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NOTE ON SOME HADAMARD-TYPE INEQUALITIES

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ABSTRACT. Some Hadamard-type inequalities involving the product of two convex functions are obtained. Our results generalize the corresponding results of B.G.Pachpatte.

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1. INTRODUCTION

Let f be a convex function on $[a,b] \subset \mathbb{R}$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality [1, p. 137], [2, p. 10] for convex functions.

Recently B.G.Pachpatte [3] considered some new integral inequalities, analogous to that of Hadamard, involving the product of two convex functions. In [3] the following theorem has been proved:

Theorem 1.1. *Let f and g be nonnegative, convex functions on $[a,b] \subset \mathbb{R}$. Then*

(i)

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)\,, \tag{1.2}$$

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(ii)

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),\tag{1.3}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ and $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$. Inequalities (1.2) and (1.3) are sharp in the sense that equalities hold for some $f(x)$ and $g(x)$ on $[a,b]$.

In the following Theorem 1.2 we give a variant of the corresponding Theorem 2 in [3].

Theorem 1.2. *Let f and g be nonnegative, convex functions on $[a,b]\subset\mathbb{R}$. Then*

(i)

$$\begin{aligned}\frac{3}{2(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdxdy\\ \leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{1}{8}\left[M(a,b)+N(a,b)\right];\end{aligned}\tag{1.4}$$

(ii)

$$\begin{aligned}\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx\\ \leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{1}{2}\left[M(a,b)+N(a,b)\right],\end{aligned}\tag{1.5}$$

where $M(a,b)$ and $N(a,b)$ are as in Theorem 1.1.

It should be noted that in [3, Theorem 2] inequalities (3) and (4) are established. Inequality (3) from [3, Theorem 2] is a variant of our inequality (1.4) in which

$$\frac{1}{8}\left[\frac{M(a,b)+N(a,b)}{(b-a)^2}\right]$$

stands in place of the term $\frac{1}{8}\left[M(a,b)+N(a,b)\right]$. Analogously, inequality (4) from [3, Theorem 2] is a variant of our inequality (1.5) in which

$$\frac{1}{4}\left(\frac{1+b-a}{b-a}\right)\left[M(a,b)+N(a,b)\right]$$

stands in place of the term $\frac{1}{2}\left[M(a,b)+N(a,b)\right]$.

However, one can compare inequalities (3) and (4) with (1.4) and (1.5), respectively, to find out that estimates given by (1.4) and (1.5) are better (worse) than those given by (3) and (4) in [3, Theorem 2] in case of $b-a<1$ $(b-a>1)$.

But on careful inspection of the proof in [3, Theorem 2], the reader can notice some errors in Pachpatte's calculation, so inequalities (3) and (4) in [3, Theorem 2] are in fact incorrect.

The aim of this paper is to prove some simple generalizations of Theorem 1.1 and Theorem 1.2, which additionally involve weight functions and also nonlinear transformations of the base interval $[a,b]$. Those generalizations are established in Theorem 2.1 and Theorem 2.3. The above cited Theorem 1.1 is a special case of Theorem 2.1, while the above Theorem 1.2 is a special case of our Theorem 2.3 (see Remark 2.4).

2. RESULTS

Throughout the rest of the paper we shall use the following notation

$$[h;x,y]=\frac{h(y)-h(x)}{y-x},\quad x\neq y$$
$$\widetilde{h}(t)=th(\alpha+\beta-t),$$
$$\widehat{h}(t)=th(t)$$

where $h:[\alpha,\beta]\to\mathbb{R}$ is a function, $[\alpha,\beta]\subset\mathbb{R}$, $x,y,t\in[\alpha,\beta]$. Note that from the above equalities we get

$$\left[\widetilde{h};\alpha,\beta\right]=\frac{\beta h(\alpha)-\alpha h(\beta)}{\beta-\alpha},$$
$$\left[\widehat{h};\alpha,\beta\right]=\frac{\beta h(\beta)-\alpha h(\alpha)}{\beta-\alpha},$$

and, by simple calculation,

$$\left[\widehat{h};\alpha,\beta\right]-\left[\widetilde{h};\alpha,\beta\right]=(\alpha+\beta)[h;\alpha,\beta]. \tag{2.1}$$

