frac{1}{4} E_{1,I,f}^2 = \frac{1}{4} \left[ - \int_I |f(x)|^2 dx \right]^2,$$

holds, where  $|E_{1,I,f}| < \infty$ . Equality holds in  $(R_1)$  if and only if the Gaussians  $f(x) = c_o \exp(-c(x - x_m)^2)$  hold for some constants  $c_o \in \mathbb{C}$  and  $c > 0$ .

We note that this inequality  $(R_1)$  still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change.

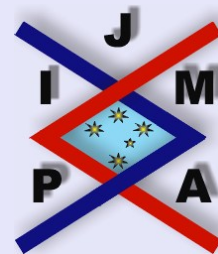
### 1.4. Fourth Moment Heisenberg-Weyl Type Inequality ([4])

For any  $f \in L^2(I)$ ,  $I = [0, \infty)$ ,  $f : I \rightarrow \mathbb{C}$ , such that  $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$ , any fixed but arbitrary constant  $x_m \in \mathbb{R}$ , and for the fourth order moment

$$(\mu_4)_{I,|f|^2} = \int_I (x - x_m)^4 |f(x)|^2 dx,$$

the fourth order moment Heisenberg – Weyl type inequality

$$(R_2) \quad (\mu_4)_{I,|f|^2} \cdot \|f''\|_{2,I}^2 \geq \frac{1}{4} E_{2,I,f}^2 = \left[ \int_I \left[ |f(x)|^2 dx - x_\delta^2 |f'(x)|^2 \right] dx \right]^2$$



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holds, where  $x_\delta = x - x_m$ , and  $|E_{2,I,f}| < \infty$ .

The “inequality”  $(R_2)$  holds, unless  $f(x) = 0$ .

We note that this inequality  $(R_2)$  still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change except that one on the following condition (2.1), where  $x \rightarrow \infty$  has to be substituted with  $|x| \rightarrow \infty$ .

We omit the proofs of the inequalities  $(R_i)$  ( $i = 1, 2$ ) as special cases of the corresponding proof of the following general Theorem 2.1 (with  $A = 0$ ) of this paper. Furthermore, we state our following four pertinent propositions. Their proofs are identical or analogous to the proofs of the corresponding propositions of [4].

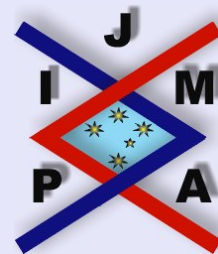
**Proposition 1.1 (Pascal type combinatorial identity, [4]).** If  $0 \leq \left[\frac{k}{2}\right]$  is the greatest integer  $\leq \frac{k}{2}$ , then

$$(C) \quad \frac{k}{k-i} \binom{k-i}{i} + \frac{k-1}{k-i} \binom{k-i}{i-1} = \frac{k+1}{k-i+1} \binom{k-i+1}{i},$$

holds for any fixed but arbitrary  $k \in \mathbb{N} = \{1, 2, \dots\}$ , and  $0 \leq i \leq \left[\frac{k}{2}\right]$  for  $i \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$  such that  $\binom{k}{-1} = 0$ .

**Proposition 1.2 (Generalized differential identity, [4]).** If  $f : I \rightarrow \mathbb{C}$  is a complex valued function of a real variable  $x$ ,  $I = [0, \infty)$ ,  $0 \leq \left[\frac{k}{2}\right]$  is the greatest integer  $\leq \frac{k}{2}$ ,  $f^{(j)} = \frac{d^j}{dx^j} f$ , and  $\overline{(\cdot)}$  is the conjugate of  $(\cdot)$ , then

$$(*) \quad f(x) \overline{f^{(k)}(x)} + f^{(k)}(x) \overline{f(x)} \\ = \sum_{i=0}^{\left[\frac{k}{2}\right]} (-1)^i \frac{k}{k-i} \binom{k-i}{i} \frac{d^{k-2i}}{dx^{k-2i}} |f^{(i)}(x)|^2,$$



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holds for any fixed but arbitrary  $k \in \mathbb{N} = \{1, 2, \dots\}$ , such that  $0 \leq i \leq \left[\frac{k}{2}\right]$  for  $i \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ .

