Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 3, Article 36, 2002

MULTIVALUED QUASI VARIATIONAL INEQUALITIES IN BANACH SPACES

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Received 22 February, 2002; accepted 22 February, 2002
Communicated by Th.M. Rassias

ABSTRACT. It is established that the multivalued quasi variational inequalities in uniformly smooth Banach spaces are equivalent to the fixed-point problem. We use this equivalence to suggest and analyze some iterative algorithms for quasi variational inequalities with noncompact sets in Banach spaces. Our results are new and represent a significant improvement of the previously known results.

Key words and phrases: Variational inequalities, Banach spaces, Iterative algorithms, Convergence analysis.

2000 *Mathematics Subject Classification.* 49J40, 90C33.

1. INTRODUCTION

Multivalued quasi variational inequalities, which were introduced and studied by Noor [9] – [12], provide us with a unified, natural, novel, innovative and general approach to study a wide class of problems arising in different branches of mathematical, physical and engineering science. In this paper, we consider the multivalued quasi variational inequalities in the setting of real Banach spaces. Using the retraction properties of the projection operator, we establish the equivalence between the quasi variational inequalities and the fixed-point problems. This alternative equivalent formulation is used to suggest and analyze an iterative methods for studying multivalued quasi variational inequalities in Banach spaces. Since multivalued quasi variational

ISSN (electronic): 1443-5756

016-02

inequalities include quasi variational inequalities, complementarity problems and nonconvex programming problems studied in [1] – [15] as special cases, the results obtained in this paper continue to hold for these problems. Our results represent an improvement and refinement of the previous results.

2. FORMULATION AND BASIC RESULTS

Let X be a real Banach space with its topological dual space X^*. Let $\langle\cdot,\cdot\rangle$ be the dual pair between X^* and X. Let 2^Xbe the family of all subsets of X and $CB(X)$ the family of all nonempty closed and bounded subsets of X. Let $T, V : X \longrightarrow CB(X)$ be two multivalued mappings and let $g : X \longrightarrow X$ be a single-valued mapping. For given point-to-set mapping $K : u \longrightarrow K(u)$, which associates a closed convex set of X with any element of X, and $N(\cdot,\cdot) : X \times X \longrightarrow X$, we consider the problem of finding $u \in X, w \in T(u), y \in V(u)$ such that

$$\langle N(w,y), J(g(v)-g(u))\rangle \geq 0, \quad \forall g(v) \in K(u), \tag{2.1}$$

where $J : X \longrightarrow X^*$ is the normalized duality mapping.

Problem (2.1) is called the multivalued quasi variational inequality in Banach spaces, which has many applications in pure and applied sciences, [1, 2, 4, 5].

I. If X is a real Hilbert space, then the duality map J reduces to the identity mapping and problem (2.1) is equivalent to finding $u \in X, w \in T(u), y \in V(u), g(u) \in K(u)$ such that

$$\langle N(w,y), g(v)-g(u)\rangle \geq 0, \quad \forall g(v) \in K(u), \tag{2.2}$$

a problem introduced and studied by Noor [9] using the projection and Wiener-Hopf equations techniques. For the applications, numerical methods and generalizations of problem (2.1), see [6, 7], [9] – [12] and the references therein.

II. If $K^*(u)$ is the polar cone of a closed convex-valued cone $K(u)$ in X, then problem (2.1) is equivalent to finding $u \in X, w \in T(u), y \in V(u)$ such that

$$g(u) \in K(u) \qquad \text{and} \qquad N(w,y) \in J(K(u)-g(u))^* \tag{2.3}$$

which is called the multivalued co-complementarity problem. Some special cases of problem (2.3) has been studied by Chen, Wong and Yao [4] in Banach spaces.

For suitable and appropriate choices of the operators and the spaces, one can obtain several new and known classes of variational inequalities and complementarity problems.

Let $D(T) \subset X$ denote the domain of T and $J : X \longrightarrow 2^{X^*}$ be the normalized duality mapping defined by

$$J(u) = \{f \in X^* : \langle u, f\rangle = \|u\|, \|f\| = \|u\|\}, \qquad u \in X.$$

Definition 2.1. [5] Let $T : D(T) \subset X \longrightarrow 2^X$ be a multi-valued mapping. For all $u, v \in D(T), w \in T(u)$ and $y \in T(v)$, the operator T is said to be:

(a) *accretive*, if there exists $j(u-v) \in J(u-v)$ such that

$$\langle w-y, j(u-v)\rangle \geq 0.$$

(b) *strongly accretive*, if there exists $j(u-v) \in J(u-v)$ and a constant $k > 0$ such that

$$\langle w-y, j(u-v)\rangle \geq k\|u-v\|^2.$$

We remark that if $X = X^* = H$ is a real Hilbert space, then the notions of accretive, strongly accretive and $m-$accretive coincide with that of monotone, strongly monotone and maximal monotone respectively, see Deimling [5].