The following results are valid:

Theorem 2.1. *Let f be a nonnegative convex function on $[m_1,M_1]$, g a nonnegative convex function on $[m_2,M_2]$, $u:[a,b]\to[m_1,M_1]$ and $v:[a,b]\to[m_2,M_2]$ continuous functions, and $p:[a,b]\to\mathbb{R}$ a positive integrable function. Then*

(i)

$$\begin{aligned}\frac{1}{P}\int_a^b p(x)f(u(x))g(v(x))\,dx &\le [f;m_1,M_1][g;m_2,M_2]\frac{1}{P}\int_a^b p(x)u(x)v(x)\,dx\\ &+[f;m_1,M_1][\widetilde{g};m_2,M_2]\frac{1}{P}\int_a^b p(x)u(x)\,dx\\ &+\left[\widetilde{f};m_1,M_1\right][g;m_2,M_2]\frac{1}{P}\int_a^b p(x)v(x)\,dx\\ &+\left[\widetilde{f};m_1,M_1\right][\widetilde{g};m_2,M_2].\end{aligned} \tag{2.2}$$

(ii)

$$\begin{aligned}f\left(\frac{m_1+M_1}{2}\right)g\left(\frac{m_2+M_2}{2}\right)&\le\frac{1}{4P}\left[\int_a^b p(x)f(u(x))g(v(x))\,dx\right.\\ &\left.+\int_a^b p(x)f(M_1+m_1-u(x))g(M_2+m_2-v(x))\,dx\right]\\ &+\frac{1}{4P}\left[-2[f;m_1,M_1][g;m_2,M_2]\int_a^b p(x)u(x)v(x)\,dx\right.\\ &+([\widehat{g};m_2,M_2]-[\widetilde{g};m_2,M_2])[f;m_1,M_1]\int_a^b p(x)u(x)\,dx\end{aligned} \tag{2.3}$$

$$+\left(\left[\widehat{f};m_1,M_1\right]-\left[\widetilde{f};m_1,M_1\right]\right)[g;m_2,M_2]\int_a^b p(x)\,v(x)\,dx\Bigg]$$
$$+\frac{1}{4}\left(\left[\widetilde{f};m_1,M_1\right][\widehat{g};m_2,M_2]+\left[\widehat{f};m_1,M_1\right][\widetilde{g};m_2,M_2]\right),$$

where $P=\int_a^b p(x)\,dx$.

Proof. For any $x\in[a,b]$ we can write

$$u(x)=\frac{M_1-u(x)}{M_1-m_1}m_1+\frac{u(x)-m_1}{M_1-m_1}M_1 \tag{2.4}$$

and

$$v(x)=\frac{M_2-v(x)}{M_2-m_2}m_2+\frac{v(x)-m_2}{M_2-m_2}M_2. \tag{2.5}$$

Since f and g are convex functions we have

$$\begin{aligned} f(u(x)) &\le \frac{M_1-u(x)}{M_1-m_1}f(m_1)+\frac{u(x)-m_1}{M_1-m_1}f(M_1)\\ &= \frac{u(x)}{M_1-m_1}\left(f(M_1)-f(m_1)\right)+\frac{M_1f(m_1)-m_1f(M_1)}{M_1-m_1}\\ &= [f;m_1,M_1]\,u(x)+\left[\widetilde{f};m_1,M_1\right] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} g(v(x)) &\le \frac{M_2-v(x)}{M_2-m_2}g(m_2)+\frac{v(x)-m_2}{M_2-m_2}g(M_2)\\ &= \frac{v(x)}{M_2-m_2}\left(g(M_2)-g(m_2)\right)+\frac{M_2g(m_2)-m_2g(M_2)}{M_2-m_2}\\ &= [g;m_2,M_2]\,v(x)+[\widetilde{g};m_2,M_2]\,. \end{aligned} \tag{2.7}$$