We note that the proof of (\*) requires the application of the new identity (C). Furthermore, we note that the above differential identity (\*) still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change.

**Proposition 1.3 ( $P^{th}$ -derivative of product, [4]).** If  $f_i : I \rightarrow \mathbb{C}$  ( $i = 1, 2$ ) are two complex valued functions of a real variable  $x$ , then the  $p^{th}$ -derivative of the product  $f_1 f_2$  is given, in terms of the lower derivatives  $f_1^{(m)}$ ,  $f_2^{(p-m)}$  by

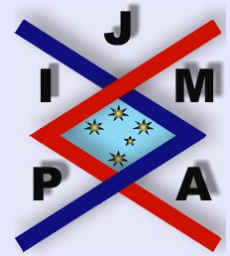
$$(1.1) \quad (f_1 f_2)^{(p)} = \sum_{m=0}^p \binom{p}{m} f_1^{(m)} f_2^{(p-m)}$$

for any fixed but arbitrary  $p \in \mathbb{N}_0$ .

**Proposition 1.4 (Generalized integral identity, [4]).** If  $f : I \rightarrow \mathbb{C}$  is a complex valued function of a real variable  $x$ ,  $I = [0, \infty)$ , and  $h : I \rightarrow \mathbb{R}$  is a real valued function of  $x$ , as well as,  $w, w_p : I \rightarrow \mathbb{R}$  are two real valued functions of  $x$ , such that  $w_p(x) = (x - x_m)^p w(x)$  for any fixed but arbitrary constant  $x_m \in \mathbb{R}$  and  $v = p - 2q$ ,  $0 \leq q \leq \left[\frac{p}{2}\right]$ , then

i)

$$(1.2) \quad \int w_p(x) h^{(v)}(x) dx = \sum_{r=0}^{v-1} (-1)^r w_p^{(r)}(x) h^{(v-r-1)}(x) + (-1)^v \int w_p^{(v)}(x) h(x) dx$$



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holds for any fixed but arbitrary  $p \in \mathbb{N}_0$  and  $v \in \mathbb{N}$ , and all  $r : r = 0, 1, 2, \dots, v - 1$ , as well as the integral identity

ii)

$$\int_I w_p(x) h^{(v)}(x) dx = (-1)^v \int_I w_p^{(v)}(x) h(x) dx$$

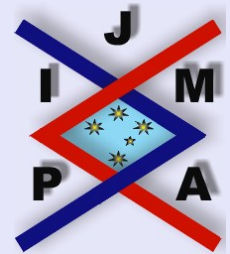
holds if the limiting condition

iii)

$$\sum_{r=0}^{v-1} (-1)^r \lim_{x \rightarrow \infty} w_p^{(r)}(x) h^{(v-r-1)}(x) = 0,$$

holds, and if all of these integrals exist.

We note that the proof of (1.2) requires the application of the differential identity (1.1). Furthermore, we note that the above *integral identity* ii) *still holds* if we replace the interval of integration  $I$  with  $\mathbb{R}$ , *without* any other change except that on the above *limiting condition* iii), where  $x \rightarrow \infty$  has to be substituted with  $|x| \rightarrow \infty$ .



