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ON ENTIRE AND MEROMORPHIC FUNCTIONS THAT SHARE SMALL FUNCTIONS WITH THEIR DERIVATIVES

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Abstract

In this paper, it is shown that if f is a non-constant entire function, f and $f^{(k)}$ share the small function $a(\not\equiv 0, \infty)$ **CM** and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$. Furthermore, if f is non-constant meromorphic, f and a do not have any common pole and $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, then the same conclusion can be obtained. Finally, some open questions are posed for the reader.

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Contents

1. Introduction and the Main Results

Given two non-constant meromorphic functions f and g, it is said that they share a finite value a **IM** (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicity, then we say that f and g share the value a **CM** (counting multiplicity). In this paper, we assume that the reader is familiar with the basic concepts of Nevanlinna value distribution theory and the notations $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$, $S(r, f)$ and etc., see e.g. [5].

L.A. Rubel and C.C. Yang [8], E. Mues and N. Steinmetz [7], G.G. Gundersen [3] and L.Z. Yang [9] have completed work on the uniqueness problem of entire functions with their first or k-th derivatives involving two **CM** or **IM** values. J.H. Zheng and S.P. Wang [12] considered the uniqueness problem of entire functions that share two small functions **CM**. In the aspect of only one **CM** value, R. Brück [1] posed the following question:

> *What results can be obtained if one assumes that f and f' share only one value **CM** plus some growth condition?*

In fact, he presented the following conjecture.

Conjecture 1.1. *Let f be a non-constant entire function. Suppose that $\rho_1(f) < \infty$, $\rho_1(f)$ is not a positive integer and f and f' share one finite value a **CM**. Then*

$$\frac{f' - a}{f - a} = c$$

for some non-zero constant c. Here $\rho_1(f)$ denotes the first iterated order of f.

He also showed in the same paper that the conjecture is true if $a = 0$ and $N\left(r, \frac{1}{f'}\right) = S(r, f)$. Furthermore in 1998, G.G. Gundersen and L.Z. Yang [4] showed that the conjecture is true if f is of finite order. Therefore, it is natural to consider whether there exist any similar results for infinite order entire, or even meromorphic, functions f and small function a of f if we keep the condition $N\left(r, \frac{1}{f'}\right) = S(r, f)$ or replace $N\left(r, \frac{1}{f'}\right)$ by $N\left(r, \frac{1}{f}\right)$ (or $\delta(0, f)$). In this paper, we answer this question and actually show that the following results hold.

Theorem 1.2. *Let $k \geq 1$. Let f be a non-constant entire function and $a(z)$ be a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $T(r, a) = o(T(r, f))$ as $r \to +\infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 **CM** and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Corollary 1.3. *Let f be a non-constant entire function and k be any positive integer. Suppose that f and $f^{(k)}$ share the value 1 **CM** and that $\delta(0, f) > \frac{3}{4}$. Then $f \equiv f^{(k)}$.*

For non-entire meromorphic functions, we have

Theorem 1.4. *Let $k \geq 1$. Let f be a non-constant, non-entire meromorphic function and $a(z)$ be a meromorphic function such that $a(z) \not\equiv 0, \infty$, f and a do not have any common pole and $T(r, a) = o(T(r, f))$ as $r \to +\infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 **CM** and $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, then $f \equiv f^{(k)}$.*

Corollary 1.5. *Let f be a non-constant, non-entire meromorphic function and k be any positive integer. Suppose that f and $f^{(k)}$ share the value 1 **CM** and that $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$. Then $f \equiv f^{(k)}$.*

If we compare our results with the conjecture, it can be seen that we do not assume any restriction on the growth of f. In fact, our results show that under the condition

$$\delta(0, f) > \frac{3}{4}$$

or

$$4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k,$$

we can prove the conjecture is true even for small functions a of even or meromorphic f and the constant c is 1.

2. Some Lemmas

In this section, we have the following lemmas which will be needed in the proofs of the main results. In the following, I is a set of infinite linear measure and may not be the same each time it occurs.

