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INEQUALITIES FOR INSCRIBED SIMPLEX AND APPLICATIONS

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ABSTRACT. In this paper, we study the problem of geometric inequalities for the inscribed simplex of an n-dimensional simplex. An inequality for the inscribed simplex of a simplex is established. Applying it we get a generalization of n-dimensional Euler inequality and an inequality for the pedal simplex of a simplex.

Key words and phrases: Simplex, Inscribed simplex, Inradius, Circumradius, Inequality.

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1. MAIN RESULTS

Let σ_n be an n-dimensional simplex in the n-dimensional Euclidean space $E^n,$ V denote the volume of $\sigma_n,$ R and r the circumradius and inradius of σ_n, respectively. Let $A_0, A_1, \ldots, A_n$ be the vertices of σ_n, $a_{ij} = |A_iA_j|$ $(0 \le i < j \le n),$ F_i denote the area of the ith face $f_i = A_0 \cdots A_{i-1}A_{i+1} \cdots A_n$ (($n-1$)-dimensional simplex) of σ_n, points O and G be the circumcenter and barycenter of σ_n, respectively. For $i = 0, 1, \ldots, n$, let A'_i be an arbitrary interior point of the ith face f_i of σ_n. The n-dimensional simplex $\sigma'_n = A'_0A'_1 \cdots A'_n$ is called the inscribed simplex of the simplex σ_n. Let $a'_{ij} = |A'_iA'_j|$ $(0 \le i < j \le n),$ R' denote the circumradius of $\sigma'_n,$ P be an arbitrary interior point of $\sigma_n,$ P_i be the orthogonal projection of the point P on the ith face f_i of σ_n. The n-dimensional simplex $\sigma''_n = P_0P_1 \cdots P_n$ is called the pedal simplex of the point P with respect to the simplex σ_n [1] – [2], let V'' denote the volume of $\sigma''_n,$ R'' and r'' denote the circumradius and inradius of σ''_n, respectively. We note that the pedal simplex σ''_n is an inscribed simplex of the simplex σ_n. Our main results are following theorems.

Theorem 1.1. *Let σ'_n be an inscribed simplex of the simplex σ_n, then we have*

$$(R')^2 \left(R^2 - \overline{OG}^2\right)^{n-1} \ge n^{2(n-1)} r^{2n}, \tag{1.1}$$

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with equality if the simplex σ_n is regular and σ'_n is the tangent point simplex of σ_n.

Let T_i be the tangent point where the inscribed sphere of the simplex σ_n touches the ith face f_i of σ_n. The simplex $\overline{\sigma}_n = T_0T_1\cdots T_n$ is called the tangent point simplex of σ_n [3]. If we take $A'_i \equiv T_i$ $(i = 0, 1, \dots, n)$ in Theorem 1.1, then σ'_n and $\overline{\sigma}_n$ are the same and $R' = r$, we get a generalization of the n-dimensional Euler inequality [4] as follows.

Corollary 1.2. *For an n-dimensional simplex σ_n, we have*

$$R^2 \geq n^2r^2 + \overline{OG}^2, \tag{1.2}$$

with equality if the simplex σ_n is regular.

Inequality (1.2) improves the n-dimensional Euler inequality [5] as follows.

$$R \geq nr. \tag{1.3}$$

Theorem 1.3. *Let P be an interior point of the simplex σ_n, and σ'_n the pedal simplex of the point P with respect to σ_n, then*

$$R''R^{n-1} \geq n^{2n-1}\left(r''\right)^n, \tag{1.4}$$

with equality if the simplex σ_n is regular and σ''_n is the tangent point simplex of σ_n.

2. Some Lemma and Proofs of Theorems

To prove the theorems stated above, we need some lemmas as follows.

Lemma 2.1. *Let σ'_n be an inscribed simplex of the n-dimensional simplex σ_n, then we have*

$$\left(\sum_{0\leq i<j\leq n}\left(a'_{ij}\right)^2\right)\left(\sum_{i=0}^{n}F_i^2\right) \geq n^2(n+1)V^2, \tag{2.1}$$

with equality if the simplex σ_n is regular and σ'_n is the tangent point simplex of σ_n.