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## 2. Refined Heisenberg-Weyl Type Inequality

We assume that  $f : I \rightarrow \mathbb{C}$  is a complex valued function of a real variable  $x$ , and  $w : I \rightarrow \mathbb{R}$  a real valued weight function of  $x$ , as well as  $x_m$  any fixed but arbitrary real constant. Also we denote

$$(\mu_{2p})_{w,I,|f|^2} = \int_I w^2(x) (x - x_m)^{2p} |f(x)|^2 dx$$

the  $2p^{th}$  weighted moment of  $x$  for  $|f|^2$  with weight function  $w : I \rightarrow \mathbb{R}$ . Besides we denote

$$C_q = (-1)^q \frac{p}{p-q} \binom{p-q}{q},$$

if  $0 \leq q \leq [\frac{p}{2}]$  (= the greatest integer  $\leq \frac{p}{2}$ ),

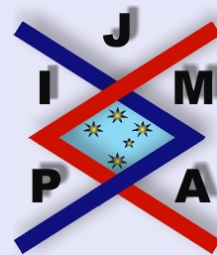
$$I_{ql} = (-1)^{p-2q} \int_I w_p^{(p-2q)}(x) |f^{(l)}(x)|^2 dx,$$

if  $0 \leq l \leq q \leq [\frac{p}{2}]$ , and  $w_p = (x - x_m)^p w$ . We assume that all these integrals exist. Finally we denote  $D_q = \sum_{l=0}^q I_{ql}$ , if  $|D_q| < \infty$  holds for  $0 \leq q \leq [\frac{p}{2}]$ , and

$$E_{p,I,f} = \sum_{q=0}^{[p/2]} C_q D_q,$$

if  $|E_{p,I,f}| < \infty$  holds for  $p \in \mathbb{N}$ . In addition, we assume *the condition*:

$$(2.1) \quad \sum_{r=0}^{p-2q-1} (-1)^r \lim_{x \rightarrow \infty} w_p^{(r)}(x) \left( |f^{(l)}(x)|^2 \right)^{(p-2q-r-1)} = 0,$$



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for  $0 \leq l \leq q \leq \lceil \frac{p}{2} \rceil$ . Furthermore,

$$(2.2) \quad |E_{p,I,f}^*| = \sqrt{E_{p,I,f}^2 + 4A^2},$$

where  $A = \|u\|_{x_0} - \|v\|_{y_0}$ , with  $L^2$ -norm  $\|\cdot\|^2 = \int_I |\cdot|^2$ , inner product  $(|u|, |v|) = \int_I |u| |v|$ , and

$$u = w(x)x_\delta^p f(x), \quad v = f^{(p)}(x);$$

$$x_0 = \int_I |v(x)h(x)| dx, \quad y_0 = \int_I |u(x)h(x)| dx,$$

as well as

$$h(x) = \frac{1}{\sqrt{\sigma}} \sqrt[4]{\frac{2}{\pi}} e^{-\frac{1}{4}(\frac{x-\mu}{\sigma})^2},$$

or

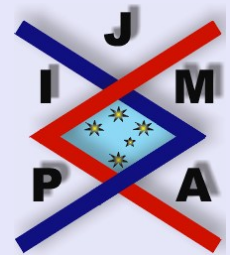
$$(H_I) \quad h(x) = \sqrt{2} \frac{1}{\sqrt[4]{n\pi}} \sqrt{\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}} \cdot \frac{1}{(1 + \frac{x^2}{n})^{\frac{n+1}{4}}},$$

where  $\mu$  is the mean,  $\sigma$  the standard deviation, and  $n \in \mathbb{N}$ , and

$$\|h(x)\|^2 = \int_I |h(x)|^2 dx = 1.$$

**Theorem 2.1.** *If (2.1) holds and  $f \in L^2(\mathbb{R})$ , then*

$$(R_p^*) \quad \sqrt[2p]{(\mu_{2p})_{w,I,|f|^2}} \sqrt[p]{\|f^{(p)}\|_{2,I}} \geq \frac{1}{\sqrt[2]{p}} \sqrt[p]{|E_{p,I,f}^*|},$$



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holds for any fixed but arbitrary  $p \in \mathbb{N}$ .