Lemma 2.1. *Let f be a meromorphic function in the complex plane. For any positive integer k, we have*

$$N\left(r,\frac{1}{f^{(k)}}\right)\leq N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

Lemma 2.2. *[10] Let f_1, f_2 be non-constant meromorphic functions and let c_1, c_2 and c_3 be non-zero constants. If $c_1f_1+c_2f_2=c_3$ holds, then*

$$T(r,f_1)<\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}(r,f_1)+S(r,f_1),$$

$r\in I$.

Lemma 2.3. *[2] Let f_j $(j=1,2,\ldots,n)$ be n linearly independent meromorphic functions. If they satisfy*

$$\sum_{j=1}^{n}f_j\equiv 1,$$

then for $1\leq j\leq n$, we have

$$T(r,f_j)<\sum_{k=1}^{n}N\left(r,\frac{1}{f_k}\right)+N(r,f_j)+N(r,D)-\sum_{k=1}^{n}N(r,f_k)-N\left(r,\frac{1}{D}\right)+S(r),$$

where D *is the Wronskian determinant* $W(f_1, f_2, \dots, f_n)$, $S(r) = o(T(r))$, *as* $r \to +\infty$, $r \in I$ *and* $T(r) = \max_{1 \le k \le n} T(r, f_k)$.

The following lemma was proven by H.X. Yi in [11].

Lemma 2.4. *Let* f_j $(j = 1, 2, 3)$ *be meromorphic and* f_1 *be non-constant. Suppose that*

$$\sum_{j=1}^{3} f_j \equiv 1 \tag{2.1}$$

and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \tag{2.2}$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$ *and* $T(r) = \max_{1 \le j \le 3} T(r, f_j)$. *Then* $f_2 \equiv 1$ *or* $f_3 \equiv 1$.

Lemma 2.5. *[6] Let* f *be a transcendental meromorphic function and* $K > 1$, *then there exists a set* $M(K)$ *of upper logarithmic density at most*

$$\delta(K) = \min\left\{(2e^{K-1} - 1)^{-1}, (1 + e(K-1))e^{e(1-K)}\right\}$$

such that for every positive integer k,

$$\limsup_{r \to +\infty, r \notin M(K)} \frac{T(r, f)}{T(r, f^{(k)})} \le 3eK.$$

If f *is entire, then* $3eK$ *can be replaced by* $2eK$ *in the above inequality.*

3. Proofs of Theorem 1.2 and Theorem 1.4

Proof of Theorem 1.2. First of all, we write

$$F = \frac{f^{(k)} - a}{f - a}. \tag{3.1}$$

Now a pole of F must be a zero of $f - a$ or a pole of $f^{(k)} - a$. Since $f - a$ and $f^{(k)} - a$ share the value 0 **CM**, poles of F cannot be zeros of $f - a$. Furthermore, f is assumed to be entire, the poles of $f^{(k)} - a$ are the poles of a. It follows that if z_0 is a pole of a, then $F(z_0) = 1$. Hence, F has no pole in the complex plane. By similar reasoning, we can show that F does not have any zero.

Therefore, we deduce from (3.1) that

$$\frac{f^{(k)} - a}{f - a} = e^g \tag{3.2}$$

where g is an entire function. Let $f_1 = \frac{f^{(k)}}{a}$, $f_2 = -\frac{e^g f}{a}$ and $f_3 = e^g$. Thus from (3.2), we have

$$f_1 + f_2 + f_3 = 1. \tag{3.3}$$

By Lemma 2.5, we see that $f_1 = \frac{f^{(k)}}{a}$ is non-constant. Hence, by Lemma 2.1,