Functions f and g are nonnegative by assumption, so after multiplying (2.6) and (2.7) we obtain

$$\begin{aligned} &f(u(x))\,g(v(x))\\ &\le [f;m_1,M_1][g;m_2,M_2]\,u(x)\,v(x)+[f;m_1,M_1][\widetilde{g};m_2,M_2]\,u(x)\\ &\quad+[g;m_2,M_2]\left[\widetilde{f};m_1,M_1\right]v(x)+\left[\widetilde{f};m_1,M_1\right][\widetilde{g};m_2,M_2]\,. \end{aligned} \tag{2.8}$$

Now, multiplying (2.8) by weight $p(x)$, integrating over $[a,b]$ and dividing by $P>0$ we get (i).

To obtain (ii) we can write

$$\begin{aligned} \frac{m_1+M_1}{2} &= \frac{1}{2}\left(\frac{M_1-u(x)}{M_1-m_1}m_1+\frac{u(x)-m_1}{M_1-m_1}M_1\right.\\ &\qquad\left.+\frac{u(x)-m_1}{M_1-m_1}m_1+\frac{M_1-u(x)}{M_1-m_1}M_1\right),\\ \frac{m_2+M_2}{2} &= \frac{1}{2}\left(\frac{M_2-v(x)}{M_2-m_2}m_2+\frac{v(x)-m_2}{M_2-m_2}M_2\right.\\ &\qquad\left.+\frac{v(x)-m_2}{M_2-m_2}m_2+\frac{M_2-v(x)}{M_2-m_2}M_2\right). \end{aligned}$$

Using the Hadamard inequality (1.1) and the convexity of functions f and g, we get

$$\begin{aligned}
& f\left(\frac{m_1+M_1}{2}\right) g\left(\frac{m_2+M_2}{2}\right) \\
& \leq \frac{1}{4}\left[f\left(\frac{M_1-u(x)}{M_1-m_1} m_1+\frac{u(x)-m_1}{M_1-m_1} M_1\right)+f\left(\frac{u(x)-m_1}{M_1-m_1} m_1+\frac{M_1-u(x)}{M_1-m_1} M_1\right)\right] \\
& \quad \times\left[g\left(\frac{M_2-v(x)}{M_2-m_2} m_2+\frac{v(x)-m_2}{M_2-m_2} M_2\right)+g\left(\frac{v(x)-m_2}{M_2-m_2} m_2+\frac{M_2-v(x)}{M_2-m_2} M_2\right)\right].
\end{aligned}$$

According to (2.4) and (2.5) , after some simple calculus we obtain

$$\begin{aligned}
(2.9)\quad & f\left(\frac{m_1+M_1}{2}\right) g\left(\frac{m_2+M_2}{2}\right) \\
& \leq \frac{1}{4}\left[f(u(x)) g(v(x))+f\left(M_1+m_1-u(x)\right) g\left(M_2+m_2-v(x)\right)\right] \\
& +\frac{1}{4}\left[f\left(\frac{M_1-u(x)}{M_1-m_1} m_1+\frac{u(x)-m_1}{M_1-m_1} M_1\right) \times g\left(\frac{v(x)-m_2}{M_2-m_2} m_2+\frac{M_2-v(x)}{M_2-m_2} M_2\right)\right. \\
& \left.+f\left(\frac{u(x)-m_1}{M_1-m_1} m_1+\frac{M_1-u(x)}{M_1-m_1} M_1\right) \times g\left(\frac{M_2-v(x)}{M_2-m_2} m_2+\frac{v(x)-m_2}{M_2-m_2} M_2\right)\right].
\end{aligned}$$

