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LINEAR ELLIPTIC EQUATIONS AND GAUSS MEASURE

G. DI BLASIO

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"
UNIVERSITÀ DEGLI STUDI DI NAPOLI "FEDERICO II"
COMPLESSO MONTE S. ANGELO - VIA CINTIA. NAPOLI-ITALY
giuseppina.diblasio@dma.unina.it

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ABSTRACT. In this paper we study a Dirichlet problem relative to a linear elliptic equation with lower-order terms, whose ellipticity condition is given in terms of the function $\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp(-|x|^2/2)$, the density in the Gaussian measure. Using the notion of rearrangement with respect to the Gauss measure, we prove a comparison result with a problem of the same type defined in a half space, with data depending only on the first variable.

Key words and phrases: Linear elliptic equation, Comparison theorem, Rearrangements of functions, Gauss measure.

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1. INTRODUCTION

The object of this paper is to give comparison results for the solution of the problem

$$\begin{cases} -\left(a_{ij}(x)u_{x_i}\right)_{x_j} + b_i(x)u_{x_i} + c(x)u = f(x)\varphi(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-|x|^2/2\right)$ is the density in the Gaussian measure, Ω is an open set of $\mathbb{R}^n$ $(n \geq 2)$ which has Gauss measure less than one, $a_{ij}(x)$, $b_i(x)$ and $c(x)$, $i, j = 1, \ldots, n$, are measurable functions on Ω such that

(i) $a_{ij}(x)\xi_i\xi_j \geq \varphi(x)|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, a.e.\ x \in \Omega,$
(ii) $\left(\sum b_i^2\right)^{\frac{1}{2}} \leq \varphi(x)B,$
(iii) $c(x) \geq c_0(x)\varphi(x), \qquad c_0(x) \in L^\infty(\Omega),$

and f is taken in a suitable weighted L^p space in order to guarantee the existence of a solution u of the problem (1.1).

We recall that $u \in H_0^1(\varphi, \Omega)$ is a weak solution of the problem (1.1), if

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$$(1.2)\quad \int_\Omega (a_{ij}(x)u_{x_i}\psi_{x_j} + b_i(x)u_{x_i}\psi + c(x)u(x)\psi)dx = \int_\Omega f(x)\varphi(x)\psi dx, \quad \forall \psi \in H_0^1(\varphi,\Omega).$$

Let us observe that the operator in (1.1) is uniformly elliptic if Ω is bounded. It is well known that when Ω is bounded, comparison results for elliptic problems have been obtained via Schwarz symmetrization, for a simpler problem which is defined in a ball and has spherically symmetric data (see for example [1], [2], [4], [3], [6], [17], [20], [19], [18], [21]).

In our case Ω can be not bounded and ellipticity condition (i) is given in terms of the density in the Gaussian measure. We consider solutions of problem (1.1) in the weighted Sobolev space $H_0^1(\varphi,\Omega)$ (see § 2) and we compare the solution of the problem (1.1) with the solution of a problem in which the data depend only on the first variable and the domain is a half-space which has the same Gauss measure as Ω.

More precisely let $\Omega^\star = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$, and let $f^\star(x)$ be the rearrangement with respect to the Gauss measure of the function $f(x)$ (see § 2 for the definition). To give an example of the obtained results we consider the case $c_0(x) = 0$.

Let $w(x) = w(x_1)$ be the solution of

$$(1.3)\quad \begin{cases} -\left(w_{x_1}\varphi(x)\right)_{x_1} - Bw_{x_1}\varphi(x) = f^\star(x_1)\varphi(x) & \text{in } \Omega^\star, \\ w = 0 & \text{on } \partial\Omega^\star. \end{cases}$$

We prove that the pointwise comparison

$$(1.4)\quad u^\star(x) = u^\star(x_1) \le w(x_1) = w^\star(x) \qquad \text{for a.e } x = (x_1, x_2, \dots, x_n) \in \Omega^\star$$

holds, where $u^\star$ and $w^\star$ are the rearrangements with respect to the Gauss measure of u and w respectively. In this case the comparison also gives an explicit estimate of $u^\star$ that is also a condition for the existence of the solution for the problem (1.1) in terms of the existence of the solution of (1.3).