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} N(r, f_j)$$
$$= N\left(r, \frac{a}{f^{(k)}}\right) + N\left(r, \frac{a}{fe^g}\right) + N\left(r, \frac{1}{e^g}\right)$$

$$\leq 2N\left(r, \frac{1}{f}\right) + S(r, f).$$

as $r \to +\infty,\ r \in I$. On the other hand, since

$$\begin{aligned} T(r, f) &= T\left(r, \frac{af_2}{-f_3}\right) \\ &\leq T(r, f_2) + T(r, a) + T(r, f_3) \\ &\leq 2T(r) + S(r, f), \end{aligned}$$

where $T(r) = \max_{1\leq j\leq 3} T(r, f_j)$, it follows from $\delta(0, f) > \frac{3}{4}$ that

$$\begin{aligned} 2N\left(r, \frac{1}{f}\right) &< (\lambda + o(1))\frac{T(r, f)}{2} \\ &\leq (\lambda + o(1))T(r) \end{aligned}$$

as $r \to +\infty$, $r \in I$ and $\lambda < 1$. So by Lemma 2.4, $\frac{fe^g}{a} \equiv -1$ or $e^g \equiv 1$.

Case 3.1. *If $e^g \equiv 1$, then we have $f \equiv f^{(k)}$ by (3.2).*

Case 3.2. *If $fe^g \equiv -a$, then*

(3.4) $$f = -ae^{-g}.$$

By (3.2),

(3.5) $$ff^{(k)} = a^2.$$

By differentiating both sides of (3.4) k times, we obtain

$$f^{(k)} = Q(g)e^{-g}, \tag{3.6}$$

where $Q(g)$ is a differential polynomial of g with small functions with respect to f, and hence to e^g by (3.4). Therefore, by (3.4), (3.5) and (3.6), we get a contradiction that $T(r,f) = o(T(r,f))$ as $r \to +\infty, r \in I$ in this case.

□

Proof of Theorem 1.4. To prove Theorem1.4, we first need to reconsider (3.1). Since f is non-entire meromorphic, we can use the same argument to show that the function F in (3.1) does not have any zero. Hence, F has the form he^g, where g is an entire function and h is a non-zero meromorphic function. Now it is clear that the poles of h come from the poles of f or a and furthermore, we have the following

$$\overline{N}(r,h) \leq \overline{N}(r,f) + S(r,f). \tag{3.7}$$

Therefore, instead of (3.2), we have

$$\frac{f^{(k)} - a}{f - a} = he^g$$

and thus

$$f_1 + f_2 + f_3 = 1,$$

where $f_1 = \frac{f^{(k)}}{a}$, $f_2 = \frac{-he^g f}{a}$ and $f_3 = he^g$.

By Lemma 2.1 and (3.7), we have

$$\begin{aligned}
N\left(r,\frac{a}{f^{(k)}}\right)&+N\left(r,\frac{a}{hfe^g}\right)+N\left(r,\frac{1}{he^g}\right)\\
&\quad+2\left[\overline{N}\left(r,\frac{f^{(k)}}{a}\right)+\overline{N}\left(r,\frac{he^g f^{(k)}}{a}\right)+\overline{N}(r,he^g)\right]\\
&\le N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+2\left[2\overline{N}(r,f)+2\overline{N}(r,h)\right]+S(r,f)\\
&\le N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+8\overline{N}(r,f)+S(r,f)\\
&=2N\left(r,\frac{1}{f}\right)+(8+k)\overline{N}(r,f)+S(r,f)
\end{aligned}$$

as $r\to+\infty,\ r\in I$. On the other hand, it follows from the condition $4\delta(0,f)+2(8+k)\Theta(\infty,f)>19+2k$ that

$$\begin{aligned}
N\left(r,\frac{a}{f^{(k)}}\right)&+N\left(r,\frac{a}{hfe^g}\right)+N\left(r,\frac{1}{he^g}\right)\\
&\quad+2\left[\overline{N}\left(r,\frac{f^{(k)}}{a}\right)+\overline{N}\left(r,\frac{he^g f^{(k)}}{a}\right)+\overline{N}(r,he^g)\right]\\
&<(\lambda+o(1))\frac{T(r,f)}{2}\\
&\le(\lambda+o(1))T(r)
\end{aligned}$$

as $r\to+\infty,\ r\in I$ and $\lambda<1$. Therefore, as in the proof of Theorem 1.2, we have $\frac{fhe^g}{a}\equiv-1$ or $he^g\equiv 1$.

