Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 2, Issue 1, Article 7, 2001

INEQUALITIES ON POLYNOMIAL HEIGHTS

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Received 21 June, 2000; accepted 26 October, 2000
Communicated by H. Gauchman

ABSTRACT. We give explicit bounds for the absolute values of the coefficients of the divisors of a complex polynomial. They are expressed in function of the coefficients and of upper and lower bounds for the roots. These bounds are compared with other estimates, in particular with the inequality of Beauzamy [B. Beauzamy, Products of polynomials and a priori estimates for coefficients in polynomial decompositions: A sharp result, *J. Symbolic Comput.*, **13** (1992), 463–472]. Through examples it is proved that for some cases our evaluations give better upper limits.

Key words and phrases: Inequalities, Polynomials.

2000 *Mathematics Subject Classification.* 12D05, 12D10, 12E05, 26C05.

1. INTRODUCTION

If $P(X) = \sum_{i=0}^{d} a_i X^i \in \mathbb{C}[X]$, the height of the polynomial P is defined by $\mathrm{H}(P) = \max(|a_0|, |a_1|, \dots, |a_d|)$. Other polynomial sizes are the norm $||P|| = \sqrt{\sum_{j=0}^{d} |a_j|}$, the measure $\mathrm{M}(P) = \exp\left\{\int_0^1 \frac{1}{2\pi} \log\left|P(e^{2i\pi\theta})\right| \mathrm{d}\theta\right\}$ and Bombieri's norm $[P]_2 = \sqrt{\sum_{j=0}^{d} |a_j|^2 / \binom{d}{j}}$.

There exist many estimates for the height of an arbitrary polynomial divisor Q of P. They can be expressed in function of polynomial sizes as the norm, the measure or Bombieri's norm. We mention, for example, the estimate

$$\mathrm{H}(Q) \le \binom{d}{\lfloor d/2 \rfloor} \mathrm{M}(P),$$

where the measure can be easily computed by the method of M. Mignotte [5].

ISSN (electronic): 1443-5756

We are grateful to the referee for the pertinent suggestions concerning Propositions 2.1 and 2.3.
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For integer polynomials one of the best height estimates uses the norm of Bombieri [2] and was obtained by Beauzamy [1]

$$\mathrm{H}(Q) \leq l_n [P]_2, \quad \text{with} \quad l_n = \frac{3^{3/4} \cdot 3^{n/2}}{2(\pi n)^{1/2}}.$$

2. A HEIGHT ESTIMATION

We present another estimate for the heights of proper divisors of a complex polynomial. It makes use of an inequality of M. Mignotte [6]. A key step is the consideration of complex polynomials with roots of moduli greater than 2.

Proposition 2.1. *Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{C}[X] \setminus \mathbb{C}$, such that $P(0) \neq 0$ and let μ be a lower bound of the absolute values of the roots of P. If Q is a monic proper divisor of P in $[X]$, then*

$$\mathrm{H}(Q) < \mathrm{H}(P) \qquad \textit{if} \quad \mu \geq 2,$$

$$\mathrm{H}(Q) < \max_{0 \leq i \leq n} \left| a_i \left(\frac{2}{\mu}\right)^{n-i} \right| \quad \textit{if} \quad \mu < 2.$$

Proof. We suppose

$$Q(X) = X^k + b_{k-1}X^{k-1} + \cdots + b_1 X + b_0.$$

Let $\xi_1, \dots, \xi_n \in \mathbb{C}$ be the roots of P. Without loss of generality we may suppose that $\xi_1, \dots, \xi_k$ are the roots of Q. Note that

$$P = (X - \xi_{k+1}) \cdots (X - \xi_n) Q. \tag{2.1}$$

By an inequality of M. Mignotte [6], we have

$$\big| |\xi_{k+1}| - 1 \big| \cdots \big| |\xi_n| - 1 \big| \, \mathrm{H}(Q) < \mathrm{H}(P). \tag{2.2}$$

We look to $\mu > 0$, which is a lower bound for the absolute values of the roots of P.