Equality holds in  $(R_p^*)$  iff  $v(x) = -2c_p u(x)$  holds for constants  $c_p > 0$ , and any fixed but arbitrary  $p \in \mathbb{N}$ ;  $c_p = k_p^2/2 > 0$ ,  $k_p \in \mathbb{R}$  and  $k_p \neq 0$ ,  $p \in \mathbb{N}$ , and  $A = 0$ , or  $h(x) = c_{1p}u(x) + c_{2p}v(x)$  and  $x_0 = 0$ , or  $y_0 = 0$ , where  $c_{ip}$  ( $i = 1, 2$ ) are constants and  $A^2 > 0$ .

We note that this inequality  $(R_p^*)$  still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change except that one on the above condition (2.1), where  $x \rightarrow \infty$  has to be substituted with  $|x| \rightarrow \infty$ , and the choice of  $h$  from  $(H_I)$  must be replaced with

$$h(x) = \frac{1}{\sqrt[4]{2\pi}\sqrt{\sigma}} e^{-\frac{1}{4}\left(\frac{x-\mu}{\sigma}\right)^2},$$

or

$$(H_{\mathbb{R}}) \quad h(x) = \frac{1}{\sqrt[4]{n\pi}} \sqrt{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}} \cdot \frac{1}{\left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{4}}},$$

where  $\mu$  is the mean,  $\sigma$  the standard deviation, and  $n \in \mathbb{N}$ .

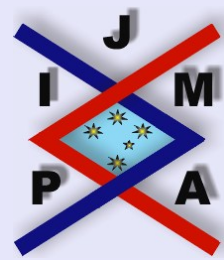
*Proof.* In fact, one gets

$$(2.3) \quad M_p^* = M_p - A^2$$

$$= (\mu_{2p})_{w,I,|f|^2} \cdot \|f^{(p)}\|_{2,I}^2 - A^2$$

$$= \left( \int_I w^2(x) (x - x_m)^{2p} |f(x)|^2 dx \right) \cdot \left( \int_I |f^{(p)}(x)|^2 dx \right) - A^2$$

$$(2.4) \quad = \|u\|^2 \|v\|^2 - A^2$$



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with  $u = w(x)x_\delta^p f(x)$ ,  $v = f^{(p)}(x)$ , where  $x_\delta = x - x_m$ .

From (2.3) – (2.4), the Cauchy-Schwarz inequality  $(|u|, |v|) \leq \|u\| \|v\|$  and the non-negativeness of the following Gram determinant [3] or

$$(2.5) \quad 0 \leq \begin{vmatrix} \|u\|^2 & (|u|, |v|) & y_0 \\ (|v|, |u|) & \|v\|^2 & x_0 \\ y_0 & x_0 & 1 \end{vmatrix}$$

$$= \|u\|^2 \|v\|^2 - (|u|, |v|)^2 - [\|u\|^2 x_0^2 - 2(|u|, |v|)x_0 y_0 + \|v\|^2 y_0^2],$$

$$0 \leq \|u\|^2 \|v\|^2 - (|u|, |v|)^2 - A^2$$

with

$$A = \|u\| x_0 - \|v\| y_0,$$

$$x_0 = \int_I |v(x)h(x)| dx,$$

$$y_0 = \int_I |u(x)h(x)| dx,$$

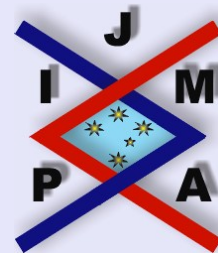
$$\|h(x)\|^2 = \int_I |h(x)|^2 dx = 1,$$

we find

$$(2.6) \quad M_p^* \geq (|u|, |v|)^2 = \left( \int_I |u| |v| \right)^2 = \left( \int_I |w_p(x) f(x) f^{(p)}(x)| dx \right)^2,$$

where  $w_p = (x - x_m)^p w$ . In general, if  $\|h\| \neq 0$ , then one gets

$$(u, v)^2 \leq \|u\|^2 \|v\|^2 - R^2,$$



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where

$$R = A / \|h\| = \|u\| x - \|v\| y,$$

such that  $x = x_0 / \|h\|$ ,  $y = y_0 / \|h\|$ .