Remark 2.1. Let $G : X \longrightarrow CB(X)$, $\varepsilon > 0$ be any real number, then for every $u_1, u_2 \in X$ and $v_1 \in G(u_1)$, there exists $v_2 \in G(u_2)$, such that

$$\|v_1 - v_2\| \leq M(G(u_1), G(u_2)) + \varepsilon \|u_1 - u_2\|, \tag{2.4}$$

where $M(\cdot, \cdot)$ is the Hausdorff metric defined on $CB(X)$ by

$$M(B, C) = \max \left\{ \sup_{v \in C} d(v, B), \sup_{u \in B} d(u, C) \right\},$$

for $B, C \in CB(X)$ and $d(v, B) = \min_{u \in B} d(v, u)$.

We note that if $G : X \longrightarrow C(X)$, where $C(X)$ denotes the family of all nonempty compact subsets of X, then it is also true for $\varepsilon = 0$.

From now onward, we assume that X is a uniformly smooth Banach space, unless otherwise specified.

Definition 2.2. [1, 5]. Let X be a real uniformly smooth Banach spaces and K be a nonempty closed convex subset of X. A mapping $P_K : X \longrightarrow K$ is said to be:

(i) retraction , if
$$P_K^2 = P_K.$$

(ii) nonexpansive retraction , if
$$\|P_K u - P_K v\| \leq \|u - v\|, \quad \forall u, v, X.$$

(iii) sunny retraction, if
$$P_K(P_K(u) + t(u - P_K(u)) = P_K(u), \quad \forall u \in X,\ t \in \mathbb{R}.$$

Lemma 2.2. [4, 5]. *P_K is a nonexpansive retraction if and only if*
$$\langle u - P_K(u), J(P_K(u) - v) \rangle \geq 0, \quad \forall u, v \in X.$$

Note that if X is a real Hilbert space, then Lemma 2.2 is well known [13], which has played a fundamental and significant role in suggesting and analyzing the iterative methods for solving variational inequalities and related optimization problems.

Invoking Lemma 2.2, we can show that the multivalued quasi variational inequalities (2.1) are equivalent to the fixed point problem.

Lemma 2.3. *The multivalued quasi variational inequalities (2.1) has a solution $u \in X, w \in T(u), y \in V(u), g(u) \in K(u)$ if and only if $u \in X, w \in T(u), y \in V(u), g(u) \in K(u)$ satisfies the relation*

$$g(u) = P_{K(u)}[g(u) - \rho N(w, y)], \tag{2.5}$$

where $\rho > 0$ is a constant.

Lemma 2.3 establishes the equivalences between the variational inequalities (2.1) and the fixed-point problem (2.5). We use this alternative equivalent formulation to suggest the following iterative algorithm for solving multivalued quasi variational inequalities (2.1) in Banach spaces.

Algorithm 2.1. For given $u_0 \in X, w_0 \in T(u_0), y_0 \in V(u_0)$, and $0 < \varepsilon < 1$, compute the sequences $\{u_n\}, \{w_n\}, \{y_n\}$ by the iterative schemes:

$$g(u_{n+1}) = P_{K(u_n)}[g(u_n) - \rho N(w_n, y_n)], \quad n = 0, 1, 2, \ldots \tag{2.6}$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)) + \varepsilon^{n+1} \|u_{n+1} - u_n\| \tag{2.7}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) + \varepsilon^{n+1} \|y_{n+1} - y_n\|, \tag{2.8}$$

where $M(\cdot, \cdot)$ is the Hausdorff metric defined on $CB(X)$.

If $X = H$, the real Hilbert space, and $\varepsilon = 0$, Algorithm 2.1 is due to Noor [9] – [12] for solving the multivalued quasi variational inequalities (2.1).

For suitable and appreciate choice of the operators T, V, N, g and the space X, one can obtain a number of known and new algorithms for solving variational inclusions and variational inequalities.

3. CONVERGENCE ANALYSIS

In this section, we study the convergence analysis of Algorithm 2.1. For this purpose, we recall the following concepts and notions.