If $\mu \geq 2$ then $|\xi_{k+1}| - 1, \dots, |\xi_n| - 1 \geq 1$ and by (2.2) we obtain

$$\mathrm{H}(Q) < \mathrm{H}(P). \tag{2.3}$$

If $\mu < 2$ we associate the polynomials

$$P_\mu(X) = (2/\mu)^n P(\mu X/2) = X^n + a_{n-1} 2/\mu \, X^{n-1} + \cdots + a_0 (2/\mu)^n,$$

$$Q_\mu(X) = (2/\mu)^k Q(\mu X/2) = X^k + b_{k-1} 2/\mu \, X^{k-1} + \cdots + b_0 (2/\mu)^k.$$

Let $\eta_1, \dots, \eta_n \in \mathbb{C}$ be the roots of P_μ. Then

$$|\eta_i| = 2\big|\xi_i\big|/\mu \geq 2$$

and from (2.2) it follows that

$$\mathrm{H}(Q) = \max_j |b_j| \leq \max_j \left| b_j (2/\mu)^{k-j} \right| = \mathrm{H}(Q_\mu) < \mathrm{H}(P_\mu) = \max_i \left| a_i (2/\mu)^{n-i} \right|.$$

Therefore

$$\mathrm{H}(Q) < \max_i \left| a_i \left(\frac{2}{\mu}\right)^{n-i} \right|. \tag{2.4}$$

□

Corollary 2.2. *If $\mu < 2$ is a lower bound of the moduli of the roots of the complex polynomial P and Q is proper divisor of P in $[X]$, then*

$$\mathrm{H}(Q) < (2/\mu)^n \, \mathrm{H}(P).$$

Proof. Note that $2/\mu > 1$. Therefore, by Proposition 2.1,

$$\begin{aligned} \mathrm{H}(Q) &< \max_{0\le i\le n}\left|a_i\left(\frac{2}{\mu}\right)^{n-i}\right| \\ &< \max_{0\le i\le n}\left(\frac{2}{\mu}\right)^{n-i}\cdot \mathrm{H}(P) \\ &= \left(\frac{2}{\mu}\right)^{n}\mathrm{H}(P). \end{aligned}$$

□

Example 2.1. Consider $P(X) = X^6 + 2X^5 + 5X^4 + 10X^3 + 21X^2 + 42X + 83$. By the criterion of Eneström–Kakeya (see M. Marden [4], p. 137), the zeros of a polynomial $a_0 + a_1X + \cdots + a_nX^n$ with real positive coefficients lie in the ring

$$\min\{a_i/a_{i+1}\} \le |z| \le \max\{a_i/a_{i+1}\}\,.$$

Thus, the roots of the polynomial P are in the ring $83/42 \le |z| \le 5/2$. If Q is a divisor of P we obtain

$$\mathrm{H}(Q) < 89.183 \quad \text{by Proposition 2.1,}$$

$$\mathrm{H}(Q) < 602.455 \quad \text{by B. Beauzamy [1].}$$

Proposition 2.3. *Let* $P(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{C}[X]\setminus\mathbb{C}$ *such that* $P(0)\neq 0$ *and let* $\nu > 0$ *be an upper bound for the absolute values of the roots of* P. *If* Q *is a proper divisor of* P *in* $[X]$, $Q(0) = c_0$, *then*

$$\mathrm{H}(Q) < \left|\frac{c_0}{a_0}\right|\cdot \mathrm{H}(P), \qquad \textit{if} \quad \nu \le \frac{1}{2}\,,$$

$$\mathrm{H}(Q) < \left|\frac{c_0}{a_0}\right|\cdot \max_{0\le i\le n}\left|a_i(2\nu)^i\right|, \quad \textit{if} \quad \nu > \tfrac{1}{2}\,.$$

Proof. Let P^* and Q^* be the reciprocal polynomials of P and Q respectively. Considering

$$P_1(X) = \frac{1}{a_0}P^*(X) = \frac{1}{a_0}X^nP\left(\frac{1}{X}\right)$$

it is possible to obtain information about the heights with respect to the upper bounds of the moduli of the roots of P. The polynomial P_1 is monic and

$$\frac{1}{\nu} \quad \text{is a lower bound for the roots of} \quad P_1.$$

On the other hand, if $P_1(X) = \sum_{i=0}^n b_iX^i$, then $b_i = a_{n-i}/a_0$ and

$$\max_{0\le i\le n}\left|b_i\left(\frac{2}{\nu^{-1}}\right)^{n-i}\right| = \frac{1}{|a_0|}\max_{0\le i\le n}\left|a_i(2\nu)^i\right|.$$

Let $Q\in\mathbb{C}[X]$ be a proper divisor of P. Therefore

$$Q_1(X) = \frac{1}{c_0}Q^*(X) = \frac{1}{c_0}X^{\deg(Q)}Q(X^{-1})$$

is a proper divisor of P_1. However,

$$\mathrm{H}(Q_1) = \frac{1}{|c_0|}\,\mathrm{H}(Q).$$

If $\nu \leq 1/2$, by Proposition 2.1 we have $\mathrm{H}(Q_1) < \mathrm{H}(P_1)$, hence

$$\frac{1}{|c_0|} \cdot \mathrm{H}(Q) < \frac{1}{|a_0|} \cdot \mathrm{H}(P),$$

which gives the first inequality. The second relation in Proposition 2.1 gives the other inequality. □