In this case,  $A$  has to be replaced by  $R$  in all the pertinent relations of this paper.

From (2.6) and the complex inequality,

$$|ab| \geq \frac{1}{2} (a\bar{b} + \bar{a}b)$$

with  $a = w_p(x) f(x)$ ,  $b = f^{(p)}(x)$ , we get

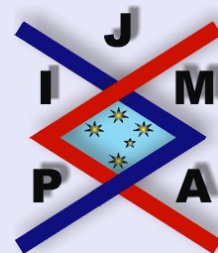
$$(2.7) \quad M_p^* = \left[ \frac{1}{2} \int_I w_p(x) (f(x) \overline{f^{(p)}(x)} + f^{(p)}(x) \overline{f(x)}) dx \right]^2.$$

From (2.7) and the generalized differential identity (\*), one finds

$$(2.8) \quad M_p^* \geq \frac{1}{2^2} \left[ \int_I w_p(x) \left( \sum_{q=0}^{[p/2]} C_q \frac{d^{p-2q}}{dx^{p-2q}} |f^{(q)}(x)|^2 \right) dx \right]^2.$$

From the generalized integral identity (1.2), the condition (2.1), and that all the integrals exist, one gets

$$\int_I w_p(x) \frac{d^{p-2q}}{dx^{p-2q}} |f^{(l)}(x)|^2 dx = (-1)^{p-2q} \int_I w_p^{(p-2q)}(x) |f^{(l)}(x)|^2 dx = I_{ql}.$$



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Thus we find

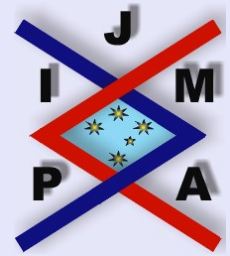
$$M_p^* \geq \frac{1}{2^2} \left[ \sum_{q=0}^{[p/2]} C_q \left( \sum_{l=0}^q I_{ql} \right) \right]^2 = \frac{1}{2^2} E_{p,I,f}^2,$$

where  $E_{p,I,f} = \sum_{q=0}^{[p/2]} C_q D_q$ , if  $|E_{p,I,f}| < \infty$  holds, or *the refined moment uncertainty formula*

$$\sqrt[p]{M_p} \geq \frac{1}{\sqrt[p]{2}} \sqrt[p]{|E_{p,I,f}^*|} \quad \left( \geq \frac{1}{\sqrt[p]{2}} \sqrt[p]{|E_{p,I,f}|} \right),$$

where  $M_p = M_p^* + A^2$ .

We note that the corresponding Gram matrix to the above Gram determinant is positive definite if and only if the above Gram determinant is positive if and only if  $u, v, h$  are linearly independent. In addition, the equality in (2.5) holds if and only if  $h$  is a linear combination of linearly independent  $u$  and  $v$  and  $u = 0$  or  $v = 0$ , completing the proof of the above theorem.  $\square$



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### 3. Applied Refined Heisenberg-Weyl Type Inequality

We apply the above Theorem 2.1 to the following simpler cases of the refined Heisenberg-Weyl type inequality.

#### 3.1. Refined Second Moment Heisenberg-Weyl Type Inequality

For any  $f \in L^2(I)$ ,  $I = [0, \infty)$ ,  $f : I \rightarrow \mathbb{C}$ , such that  $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$ , any fixed but arbitrary constant  $x_m \in \mathbb{R}$ , and for the second order moment

$$(\mu_2)_{I,|f|^2} = \sigma_{I,|f|^2}^2 = \int_I (x - x_m)^2 |f(x)|^2 dx,$$

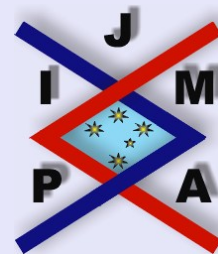
the second order moment Heisenberg-Weyl type inequality

$$(R_1^*) \quad (\mu_2)_{I,|f|^2} \cdot \|f'\|_{2,I}^2 \geq \frac{1}{4} (E_{1,I,f}^*)^2 = \frac{1}{4} \left[ \int_I |f(x)|^2 dx + 4A^2 \right]^2,$$

holds, where  $|E_{1,I,f}^*| < \infty$ .