Proof. Let B be an interior point of the simplex σ_n, and $(\lambda_0, \lambda_1, \dots, \lambda_n)$ the barycentric coordinates of the point B with respect to coordinate simplex σ_n. Here $\lambda_i = V_iV^{-1}\,(i = 0, 1, \dots, n)$, V_i is the volume of the simplex $\sigma_n(i) = BA_0\cdots A_{i-1}A_{i+1}\cdots A_n$ and $\sum_{i=0}^{n}\lambda_i = 1$. Let Q be an arbitrary point in E^n, then

$$\overrightarrow{QB} = \sum_{i=0}^{n}\lambda_i\overrightarrow{QA_i}.$$

From this we have

$$\sum_{i=0}^{n}\lambda_i\overrightarrow{BA_i} = \sum_{i=0}^{n}\lambda_i\left(\overrightarrow{QA_i} - \overrightarrow{QB}\right) = \overrightarrow{0},$$

$$\begin{aligned}\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{QA_i}\right)^2 &= \sum_{i=0}^{n}\lambda_i\left(\overrightarrow{QB} + \overrightarrow{BA_i}\right)^2 \\ &= \sum_{i=0}^{n}\lambda_i\overrightarrow{QB}^2 + 2\overrightarrow{QB}\cdot\sum_{i=0}^{n}\lambda_i\overrightarrow{BA_i} + \sum_{i=0}^{n}\lambda_i\left(\overrightarrow{BA_i}\right)^2 \\ &= \overrightarrow{QB}^2 + \sum_{i=0}^{n}\lambda_i\left(\overrightarrow{BA_i}\right)^2.\end{aligned} \tag{2.2}$$

For $j = 0, 1, \ldots, n$, taking $Q \equiv A_j$ in (2.2) we get

$$\sum_{i=0}^{n} \lambda_i \lambda_j \left(\overrightarrow{A_i A_j}\right)^2 = \lambda_j \left(\overrightarrow{BA_j}\right)^2 + \lambda_j \sum_{i=0}^{n} \lambda_i \left(\overrightarrow{BA_i}\right)^2 \quad (j = 0, 1, \ldots, n). \tag{2.3}$$

Adding up these equalities in (2.3) and noting that $\sum_{j=0}^{n} \lambda_j = 1$, we get

$$\sum_{0 \le i < j \le n} \lambda_i \lambda_j \left(\overrightarrow{A_i A_j}\right)^2 = \sum_{i=0}^{n} \lambda_i \left(\overrightarrow{BA_i}\right)^2. \tag{2.4}$$

For any real numbers $x_i > 0 \ (i = 0, 1, \ldots, n)$ and an inscribed simplex $\sigma'_n = A'_0 A'_1 \cdots A'_n$ of σ_n, we take an interior point B' of σ'_n such that $(\lambda'_0, \lambda'_1, \ldots, \lambda'_n)$ is the barycentric coordinates of the point B' with respect to coordinate simplex σ'_n, here $\lambda'_i = x_i \Big/ \sum_{j=0}^{n} x_j \ (i = 0, 1, \ldots, n)$. Using equality (2.4) we have

$$\sum_{0 \le i < j \le n} \lambda'_i \lambda'_j \left(a'_{ij}\right)^2 = \sum_{i=0}^{n} \lambda'_i \left(\overrightarrow{B' A'_i}\right)^2,$$

i.e.

$$\sum_{0 \le i < j \le n} x_i x_j \left(a'_{ij}\right)^2 = \left(\sum_{i=0}^{n} x_i\right) \left(\sum_{i=0}^{n} x_i \left(\overrightarrow{B' A'_i}\right)^2\right). \tag{2.5}$$

Since B' is an interior point of σ'_n and σ'_n is an inscribed simplex of σ_n, so B' is an interior point of σ_n. Let the point Q_i be the orthogonal projection of the point B' on the ith face f_i of σ_n, then

$$\sum_{i=0}^{n} x_i \left(\overrightarrow{B' A'_i}\right)^2 \ge \sum_{i=0}^{n} x_i \left(\overrightarrow{B' Q_i}\right)^2. \tag{2.6}$$