Definition 3.1. For all $u_1, u_2 \in X$, the operator $N(\cdot,\cdot)$ is said to be

(i) $\beta-$Lipschtz continuous with respect to the first argument, if there exists a constant $\beta > 0$ such that

$$\|N(w_1,\cdot) - N(w_2,\cdot)\| \leq \beta\|w_1 - w_2\|,$$

for all $w_1 \in T(u_1), w_2 \in T(u_2)$, and $u_1, u_2 \in X$.

(ii) $\gamma-$Lipschitz continuous with respect to the second argument, if there exists constant $\gamma > 0$ such that

$$\|N(\cdot, y_1) - N(\cdot, y_2)\| \leq \gamma\|y_1 - y_2\|,$$

for all $y_1 \in V(u_1), y_2 \in V(u_2)$, and $u_1, u_2 \in X$.

Definition 3.2. The multi-valued mapping $T : X \longrightarrow CB(X)$ is said to be $M-$Lipschitz continuous if there exits a constant $\eta > 0$ such that

$$M(T(u), T(v)) \leq \eta\|u - v\|, \qquad \text{for all } u, v \in X.$$

Lemma 3.1. [1, 3]. *Let X be a real Banach space and $J : X \longrightarrow 2^{X^*}$ be the normalized dual mapping. Then for all $u, v \in X$, there exits $j(u + v) \in J(u + v)$ such that*

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, j(u + v)\rangle.$$

We also need the following condition.

Assumption 3.1. For all $u, u, w \in X$, the operator $P_{K(u)}$ satisfies the condition

$$\|P_{K(u)}(w) - P_{K(v)}(w)\| \leq \nu\|u - v\|,$$

where $\nu > 0$ is a constant.

We now consider the convergence of the Algorithm 2.1 for the case $g \neq I$.

Theorem 3.2. *Let X be a real uniformly smooth Banach space. Let the operator $N(\cdot,\cdot)$ be a $\beta-$Lipschitz and $\gamma-$Lipschitz continuous with respect to the first argument and second argument respectively. Let the operator g be Lipschitz continuous with constant $\delta > 0$ and strongly accretive with constant $k > \frac{1}{2}$. Assume that the operators $T, V : X \longrightarrow CB(X)$ are M-Lipschitz continuous with constant $\mu > 0$ and $\eta > 0$ respectively. If the Assumption 3.1 holds and*

$$0 < \rho < \frac{\sqrt{2k - 1} - (\delta + \nu)}{\beta\mu + \gamma\eta}, \tag{3.1}$$

then there exists $u \in X, w \in T(u), y \in V(u)$ satisfying the (2.1) and the iterative sequences $\{u_n\}, \{w_n\}$, and$\{y_n\}$ generated by Algorithm 2.1 convergence to u, w, and y strongly in X, respectively.

Proof. From Lemma 3.1 and Algorithm 2.1, it follows that there exists $j(u_{n+1}-u_n) \in J(u_{n+1}-u_n)$ such that

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &= \|g(u_{n+1})-g(u_n)+u_{n+1}-u_n-(g(u_{n+1})-g(u_n))\|^2 \\ &\le \|g(u_{n+1})-g(u_n)\|^2 \\ &\quad + 2\langle u_{n+1}-u_n-(g(u_{n+1})-g(u_n)), j(u_{n+1}-u_n)\rangle \\ &\le \|g(u_{n+1})-g(u_n)\|^2 + 2\|u_{n+1}-u_n\|^2 - 2k\|u_{n+1}-u_n\|^2,\end{aligned}$$