Equality holds in  $(R_1^*)$  iff  $v(x) = -2c_1 u(x)$  holds for constants  $c_1 > 0$ , and any fixed  $c_1 = k_1^2/2 > 0$ ,  $k_1 \in \mathbb{R}$  and  $k_1 \neq 0$ , and  $A = 0$ , or  $h(x) = c_{11}u(x) + c_{21}v(x)$  and  $x_0 = 0$ , or  $y_0 = 0$ , where  $c_{i1}$  ( $i = 1, 2$ ) are constants and  $A^2 > 0$ .

We note that this inequality  $(R_1^*)$  still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change except that one on the choice of  $h$ , where  $(H_I)$  has to be replaced with  $(H_{\mathbb{R}})$ .



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### 3.2. Refined Fourth Moment Heisenberg-Weyl Type Inequality

For any  $f \in L^2(I)$ ,  $I = [0, \infty)$ ,  $f : I \rightarrow \mathbb{C}$ , such that  $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$ , any fixed but arbitrary constant  $x_m \in \mathbb{R}$ , and for the fourth order moment

$$(\mu_4)_{I,|f|^2} = \int_I (x - x_m)^4 |f(x)|^2 dx,$$

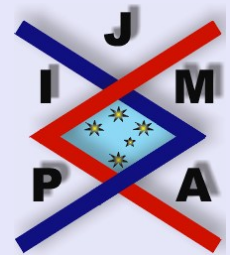
the fourth order moment Heisenberg-Weyl type inequality

$$(R_2^*) \quad (\mu_4)_{I,|f|^2} \cdot \|f''\|_{2,I}^2 \geq \frac{1}{4} (E_{2,I,f}^*)^2 \\ = \frac{1}{4} \left[ \int_I \left[ |f(x)|^2 dx - x_\delta^2 |f'(x)|^2 \right] dx + 4A^2 \right]^2$$

holds, where  $x_\delta = x - x_m$ , and  $|E_{2,I,f}^*| < \infty$ .

Equality holds in  $(R_2^*)$  iff  $v(x) = -2c_2u(x)$  holds for constants  $c_2 > 0$ , and any fixed but arbitrary  $c_2 = \frac{1}{2}k_2^2 > 0$ ,  $k_2 \in \mathbb{R}$  and  $k_2 \neq 0$ , and  $A = 0$ , or  $h(x) = c_{12}u(x) + c_{22}v(x)$  and  $x_0 = 0$ , or  $y_0 = 0$ , where  $c_{i2}$  ( $i = 1, 2$ ) are constants and  $A^2 > 0$ .

We note that this inequality  $(R_2^*)$  still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change except that one on the above condition (2.1), where  $x \rightarrow \infty$  has to be substituted with  $|x| \rightarrow \infty$ , and the choice of  $h$ , where  $(H_I)$  has to be replaced with  $(H_{\mathbb{R}})$ .