If $c_0(x) \neq 0$, a comparison with a problem which also takes the term "$c(x)\,u(x)$" into account can be found. In this case, depending on the sign of $c_0(x)$, pointwise comparison (1.4) is false in general, but a comparison between the concentrations can be found (see Theorem 3.4).

In the proofs of our results we use tools similar to the classical methods based on the isoperimetric inequality and Schwarz symmetrization (see [2]). In our case a fundamental rule is played by the isoperimetric inequality with respect to the Gauss measure.

The problem (1.1) has been studied using rearrangement with respect to the Gauss measure in [7], when $b_i(x) = c(x) = 0$.

Existence results for weak solutions of problem (1.1) can be obtained for example via the Lax Milgram theorem when (i) and (ii) hold and $\frac{a_{ij}(x)}{\varphi(x)}, \frac{c(x)}{\varphi(x)} \in L^\infty(\Omega)$ and $f(x) \in L^2(\varphi,\Omega)$ (see also [15], [22]).

2. Notations and Preliminary Results

In this section we recall some definitions and results which will be useful in what follows. Let Ω be an open set of $\mathbb{R}^n$ and let γ_n be the $n-$dimensional Gauss measure on $\mathbb{R}^n$ defined by

$$\gamma_n(dx) = \varphi(x)\,dx = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx, \qquad x \in \mathbb{R}^n$$

normalized by $\gamma_n(\mathbb{R}^n) = 1$.

One of the main tools used to prove the comparison result is the isoperimetric inequality with respect to the Gauss measure, to recall this result we define the perimeter with respect to the Gauss measure as (see [13])

$$P(E) = (2\pi)^{-\frac{n}{2}} \int_{\partial E} \exp\left(-\frac{|x|^2}{2}\right) \mathcal{H}_{n-1}(dx),$$

where E is a $(n-1)-$rectifiable set and $\mathcal{H}_{n-1}$ denotes the $(n-1)-$dimensional Hausdorff measure. For all $\lambda \in \mathbb{R}$, we denote by $H(\xi, \lambda)$ the half-space defined by

$$H(\xi, \lambda) = \{x \in \mathbb{R}^n : (x, \xi) > \lambda\}$$

and we set $H(\xi, \lambda) = \mathbb{R}^n$ if $\lambda = -\infty$ and $H(\xi, \lambda) = \emptyset$ if $\lambda = +\infty$.

It is well known (see [11]) that among all measurable sets of $\mathbb{R}^n$ with prescribed Gauss measure, the half-spaces take the smallest perimeter. In other words, the half-spaces are extremal in the isoperimetric problem for the Gauss measure. From an appropriate form of the Brunn - Minkowski inequality, the isoperimetric inequality follows (see [8], [11], [10], [12])

$$P(E) \geq P(H(\xi, \lambda))$$

for all subsets $E \subset \mathbb{R}^n$ such that $\gamma_n(E) = \gamma_n(H(\xi, \lambda))$. For the sake of simplicity we shall consider $\xi = (1, 0, \ldots, 0)$.