Case 3.1. *If* $he^g \equiv 1$, *then* $e^g = \frac{1}{h}$ *which is a contradiction because* h *is a non-entire meromorphic function.*

Case 3.2. *If* $\frac{fhe^g}{a} \equiv -1$, *then* $f = -\frac{ae^{-g}}{h}$ *and we still have (3.5) in this case. Since* f *is non-entire meromorphic, we let* z_0 *be a pole of* f. *Then we see that* f *and* a *have* z_0 *as their common pole which is a contradiction.*

□

Remark 3.1. *It is easily seen that Corollaries 1.3 and 1.5 are true if we take* $a(z) \equiv 1$ *in Theorems 1.2 and 1.4 respectively.*

4. Final Remarks

Remark 4.1. *By the remark pertaining to Theorem 2 in [12], we have the following example which shows that the conditions* 0 ***IM*** *and* $\delta(0,f) > \frac{3}{4}$ *are not sufficient for meromorphic functions in the above theorems and corollaries.*

Example 4.1.

$$f(z) = \frac{2A}{1-e^{-2z}}, \quad f'(z) = -\frac{4Ae^{-2z}}{(1-e^{-2z})^2},$$

where $A \neq 0$, *then*

$$f(z) - A = \frac{A(1+e^{-2z})}{1-e^{-2z}}, \quad f'(z) - A = -\frac{A(1+e^{-2z})^2}{(1-e^{-2z})^2}.$$

Here, it is easily seen that A *is an* ***IM*** *shared value of* f *and* f', 0 *is a Picard value of* f *and* f' *(i.e.* $\delta(0,f) = 1$*), but* $f \not\equiv f'$.

Remark 4.2. *Next, we extend our results to the "****CM****" shared value. Let us recall the definition first. Let* $f(z)$ *and* $g(z)$ *be non-constant meromorphic functions,* a *is any complex number. We denote* $N_E(r,a)$ *to be the reduced counting function of the common zero (with the same multiplicity) of* $f-a$ *and* $g-a$*. If*

$$\overline{N}\left(r, \frac{1}{f-a}\right) - N_E(r,a) = S(r,f)$$

and

$$\overline{N}\left(r, \frac{1}{g-a}\right) - N_E(r,a) = S(r,g),$$

*then a is said to be a "**CM**" shared value of f and g. The case for small functions of f and g is similar. In this case, the function h, mentioned in Section 3, will be allowed to have zero with $\overline{N}\left(r, \frac{1}{h}\right) = S(r, f)$. Therefore, it is easily seen that the results are also valid if we replace the **CM** shared value by the "**CM**" shared value. That is*

Theorem 4.1. *Let $k \geq 1$. Let f be a non-constant entire function and $a(z)$ be a meromorphic function such that $a(z) \not\equiv 0, \infty$, and $T(r, a) = o(T(r, f))$ as $r \to +\infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 "**CM**" and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem 4.2. *Let $k \geq 1$. Let f be a non-constant meromorphic function and $a(z)$ be a meromorphic function such that $a(z) \not\equiv 0, \infty$, f and a do not have any common pole and $T(r, a) = o(T(r, f))$ as $r \to +\infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 "**CM**" and $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, then $f \equiv f^{(k)}$.*

The proofs are similar to the ones of Theorem 1.2 and Theorem 1.4.

Remark 4.3. *One may ask what we can obtain if f and a are allowed to have a common pole in Theorem 1.4. In fact, by (3.5) we have the following.*

Theorem 4.3. *Suppose that k is an odd integer. Then Theorem 1.4 is valid for all small functions a.*

5. Four Open Questions

Finally, we pose the following natural questions for the reader.

Question 1. *Can a **CM** shared value be replaced by an **IM** shared value in Theorem 1.2 and Corollary 1.3?*

Question 2. *Is the condition $\delta(0, f) > \frac{3}{4}$ sharp in Theorem 1.2 and Corollary 1.3?*

Question 3. *Is the condition $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$ sharp in Theorem 1.4 and Corollary 1.5?*

Question 4. *Can the condition "f and a do not have any common pole" be deleted in Theorem 1.4 and Theorem 4.2?*

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