Using the convexity of functions f and g, from inequality (2.9) we get

$$\begin{aligned}
(2.10)\quad & f\left(\frac{m_1+M_1}{2}\right) g\left(\frac{m_2+M_2}{2}\right) \\
& \leq \frac{1}{4}\left[f(u(x)) g(v(x))+f\left(M_1+m_1-u(x)\right) g\left(M_2+m_2-v(x)\right)\right] \\
& +\frac{1}{4}\left[\left(\frac{M_1-u(x)}{M_1-m_1} f\left(m_1\right)+\frac{u(x)-m_1}{M_1-m_1} f\left(M_1\right)\right)\right. \\
& \times\left(\frac{v(x)-m_2}{M_2-m_2} g\left(m_2\right)+\frac{M_2-v(x)}{M_2-m_2} g\left(M_2\right)\right) \\
& +\left(\frac{u(x)-m_1}{M_1-m_1} f\left(m_1\right)+\frac{M_1-u(x)}{M_1-m_1} f\left(M_1\right)\right) \\
& \left.\times\left(\frac{M_2-v(x)}{M_2-m_2} g\left(m_2\right)+\frac{v(x)-m_2}{M_2-m_2} g\left(M_2\right)\right)\right].
\end{aligned}$$

With respect to the notation introduced at the beginning of this section, inequality (2.10) becomes

$$\begin{aligned}
& f\left(\frac{m_1+M_1}{2}\right) g\left(\frac{m_2+M_2}{2}\right) \\
& \leq \frac{1}{4}\left\{f(u(x)) g(v(x))+f\left(M_1+m_1-u(x)\right) g\left(M_2+m_2-v(x)\right)\right\} \\
& +\frac{1}{4}\left\{\left(\left[f ; m_1, M_1\right] u(x)+\left[\widetilde{f} ; m_1, M_1\right]\right)\left(\left[\widehat{g} ; m_2, M_2\right]-\left[g ; m_2, M_2\right] v(x)\right)\right. \\
& \left.+\left(\left[\widehat{f} ; m_1, M_1\right]-\left[f ; m_1, M_1\right] u(x)\right)\left(\left[g ; m_2, M_2\right] v(x)+\left[\widetilde{g} ; m_2, M_2\right]\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\Big\{ f(u(x))\,g(v(x)) + f(M_1+m_1-u(x))\,g(M_2+m_2-v(x)) \Big\} \\
&\quad + \frac{1}{4}\Bigg\{ -2\,[f;m_1,M_1]\,[g;m_2,M_2]\,u(x)\,v(x) \\
&\quad + \Big([\widehat{g};m_2,M_2] - [\widetilde{g};m_2,M_2] \Big)[f;m_1,M_1]\,u(x) \\
&\quad + \Big(\Big[\widehat{f};m_1,M_1\Big] - \Big[\widetilde{f};m_1,M_1\Big] \Big)[g;m_2,M_2]\,v(x) \\
&\quad + \Big[\widetilde{f};m_1,M_1\Big][\widehat{g};m_2,M_2] + \Big[\widehat{f};m_1,M_1\Big][\widetilde{g};m_2,M_2] \Bigg\} \qquad (2.11)
\end{aligned}
$$

Now we multiply both sides of (2.11) by $p(x)$, integrate over $[a,b]$ and divide by P. We thus obtain (ii) and the proof is completed. □

Remark 2.2. Pachpatte's results (1.2) and (1.3) can be obtained from (2.2) and (2.3) respectively if we put $p(x)=1, u(x)=v(x)=x$ for all $x\in[a,b]$ (then we have $m_1=m_2=a$ and $M_1=M_2=b$). In the case of $g(x)\equiv 1$ inequality (i) becomes the right side of Hadamard's inequality (1.1).