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**Remark 1.** Take  $w_p(x) = x^p$ , and  $w_p^{(p)}(x) = p!$  ( $p = 1, 2, 3, 4, \dots$ ). Thus

$$E_{1,I,f} = - \int_I |f(x)|^2 dx = -E_{I,|f|^2},$$

$$E_{2,I,f} = 2 \int_I \left[ |f(x)|^2 - x^2 |f'(x)|^2 \right] dx,$$

$$E_{3,I,f} = -3 \int_I \left[ 2 |f(x)|^2 - 3x^2 |f'(x)|^2 \right] dx,$$

$$E_{4,I,f} = 2 \int_I \left[ 12 |f(x)|^2 - 24x^2 |f'(x)|^2 + x^4 |f''(x)|^2 \right] dx,$$

respectively, if  $|E_{p,I,f}| < \infty$  holds for  $p = 1, 2, 3, 4$ . Therefore

$$D_q = A_{qq} I_{qq} = I_{qq} = (-1)^{p-2q} \int_I w_p^{(p-2q)}(x) |f^{(q)}(x)|^2 dx,$$

if  $|D_q| < \infty$ , for  $0 \leq q \leq \left[ \frac{p}{2} \right]$ .

Furthermore,

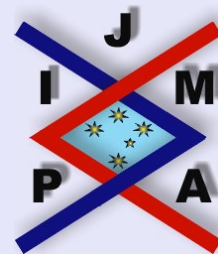
$$w_p^{(p-2q)}(x) = (x^p)^{(p-2q)} = p(p-1) \cdots (p-(p-2q)+1) x^{p-(p-2q)},$$

or

$$w_p^{(p-2q)}(x) = \frac{p!}{(p-(p-2q))!} x^{2q} = \frac{p!}{(2q)!} x^{2q}, \quad 0 \leq q \leq \left[ \frac{p}{2} \right].$$

In addition

$$D_q = (-1)^{p-2q} \frac{p!}{(2q)!} \int_I x^{2q} |f^{(q)}(x)|^2 dx,$$



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if  $|D_q| < \infty$  holds for  $0 \leq q \leq [\frac{p}{2}]$ .

Therefore

$$E_{p,I,f} = \sum_{q=0}^{[p/2]} C_q D_q$$

$$= \sum_{q=0}^{[p/2]} \left[ (-1)^q \frac{p}{p-q} \binom{p-q}{q} \right] \left[ (-1)^{p-2q} \frac{p!}{(2q)!} \int_I x^{2q} |f^{(q)}(x)|^2 dx \right],$$

or the formula

$$E_{p,I,f} = \int_I \sum_{q=0}^{[p/2]} (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q} x^{2q} |f^{(q)}(x)|^2 dx,$$

if  $|E_{p,I,f}| < \infty$  holds for  $0 \leq q \leq [\frac{p}{2}]$ , when  $w = 1$  and  $x_m = 0$ .

Let

$$(m_{2p})_{I,|f|^2} = \int_I x^{2p} |f(x)|^2 dx$$

be the  $2p^{th}$  moment of  $x$  for  $|f|^2$  about the origin  $x_m = 0$ .

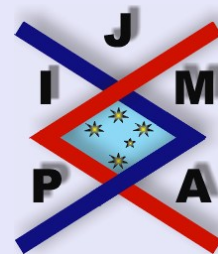
Denote

$$\varepsilon_{p,q} = (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q},$$

for  $p \in \mathbb{N}$  and  $0 \leq q \leq [\frac{p}{2}]$ .

Thus

$$E_{p,I,f} = \int_I \sum_{q=0}^{[p/2]} \varepsilon_{p,q} x^{2q} |f^{(q)}(x)|^2 dx, \quad \text{if } |E_{p,I,f}| < \infty$$



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holds for  $0 \leq q \leq \left[\frac{p}{2}\right]$ .

**Corollary 3.1.** Assume that  $f : I \rightarrow \mathbb{C}$  is a complex valued function of a real variable  $x$ ,  $w = 1$ ,  $x_m = 0$ . If  $f \in L^2(I)$ , then the following inequality

$$(S_p) \quad \sqrt[p]{(m_{2p})_{I,|f|^2}} \sqrt[p]{\|f^{(p)}\|_{2,I}} \geq \frac{1}{\sqrt[p]{2}} \sqrt[p]{\sum_{q=0}^{\left[\frac{p}{2}\right]} \varepsilon_{p,q} (m_{2q})_{I,|f^{(q)}|^2}} + 4A^2,$$