Now we consider the notion of rearrangement with respect to the Gauss measure. If u is a measurable function in Ω, we define the distribution function of u, denoted by μ, as the Gauss measure of the level set of u, i.e.

$$\mu : t \in [0, \infty[\longrightarrow \mu(t) \in [0, 1],$$

where

$$\mu(t) = \gamma_n(\{x \in \Omega : |u| > t\}).$$

We denote by u^* the decreasing rearrangement of u (with respect to the Gauss measure), i.e.

$$u^*(s) = \inf\{t \geq 0 : \mu(t) \leq s\}, \qquad s \in]0, 1],$$

the functions μ and u^* are decreasing and right - continuous. Moreover the increasing rearrangement of u is the function defined as follows

$$u_*(s) = u^*(\gamma_n(\Omega) - s), \qquad s \in [0, 1[.$$

Finally we define the rearrangement of u with respect to the Gauss measure, denoted by $u^\star$, as the function whose level sets are half-spaces having the same Gauss measure of the level sets of u. More precisely we define

$$\Phi(\tau) = \gamma_n(\{x \in \mathbb{R}^n : x_1 > \tau\}) = \frac{1}{\sqrt{2\pi}} \int_\tau^{+\infty} \exp\left(-\frac{t^2}{2}\right) d\tau$$

for all $\tau \in \mathbb{R}$. Then $u^\star$ is a map from $\Omega^\star$ into $[0, +\infty[$ defined by

$$u^\star(x) = u^*(\Phi(x_1)),$$

where $\Omega^\star = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$.

For statements about the properties of rearrangement with respect to a positive measure see, for example, [9], [20], [16].

We recall that if $f(x)$, $g(x)$ are measurable functions, a Hardy type inequality (see [9])

$$\int_\Omega |f(x)\, g(x)| \;\gamma_n(dx) \leq \int_{\Omega^\star} f^\star(x)\, g^\star(x) \;\gamma_n(dx) = \int_0^{\gamma_n(\Omega)} f^*(s)\, g^*(s) \; ds$$

and a Polya-Szëgo inequality for a Lipschitz continuous function $u(x)$ (see [20])

$$\int_{R^n} |\nabla u^\star| \ \gamma_n(dx) \le \int_{R^n} |\nabla u| \ \gamma_n(dx) \tag{2.1}$$

holds. Moreover we have

$$\int_\Omega |f(x)|^p \gamma_n(dx) = \int_{\Omega^\star} |f^\star(x)|^p \gamma_n(dx) = \int_0^{\gamma_n(\Omega)} f^*(s)^p \, ds,$$

that is, the L^p weighted norm is invariant under the rearrangement.

Now define the weighted Sobolev space $H_0^1(\varphi,\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{H_0^1(\varphi,\Omega)} = \left(\int_\Omega |\nabla u(x)|^2 \, d\gamma_n(x)\right)^{\frac{1}{2}}. \tag{2.2}$$

We remark that the following Poincarè type inequality can be proved.

Proposition 2.1. *Let Ω be a open subset of $\mathbb{R}^n$ with $\gamma_n(\Omega) < 1$. For each function $f \in H_0^1(\varphi,\Omega)$ we have*

$$\|f\|_{L^2(\varphi,\Omega)} \le C \|\nabla f\|_{L^2(\varphi,\Omega)}, \tag{2.3}$$

where C is a constant depending on n and Ω.

Proof. By (2.1) we have the following inequality

$$\begin{aligned} \frac{\|f\|^2_{L^2_{(\varphi,\Omega)}}}{\|\nabla f\|^2_{L^2_{(\varphi,\Omega)}}} &\le \frac{\|f^\star\|^2_{L^2_{(\varphi,\Omega^\star)}}}{\|\nabla f^\star\|^2_{L^2_{(\varphi,\Omega^\star)}}} \\ &= \frac{\int_\lambda^{+\infty} f^\star(x_1)^2 \exp\left(-\frac{x_1^2}{2}\right) dx_1}{\int_\lambda^{+\infty} \left(\frac{d}{dx_1} f^\star(x_1)\right)^2 \exp\left(-\frac{x_1^2}{2}\right) dx_1}, \end{aligned} \tag{2.4}$$

where λ is such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$ with $\Omega^\star = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$. The ratio in (2.4) is bounded (see e.g. [14, Theorem 1.3.1./2]). □

The previous Poincaré type inequality ensures the equivalence between the norm (2.2) and the following one

$$\|u\|_{H_0^1(\varphi,\Omega)} = \left(\int_\Omega |u(x)|^2 \, d\gamma_n(x)\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla u(x)|^2 \, d\gamma_n(x)\right)^{\frac{1}{2}}.$$

Now, we recall the following lemmas which will be useful in the following.