Theorem 2.3. *Let f be a nonnegative convex function on $[m_1,M_1]$, g a nonnegative convex function on $[m_2,M_2]$, $u:[a,b]\to[m_1,M_1]$ and $v:[a,b]\to[m_2,M_2]$ continuous functions, and $p,q:[a,b]\to\mathbb{R}$ positive integrable functions. Then*

(i)

$$
\begin{aligned}
&\frac{1}{PQ}\int_a^b\int_a^b\int_0^1 p(x)\,q(y)\,f(tu(x)+(1-t)u(y))\times g(tv(x)+(1-t)v(y))\,dtdxdy \\
&\quad\le \frac{1}{3PQ}\left[Q\int_a^b f(u(x))\,g(v(x))\,p(x)\,dx + P\int_a^b f(u(y))\,g(v(y))\,q(y)\,dy\right] \\
&\qquad + \frac{1}{3PQ}\int_a^b p(x)\,f(u(x))\,dx\int_a^b q(y)\,g(v(y))\,dy;
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&\frac{1}{P}\int_a^b\int_0^1 p(x)\,f(tu(x)+(1-t)\overline{u})\,g(tv(x)+(1-t)\overline{v})\,dtdx \\
&\quad\le \frac{1}{3P}\int_a^b p(x)\,f(u(x))\,g(v(x))\,dx + \frac{1}{3}f(\overline{u})\,g(\overline{v}) \\
&\qquad + \frac{1}{6P}\left[g(\overline{v})\int_a^b p(x)\,f(u(x))\,dx + f(\overline{u})\int_a^b p(x)\,g(v(x))\,dx\right],
\end{aligned}
$$

where $\overline{u}=\frac{1}{P}\int_a^b p(x)\,u(x)\,dx, \overline{v}=\frac{1}{Q}\int_a^b q(x)\,v(x)\,dx$.

Proof. Since f and g are convex functions, for $t\in[0,1]$ we have

$$f(tu(x)+(1-t)u(y)) \le tf(u(x))+(1-t)f(u(y)) \qquad (2.12)$$

$$g(tv(x)+(1-t)v(y)) \le tg(v(x))+(1-t)g(v(y)). \qquad (2.13)$$

Functions f and g are nonnegative, so multiplying (2.12) and (2.13) we get

$$\begin{aligned}(2.14)\quad & f\left(tu\left(x\right)+\left(1-t\right)u\left(y\right)\right)g\left(tv\left(x\right)+\left(1-t\right)v\left(y\right)\right)\\ &\le t^{2}f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)+\left(1-t\right)^{2}f\left(u\left(y\right)\right)g\left(v\left(y\right)\right)\\ &\quad+t\left(1-t\right)\left[f\left(u\left(x\right)\right)g\left(v\left(y\right)\right)+f\left(u\left(y\right)\right)g\left(v\left(x\right)\right)\right].\end{aligned}$$

Integrating (2.14) over $[0,1]$ we obtain

$$\begin{aligned}(2.15)\quad & \int_{0}^{1}f\left(tu\left(x\right)+\left(1-t\right)u\left(y\right)\right)g\left(tv\left(x\right)+\left(1-t\right)v\left(y\right)\right)dt\\ &\le\frac{1}{3}\left[f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)+f\left(u\left(y\right)\right)g\left(v\left(y\right)\right)\right]\\ &\quad+\frac{1}{6}\left[f\left(u\left(x\right)\right)g\left(v\left(y\right)\right)+f\left(u\left(y\right)\right)g\left(v\left(x\right)\right)\right].\end{aligned}$$

Now we multiply (2.15) by $p\left(x\right)q\left(y\right)$, integrate over $[a,b]\times[a,b]$ and divide by PQ, where