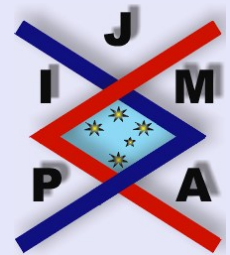
holds for any fixed but arbitrary  $p \in \mathbb{N}$  and  $0 \leq q \leq \left[\frac{p}{2}\right]$ , where

$$(m_{2q})_{I,|f^{(q)}|^2} = \int_I x^{2q} |f^{(q)}(x)|^2 dx$$

and  $A$  is analogous to the one in the above theorem.

Similar conditions are assumed for the “equality” in  $(S_p)$  with respect to those in the above theorem. We note that this inequality  $(S_p)$  still holds if we replace the interval of integration  $I$  with  $\mathbb{R}$ , without any other change except that one on the above condition (2.1), where  $x \rightarrow \infty$  has to be substituted with  $|x| \rightarrow \infty$ , and the choice of  $h$ , where  $(H_I)$  has to be replaced with  $(H_{\mathbb{R}})$ .

**Problem 1.** Concerning our inequality  $(H_2)$  further investigation is needed for the case of the “equality”. As a matter of fact, our function  $f$  is not in  $L^2(\mathbb{R})$ , leading the left-hand side to be infinite in that “equality”. A limiting argument is required for this problem. On the other hand, why does not the corresponding “inequality”  $(H_2)$  attain an extremal in  $L^2(\mathbb{R})$ ?



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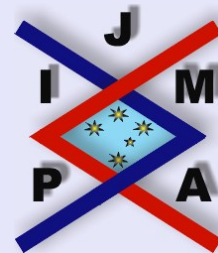
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Here are some of our old results [4] related to the above problem. In particular, if we take into account these results contained in Section 9 on pp. 46-70 [4], where the Gaussian function and the Euler gamma function  $\Gamma$  are employed, then via Corollary 9.1 on pp 50-51 of [4] we conclude that “equality” in  $(H_p)$  of [4, p. 22],  $p \in \mathbb{N} = \{1, 2, 3, \dots\}$ , holds only for  $p = 1$ . Furthermore, employing the above Gaussian function, we established the following *extremum principle* (via (9.33) on p. 51 [4]):

$$(R) \quad R(p) \geq 1/2\pi, \quad p \in \mathbb{N}$$

for the corresponding “inequality” in  $(H_p)$  of [4, p. 22],  $p \in \mathbb{N}$ , where the constant  $1/2\pi$  “on the right-hand side” is the best lower bound for  $p \in \mathbb{N}$ . Therefore “equality” in  $(H_p)$  of [4, p. 22],  $p \in \mathbb{N}$  and  $p \neq 1$ , in Section 8.1 on pp 19-46 [4] cannot occur under the afore-mentioned well-known functions. On the other hand, there is a lower bound “on the right-hand side” of the corresponding “inequality”  $(H_2)$  if we employ the above Gaussian function, which bound equals to  $\frac{1}{64\pi^4} E_{2,\mathbb{R},f}^2 = \frac{1}{512\pi^3} \frac{|c_0|^4}{c}$ , with  $c_0, c$  constants and  $c_0 \in \mathbb{C}, c > 0$ , because  $E_{\mathbb{R},|f|^2} = |c_0|^2 \sqrt{\frac{\pi}{2c}}$  and  $E_{2,\mathbb{R},f} = \frac{1}{2} E_{\mathbb{R},|f|^2}$ .

Analogous pertinent results are investigated via our Corollaries 9.2-9.6 on pp 53-68 [4].



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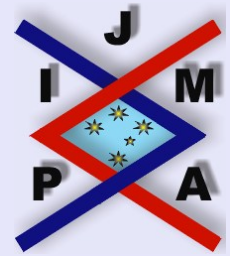
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