Equality in (2.6) holds if and only if $Q_i \equiv A'_i \ (i = 0, 1, \ldots, n)$. In addition, we have

$$\sum_{i=0}^{n} \left|\overrightarrow{B' Q_i}\right| F_i = nV. \tag{2.7}$$

By the Cauchy's inequality and (2.7) we have

$$\left(\sum_{i=0}^{n} x_i \overrightarrow{B' Q_i}^2\right) \left(\sum_{i=0}^{n} x_i^{-1} F_i^2\right) \ge \left(\sum_{i=0}^{n} \left|\overrightarrow{B' Q_i}\right| \cdot F_i\right)^2 = (nV)^2. \tag{2.8}$$

Using (2.5), (2.6) and (2.8), we get

$$\left(\sum_{0 \le i < j \le n} x_i x_j \left(a'_{ij}\right)^2\right) \left(\sum_{i=0}^{n} x^{-1} F_i^2\right) \ge n^2 \left(\sum_{i=0}^{n} x_i\right) V^2. \tag{2.9}$$

Taking $x_0 = x_1 = \cdots = x_n = 1$ in (2.9), we get inequality (2.1). It is easy to prove that equality in (2.1) holds if the simplex σ_n is regular and σ'_n is the tangent point simplex of σ_n. □

Lemma 2.2 ([1, 6]). *For the n-dimensional simplex σ_n, we have*

$$\sum_{i=0}^{n} F_i^2 \le [n^{n-4} (n!)^2 (n+1)^{n-2}]^{-1} \left(\sum_{0 \le i < j \le n} a_{ij}^2\right), \tag{2.10}$$

with equality if the simplex σ_n is regular.

Lemma 2.3 ([2]). *Let P be an interior point of the simplex σ, σ_n'' the pedal simplex of the point P with respect to σ_n, then*

$$V \geq n^n V'', \tag{2.11}$$

with equality if the simplex σ_n is regular.

Lemma 2.4 ([1]). *For the n-dimensional simplex σ_n, we have*

$$V \geq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} r^n, \tag{2.12}$$

with equality if the simplex σ_n is regular.

Lemma 2.5 ([4]). *For the n-dimensional simplex σ_n, we have*

$$\sum_{0\leq i<j\leq n} a_{ij}^2 = (n+1)^2 \left(R^2 - \overline{OG}^2 \right). \tag{2.13}$$

Here O and G are the circumcenter and barycenter of the simplex σ_n, respectively.

Proof of Theorem 1.1. Using inequalities (2.1) and (2.10), we get

$$\left(\sum_{0\leq i<j\leq n} \left(a_{ij}'\right)^2 \right) \left(\sum_{0\leq i<j\leq n} a_{ij}^2 \right)^{n-1} \geq n^{n-2}(n!)^2(n+1)^{n-1}V^2. \tag{2.14}$$

By Lemma 2.5 we have

$$\sum_{0\leq i<j\leq n} \left(a_{ij}'\right)^2 \leq (n+1)^2 \left(R'\right)^2. \tag{2.15}$$

From (2.13), (2.14) and (2.15) we get

$$\left(R'\right)^2 \left(R^2 - \overline{OG}^2 \right)^{n-1} \geq \frac{n^{n-1}(n!)^2}{(n+1)^{n+1}} V^2. \tag{2.16}$$

Using inequalities (2.16) and (2.12), we get inequality (1.1). It is easy to prove that equality in (1.1) holds if the simplex σ_n is regular and σ_n' is the tangent point simplex of σ_n. □

Proof of Theorem 1.3. Since the pedal simplex σ_n'' is an inscribed simplex of the simplex σ_n, thus inequality (2.16) holds for the pedal simplex σ_n'', i.e.

$$\left(R''\right)^2 \left(R^2 - \overline{OG}^2 \right)^{n-1} \geq \frac{n^{n-2}(n!)^2}{(n+1)^{n+1}} V^2. \tag{2.17}$$

Using inequalities (2.17) and (2.11), we get

$$\left(R''\right)^2 R^{2(n-1)} \geq \left(R''\right)^2 \left(R^2 - \overline{OG}^2 \right)^{n-1} \geq \frac{n^{3n-2}(n!)^2}{(n+1)^{n+1}} \left(V''\right)^2 \tag{2.18}$$

By Lemma 2.4 we have

$$V'' \geq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} \left(r''\right)^n. \tag{2.19}$$

From (2.18) and (2.19) we obtain inequality (1.4). It is easy to prove that equality in (1.4) holds if the simplex σ_n is regular and σ_n'' is the tangent point simplex of σ_n. □

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