Lemma 2.2. *Let $f(x), g(x)$ be measurable, positive functions such that*

$$\int_0^\alpha f(x)\, dx \le \int_0^\alpha g(x)\, dx, \qquad \alpha \in [0,a].$$

If $h(x) \ge 0$ is a decreasing function then

$$\int_0^\alpha f(x)\, h(x) dx \le \int_0^\alpha g(x)\, h(x) dx, \qquad \alpha \in [0,a].$$

Lemma 2.3. *Let z be a bounded function, K a nonnegative integrable function, and ψ a function with bounded variation that vanishes at $+\infty$. If*

$$z(t) \le \int_t^\infty K(s)\, z(s)\, ds + \psi(t)$$

for almost every $t > 0$, then

$$z(t) \leq \int_t^\infty \exp\left[\int_t^s K(\tau)\, d\tau\right] [-d\psi(s)]$$

for almost every $t > 0$.

3. Comparison Results

In this section we prove some comparison results for the solution of the problem (1.1). We consider first the case $c_0(x) = 0$ and then the case $c_0(x) \neq 0$.

Theorem 3.1. *Let Ω be an open set of $\mathbb{R}^n$ with $\gamma_n(\Omega) < 1$ and let $u \in H_0^1(\varphi, \Omega)$ be the solution of (1.1) with the assumptions (i), (ii) and (iii). Let $c_0(x) = 0$ and*

$$\int_\lambda^{+\infty} \exp\left(\frac{\tau^2}{2}\right) \left(\int_\tau^{+\infty} f^\star(\sigma) \exp\left(B(\sigma - \tau) - \frac{\sigma^2}{2}\right) d\sigma\right)^2 d\tau < +\infty, \tag{3.1}$$

where λ is such that $\Omega^\star = \{x_1 > \lambda\}$. Then we have

$$u^\star(x_1) \leq w(x) \qquad \textit{for a.e. } x \in \Omega^\star \tag{3.2}$$

and

$$\int_\Omega |\nabla u|^q \varphi(x)\, dx \leq \int_{\Omega^\star} |\nabla w|^q \varphi(x)\, dx \qquad \textit{for all } 0 < q \leq 2, \tag{3.3}$$

where

$$w(x) = w(x_1) = \int_\lambda^{x_1} \exp\left(\frac{\tau^2}{2}\right) \left(\int_\tau^{+\infty} f^\star(\sigma) \exp\left(B(\sigma - \tau) - \frac{\sigma^2}{2}\right) d\sigma\right) d\tau$$

is the solution of the problem (1.3).

Remark 3.2. Condition (3.1) ensures the existence of a solution $w(x_1) = w^\star(x) \in H_0^1(\varphi, \Omega)$ of (1.3). It is satisfied for a wide class of functions, for instance for functions f such that

$$f^\star \leq \exp\left(-B(\sigma - \tau) + \frac{\sigma^2}{4}\right) (1 + |\tau|)^{\frac{1}{2} - \epsilon} \qquad \forall \tau \geq \lambda$$

for some constants $C > 0$, $\epsilon > 0$.

Remark 3.3. The assumption $\gamma_n(\Omega) < 1$ is made to guarantee that the Poincarè type inequality (2.3) holds.