$$P=\int_{a}^{b}p\left(x\right)dx,\quad Q=\int_{a}^{b}q\left(x\right)dx,$$

so we get

$$\begin{aligned}&\frac{1}{PQ}\int_{a}^{b}\int_{a}^{b}\int_{0}^{1}p\left(x\right)q\left(y\right)f\left(tu\left(x\right)+\left(1-t\right)u\left(y\right)\right)\\ &\qquad\times g\left(tv\left(x\right)+\left(1-t\right)v\left(y\right)\right)dtdxdy\\ &\le\frac{1}{3PQ}\left[\int_{a}^{b}p\left(x\right)f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)dx\int_{a}^{b}q\left(y\right)dy\right.\\ &\qquad\left.+\int_{a}^{b}q\left(y\right)f\left(u\left(y\right)\right)g\left(v\left(y\right)\right)dy\int_{a}^{b}p\left(x\right)dx\right]\\ &\qquad+\frac{1}{6PQ}\left[\int_{a}^{b}p\left(x\right)f\left(u\left(x\right)\right)dx\int_{a}^{b}q\left(y\right)g\left(v\left(y\right)\right)dy\right.\\ &\qquad\left.+\int_{a}^{b}p\left(y\right)f\left(u\left(y\right)\right)dy\int_{a}^{b}q\left(x\right)g\left(v\left(x\right)\right)dx\right]\\ &=\frac{1}{3PQ}\left[\int_{a}^{b}p\left(x\right)f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)dx\int_{a}^{b}q\left(y\right)dy\right.\\ &\qquad\left.+\int_{a}^{b}q\left(y\right)f\left(u\left(y\right)\right)g\left(v\left(y\right)\right)dy\int_{a}^{b}p\left(x\right)dx\right]\\ (2.16)\quad&\qquad+\frac{1}{3PQ}\int_{a}^{b}p\left(x\right)f\left(u\left(x\right)\right)dx\int_{a}^{b}q\left(y\right)g\left(v\left(y\right)\right)dy.\end{aligned}$$

This is the desired inequality (i).

To prove inequality (ii), in (2.12) and (2.13) we substitute $u\left(y\right)$ and $v\left(y\right)$ with $\overline{u}$ and $\overline{v}$ respectively.

Then we obtain

$$\begin{aligned}(2.17)\quad & f\left(tu\left(x\right)+\left(1-t\right)\overline{u}\right)g\left(tv\left(x\right)+\left(1-t\right)\overline{v}\right)\\ &\le t^{2}f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)+\left(1-t\right)^{2}f\left(\overline{u}\right)g\left(\overline{v}\right)\\ &\quad+t\left(1-t\right)\left[f\left(u\left(x\right)\right)g\left(\overline{v}\right)+f\left(\overline{u}\right)g\left(v\left(x\right)\right)\right].\end{aligned}$$

Integrating (2.17) in respect to t over $[0,1]$ we obtain

$$\begin{aligned}(2.18)\quad &\int_0^1 f\left(tu\left(x\right)+\left(1-t\right)\overline{u}\right)g\left(tv\left(x\right)+\left(1-t\right)\overline{v}\right)dt\\ &\leq \frac{1}{3}\left[f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)+f\left(\overline{u}\right)g\left(\overline{v}\right)\right]+\frac{1}{6}\left[f\left(u\left(x\right)\right)g\left(\overline{v}\right)+f\left(\overline{u}\right)g\left(v\left(x\right)\right)\right].\end{aligned}$$

Similarly as before, from (2.18) we get

$$\begin{aligned}&\frac{1}{P}\int_a^b\int_0^1 p\left(x\right)f\left(tu\left(x\right)+\left(1-t\right)\overline{u}\right)g\left(tv\left(x\right)+\left(1-t\right)\overline{v}\right)dtdx\\ &\leq \frac{1}{3P}\int_a^b p\left(x\right)f\left(u\left(x\right)\right)g\left(v\left(x\right)\right)dx+\frac{1}{3}f\left(\overline{u}\right)g\left(\overline{v}\right)\\ &\quad+\frac{1}{6P}\left[g\left(\overline{v}\right)\int_a^b p\left(x\right)f\left(u\left(x\right)\right)dx+f\left(\overline{u}\right)\int_a^b p\left(x\right)g\left(v\left(x\right)\right)dx\right].\end{aligned}$$