which implies that

$$\|u_{n+1}-u_n\|^2 \le \frac{1}{2k-1}\|g(u_{n+1})-g(u_n)\|^2,$$

that is

$$\|u_{n+1}-u_n\| \le \frac{1}{\sqrt{2k-1}}\|g(u_{n+1})-g(u_n)\|. \tag{3.2}$$

Now, using Assumption 3.1, we have

$$\begin{aligned}&\|g(u_{n+1})-g(u_n)\| \\ &\quad = \left\|P_{K(u_n)}[g(u_n)-\rho N(w_n,y_n)] - P_{K(u_{n-1})}[g(u_{n-1})-\rho N(w_{n-1},y_{n-1})]\right\| \\ &\quad \le \|P_{K(u_n)}[g(u_n)-\rho N(w_n,y_n)] - P_{K(u_n)}[g(u_{n-1})-\rho N(w_{n-1},y_{n-1})]\| \\ &\qquad + \|P_{K(u_n)}[g(u_{n-1})-\rho N(w_{n-1},y_{n-1})] - P_{K(u_{n-1})}[g(u_{n-1})-\rho N(w_{n-1},y_{n-1})]\| \\ &\quad \le \|g(u_n)-g(u_{n-1})-\rho(N(w_n,y_n)-N(w_{n-1},y_{n-1}))\| + \nu\|u_n-u_{n-1}\| \\ &\quad \le \|g(u_n)-g(u_{n-1})\| + \rho\|N(w_n,y_n)-N(w_{n-1},y_{n-1})\| + \nu\|u_n-u_{n-1}\| \\ &\quad \le \delta\|u_n-u_{n-1}\| + \rho\|N(w_n,y_n)-N(w_{n-1},y_n)\| \\ &\qquad + \rho\|N(w_{n-1},y_n)-N(w_{n-1},y_n)\| + \nu\|u_n-u_{n-1}\|.\end{aligned} \tag{3.3}$$

Using the Lipschitz continuity of $M(\cdot,\cdot)$ with respect to the first argument and M-Lipschitz continuity of T, we have

$$\begin{aligned}\|N(w_n,y_n)-N(w_{n-1},y_n)\| &\le \beta\|w_n-w_{n-1}\| \\ &\le \beta(M(T(u_n),T(u_{n-1}))+\varepsilon^n\|u_n-u_{n-1}\|) \\ &\le \beta(\mu+\varepsilon^n)\|u_n-u_{n-1}\|.\end{aligned} \tag{3.4}$$

In a similar way,

$$\begin{aligned}\|N(w_{n-1},y_n)-N(w_{n-1},y_{n-1})\| &\le \gamma\|y_n-y_{n-1}\| \\ &\le \gamma(M(V(u_n),V(u_{n-1}))+\varepsilon^n\|u_n-u_{n-1}\|) \\ &\le \gamma(\eta+\varepsilon^n)\|u_n-u_{n-1}\|.\end{aligned} \tag{3.5}$$

From (3.2) – (3.5) we have

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \frac{(\delta+\gamma)+\rho\{\beta\mu+\gamma\eta+(\beta+\eta)\varepsilon^n\}}{\sqrt{2k-1}}\|u_n-u_{n-1}\| \\ &= \theta(\varepsilon^n)\|u_n-u_{n-1}\|,\end{aligned} \tag{3.6}$$

where

$$\theta(\varepsilon^n) = \frac{(\delta+\gamma)+\rho\{\beta\mu+\gamma\eta+(\beta+\eta)\varepsilon^n\}}{\sqrt{2k-1}}. \tag{3.7}$$

Since $0 < \varepsilon < 1$, it follows that

$$\theta(\varepsilon^n) \longrightarrow \theta \equiv \frac{(\delta+\gamma)+\rho(\beta\mu+\gamma\eta)}{\sqrt{2k-1}}, \qquad \text{as } n \longrightarrow \infty. \tag{3.8}$$

From (3.1), we have $\theta < 1$. Consequently, the sequence $\{u_n\}$ is a Cauchy sequence in X. Since X is a Banach space, there exists $u \in X$, such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$.

From (3.4) and (3.5) we see that w_n, y_n are Cauchy sequences in X, that is, there exist $w, y \in H$ such that $w_n \longrightarrow w$, $y_n \longrightarrow y$. Now by using the continuity of the operators $N, T, V, g, P_{K(u)}$ and Algorithm 2.1, we have

$$g(u) = P_{K(u)}[g(u) - \rho N(w, y)].$$

Finally, we prove that$w \in T(u)$ and $y \in V(u)$. In fact, since $w \in T(u_n)$ we have

$$\begin{aligned} d(w, T(u)) &\le \|w - w_n\| + d(w_n, T(u)) \\ &\le \|w - w_n\| + M(T(u_n), T(u)) \\ &\le \|w - w_n\| + \mu\|u_n - u\| \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty, \end{aligned}$$

which implies that $d(w, T(u)) = 0$, and since $T(u)$ is a closed bounded subset of X, it follows that $w \in T(u)$. In a similar way, we can also prove that $y \in V(u)$.

By Lemma 2.2, it follows that (u, w, y) is a solution of the multivalued quasi variational inequalities problem (2.1), and $u_n \longrightarrow u, w_n \longrightarrow w, y_n \longrightarrow y$ strongly in X, the required result. □

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