Remark 2.4. If $|P(0)| = 1$, the inequalities in Proposition 2.3 become

$$\begin{aligned} \mathrm{H}(Q) &< \mathrm{H}(P), && \text{if } \nu \leq \tfrac{1}{2}, \\ \mathrm{H}(Q) &< \max_{0\leq i\leq n} \left|a_i(2\nu)^i\right|, && \text{if } \nu > \tfrac{1}{2}. \end{aligned}$$

Indeed, if $|P(0)| = 1$ we have $|a_0| = |c_0| = 1$.

The same inequalities are valid if $P \in [X]$ and Q is a proper divisor of P over $\mathbb{Z}$. In this case a_0 and c_0 are integers and c_0 divides a_0, therefore $\left|\frac{c_0}{a_0}\right| \leq 1$.

As in Corollary 2.2 we deduce

Corollary 2.5. *If $\nu > \frac{1}{2}$ is an upper bound of the moduli of the roots of the complex polynomial P and Q is proper divisor of P in $\mathbb{C}[X]$, then*

$$\mathrm{H}(Q) < \left|\frac{c_0}{a_0}\right| (2\nu)^n \, \mathrm{H}(P).$$

Example 2.2. Let $P = 2381X^5 - 597X^4 - 150X^3 - 37X^2 + 9X + 2$.
If $z \in$ is a root of P, then $|z| \leq 2 \max |\frac{a_{i-1}}{a_i}| = \frac{100}{199}$, by the criterion of T. Kojima [3]. If Q is a possible divisor of P, then

$$\begin{aligned} \mathrm{H}(Q) &< 2440.403 && \text{By Proposition 2.3,} \\ \mathrm{H}(Q) &< 10745.533 && \text{by B. Beauzamy [1].} \end{aligned}$$

Remark 2.6. The estimates from Propositions 2.1 and 2.3 apply to any complex polynomial, while the estimate of Beauzamy refers only to integer polynomials.

Example 2.3. Let $P = 381X^5 - 95iX^4 + (45 - 9i)X^3 + 17iX^2 + (2 + 7i)X + 3$.
If $z \in$ is a root of P, then $|z| \leq 2 \max |\frac{a_{i-1}}{a_i}| = \frac{\sqrt{2106}}{95}$, again by the criterion of T. Kojima [3]. If Q is a possible divisor of P, then

$$\begin{aligned} \mathrm{H}(Q) &< 320.703 && \text{By Proposition 2.3,} \\ \mathrm{H}(Q) &< 2374.689 && \text{by M. Mignotte [5].} \end{aligned}$$

Example 2.4. Let $P = 127X^7 + 64X^6 + 32X^5 + 16X^4 + 8X^3 + 4X^2 + 2X + 1$.
The roots of P are in the ring $1/2 \leq |z| \leq 64/127$, by the criterion of Eneström–Kakeya (M. Marden, loc. cit.). If Q is a possible divisor of P, then

$$\begin{aligned} \mathrm{H}(Q) &< 134.167 && \text{By Proposition 2.3,} \\ \mathrm{H}(Q) &< 1472.464 && \text{by Beauzamy.} \end{aligned}$$

Corollary 2.7. *Let $P[X] = \sum_{i=0}^{n} a_i X^i \in \mathbb{R}[X] \setminus \mathbb{R}$ be monic and let $Q \in [X]$ be a proper monic divisor of P. If $a_i > 0$ for all i, there exists $\mu \in$ such that $2^{1-\frac{1}{n}} < \mu < 2$ and $a_{i-1} \geq \mu a_i$ for $i = 1, 2, \ldots, n$, then*

$$\mathrm{H}(Q) < 2\,\mathrm{H}(P).$$

Proof. From the condition $a_{i-1} \geq \mu a_i$ for all i and the criterion of Eneström–Kakeya (M. Marden, loc. cit.) it follows that μ is a lower bound for the absolute values of the roots of P. Because $\mu < 2$, the inequality (2.4) from Proposition 2.1 is verified. Noting that

$$\frac{1}{2} < \frac{1}{\mu} < 2^{\frac{1}{n}-1}$$

we obtain

$$\begin{aligned} \mathrm{H}(Q) &< \max_{1\leq i\leq n} |a_i| \cdot \left|\left(\frac{2}{\mu}\right)^{n-i}\right| \\ &= \mathrm{H}(P) \cdot \max_{1\leq i\leq n} \left(\frac{2}{\mu}\right)^{n-i} \\ &< \mathrm{H}(P) \cdot \max_{1\leq i\leq n} 2^{\frac{n-i}{n}} < 2\,\mathrm{H}(P). \end{aligned}$$