This completes the proof. □

Remark 2.4. If in (i) we put $u\left(x\right)=v\left(x\right)=x$ for all $x\in[a,b]$, it becomes

$$\begin{aligned}(2.19)\quad &\frac{1}{PQ}\int_a^b\int_a^b\int_0^1 p\left(x\right)q\left(y\right)f\left(tx+\left(1-t\right)y\right)g\left(tx+\left(1-t\right)y\right)dtdxdy\\ &\leq \frac{1}{3PQ}\left[Q\int_a^b p\left(x\right)f\left(x\right)g\left(x\right)dx+P\int_a^b q\left(y\right)f\left(y\right)g\left(y\right)dy\right]\\ &\quad+\frac{1}{3PQ}\int_a^b p\left(x\right)f\left(x\right)dx\int_a^b q\left(y\right)g\left(y\right)dy,\end{aligned}$$

so by using a generalization of Hadamard's inequality [1, p.138]

$$(2.20)\qquad f\left(\frac{a+b}{2}\right)\leq\frac{1}{P}\int_a^b p\left(x\right)f\left(x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2}$$

which holds for $p\left(a+t\right)=p\left(b-t\right),0\leq t\leq\frac{1}{2}\left(a+b\right)$, we obtain from (2.19) the following inequality

$$\begin{aligned}&\frac{1}{PQ}\int_a^b\int_a^b\int_0^1 p\left(x\right)q\left(y\right)f\left(tx+\left(1-t\right)y\right)g\left(tx+\left(1-t\right)y\right)dtdxdy\\ &\leq \frac{1}{3PQ}\left[Q\int_a^b p\left(x\right)f\left(x\right)g\left(x\right)dx+P\int_a^b q\left(y\right)f\left(y\right)g\left(y\right)dy\right]\\ &\quad+\frac{1}{3}\frac{f\left(a\right)+f\left(b\right)}{2}\frac{g\left(a\right)+g\left(b\right)}{2}\\ &=\frac{1}{3PQ}\left[Q\int_a^b p\left(x\right)f\left(x\right)g\left(x\right)dx+P\int_a^b q\left(y\right)f\left(y\right)g\left(y\right)dy\right]\\ (2.21)\quad &\quad+\frac{1}{12}\left[M\left(a,b\right)+N\left(a,b\right)\right].\end{aligned}$$

Now it is easy to observe that if $p\left(x\right)=q\left(x\right)=1$ for all $x\in[a,b]$ inequality (2.21) becomes the corrected Pachpatte's result (1.4).

If we do the same in (ii) we get

$$\frac{1}{P}\int_a^b\int_0^1 p(x)\, f\left(tx+(1-t)\frac{a+b}{2}\right) g\left(tx+(1-t)\frac{a+b}{2}\right) dtdx$$
$$\leq \frac{1}{3P}\int_a^b p(x)\, f(x)\, g(x)\, dx + \frac{1}{3} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)$$
$$+\frac{1}{6P}\left[g\left(\frac{a+b}{2}\right)\int_a^b p(x)\, f(x)\, dx + f\left(\frac{a+b}{2}\right)\int_a^b p(x)\, g(x)\, dx\right].$$

Using again (2.20) we obtain

$$\frac{1}{P}\int_a^b\int_0^1 p(x)\, f\left(tx+(1-t)\frac{a+b}{2}\right) g\left(tx+(1-t)\frac{a+b}{2}\right) dtdx$$
$$\leq \frac{1}{3P}\int_a^b p(x)\, f(x)\, g(x)\, dx + \frac{1}{3}\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}$$
$$+\frac{1}{6}\left[\frac{g(a)+g(b)}{2}\frac{f(a)+f(b)}{2} + \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}\right]$$
$$= \frac{1}{3P}\int_a^b p(x)\, f(x)\, g(x)\, dx + \frac{1}{6}\left(M(a,b)+N(a,b)\right)$$

Furthermore, in the case $p(x)=1$ for all $x\in[a,b]$ we get

$$\frac{1}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right) g\left(tx+(1-t)\frac{a+b}{2}\right) dtdx$$
$$\leq \frac{1}{3(b-a)}\int_a^b f(x)\, g(x)\, dx + \frac{1}{6}\left(M(a,b)+N(a,b)\right),$$

which is the corrected Pachpatte's result (1.5).

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