□

Example 2.5. Let $P(X) = X^7 + 2X^6 + 4X^5 + 8X^4 + 15X^3 + 28X^2 + 51X + 92$. Then, by the theorem of Eneström–Kakeya (M. Marden loc. cit.), $\mu = 51/28$ is a lower bound for the absolute values of roots of P. We have $2^{1-\frac{1}{7}} < 51/28 < 2$, so the hypotheses of Corollary 2.7 are fulfilled.

Remark 2.8. The same conclusion as in Corollary 2.7 holds if $a_i > 0$ and there exists $\nu > 0$ such that $a_{i-1} \leq \nu a_i$ and $\frac{1}{2} < \nu \leq 2^{\frac{1}{n}-1}$.

3. A SMALLEST DIVISOR

In this section we give a limit for the smallest height of a proper divisor of a complex polynomial.

Theorem 3.1. *Let P be a nonconstant complex polynomial and suppose that its roots $\xi_1, \dots, \xi_n \in \mathbb{C}$ are such that*

$$|\xi_1| \geq \cdots \geq |\xi_k| > 1 \geq |\xi_{k+1}| \geq \cdots \geq |\xi_n|.$$

If $P = P_1P_2$ is a factorization of P over $\mathbb{Z}$, we have

$$\mathrm{H}(P) > \min\{\mathrm{H}(P_1), \mathrm{H}(P_2)\} \cdot \big(|\xi_k| - 1\big)^{k/2} \cdot \big(1 - |\xi_{k+1}|\big)^{(n-k)/2}.$$

Proof. We observe that

$$\mathrm{H}(P) > \mathrm{H}(P_1) \prod_{s\in J_1} \big||\xi_s| - 1\big|$$

and

$$\mathrm{H}(P) > \mathrm{H}(P_2) \prod_{t\in J_2} \big||\xi_t| - 1\big|,$$

where $\{J_1, J_2\}$ is a partition of $\{1, 2, \dots, n\}$. It follows that

$$\mathrm{H}(P)^2 > \mathrm{H}(P_1)\,\mathrm{H}(P_2) \prod_{j=1}^{n} \big||\xi_j| - 1\big|,$$

therefore

$$\mathrm{H}(P) > \min\big(\mathrm{H}(P_1), \mathrm{H}(P_2)\big) \cdot \sqrt{\prod_{j=1}^{n} \big||\xi_j| - 1\big|}.$$

We finally observe that

$$\prod_{j=1}^{n}\big||\xi_j|-1\big| \;\geq\; \big(|\xi_k|-1\big)^{k/2}\cdot\big(1-|\xi_{k+1}|\big)^{(n-k)/2},$$

which ends the proof. □

Example 3.1. Let $P(X) = X^5 - 49/6\,X^4 + 59/3\,X^3 - 6X^2 - 18X - 9/2$. We have $\xi_1 = \xi_2 = \xi_3 = 3$, $\xi_4 = -1/2$ and $\xi_5 = -1/3$. Therefore

$$\mathrm{H}(P) \;>\; \min\{\mathrm{H}(P_1), \mathrm{H}(P_2)\}\cdot(3-1)^{3/2}\cdot(1-1/2)^{2/2}.$$

But $(3-1)^{3/2}\cdot(1-1/2)^{2/2} > 1.415$, hence

$$\min\{\mathrm{H}(P_1), \mathrm{H}(P_2)\} \;<\; \frac{1}{1.415}\,\mathrm{H}(P) \;<\; \frac{8}{11}\,\mathrm{H}(P).$$

Remark 3.2. The index k from Theorem 3.1 can be computed by the Schur–Cohn criterion (see M. Marden [4], p. 198). Since it is usually not possible to find the roots ξ_k and ξ_{k+1}, we need to know lower bounds for roots outside the unit circle, respectively upper bounds for roots outside the unit circle.

Corollary 3.3. *If P has no roots on the unit circle, we have*

$$\min\{\mathrm{H}(P_1), \mathrm{H}(P_2)\} \;<\; \mathrm{H}(P)\Big/\big(|\xi_k|-1\big)^{k/2}\cdot\big(1-|\xi_{k+1}|\big)^{(n-k)/2}.$$

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