Proof. If we choose in (1.2), for $h > 0$ and $t \in [0, \sup |u|[$

$$\psi(x) = \begin{cases} h \operatorname{sgn} u & \text{if } |u| > t + h, \\ (|u| - t) \operatorname{sgn} u & \text{if } t < |u| \leq t + h, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\begin{aligned}\frac{1}{h}\int_{t<|u|\le t+h} & a_{ij}(x)u_{x_i}u_{x_j}dx \\ & + \frac{1}{h}\int_{t<|u|\le t+h} b_i(x)u_{x_i}\left(|u|-t\right)\operatorname{sgn} u dx + \int_{|u|>t+h} b_i(x)u_{x_i}\operatorname{sgn} u dx \\ & + \frac{1}{h}\int_{t<|u|\le t+h} c(x)\left(|u|-t\right)\operatorname{sgn} u dx + \int_{|u|>t+h} c(x)u\left(x\right)\operatorname{sgn} u dx \\ & = \frac{1}{h}\int_{t<|u|\le t+h} f(x)\varphi(x)\left(|u|-t\right)\operatorname{sgn} u dx + \int_{|u|>t+h} f(x)\varphi(x)\operatorname{sgn} u dx.\end{aligned}$$

By the ellipticity condition (i), (ii) and (iii) and letting h go to 0, we obtain

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\varphi(x)dx \le B\int_{|u|>t}|\nabla u|\,\varphi\left(x\right)dx + \int_{|u|>t} f(x)\operatorname{sgn} u\varphi(x)dx \tag{3.4}$$

On the other hand, the coarea formula (see [13]) and the isoperimetric inequality with respect to the Gauss measure give

$$\begin{aligned}-\frac{d}{dt}\int_{|u|>t}|\nabla u|\,\varphi(x)dx &\ge \int_{\partial\{|u|>t\}^\star}\varphi(x)H_{n-1}(dx) \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\Phi^{-1}\left(\mu\left(t\right)\right)^2}{2}\right),\end{aligned} \tag{3.5}$$

where $\{|u|>t\}^\star$ is the half space having Gauss measure $\mu(t)$.

Then using (3.5) and the Hölder inequality we obtain

$$1 \le (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}\left(\mu\left(t\right)\right)^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\varphi(x)\,dx\right)^{\frac{1}{2}}. \tag{3.6}$$

Using (3.6) and the Hölder inequality, (3.4) becomes

$$\begin{aligned}-\frac{d}{dt}&\int_{|u|>t}|\nabla u|^2\varphi(x)dx \\ &\le B\left(2\pi\right)^{\frac{1}{2}}\int_t^\infty\left(-\frac{d}{ds}\int_{|u|>s}|\nabla u|^2\varphi(x)dx\right)\exp\left(\frac{\Phi^{-1}\left(\mu\left(s\right)\right)^2}{2}\right) \\ &\qquad\qquad \times\left(-\mu'\left(s\right)\right)ds + \int_0^{\mu(t)} f^*\left(s\right)ds.\end{aligned}$$

By Gronwall's Lemma 2.3 we obtain

$$\begin{aligned}-\frac{d}{dt}&\int_{|u|>t}|\nabla u|^2\varphi(x)dx \\ &\le \int_0^{\mu(t)}\exp\left[B\left(2\pi\right)^{\frac{1}{2}}\int_r^{\mu(t)}\exp\left(\frac{\Phi^{-1}\left(\tau\right)^2}{2}\right)d\tau\right]f^*\left(r\right)dr \\ &= \int_0^{\mu(t)}\exp\left[B(\Phi^{-1}\left(r\right)-\Phi^{-1}\left(\mu\left(t\right)\right))\right]f^*\left(r\right)dr.\end{aligned} \tag{3.7}$$

Using again (3.6) we have

$$1 \le 2\pi \exp\left[\Phi^{-1}(\mu(t))^2\right](-\mu'(t)) \int_0^{\mu(t)} \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(\mu(t)))\right] f^*(r)\, dr. \tag{3.8}$$

Then using (3.8) and integrating between 0 and t, (3.7) becomes

$$t \le 2\pi \int_{\mu(t)}^{\gamma_n(\Omega)} \exp\left[\Phi^{-1}(\sigma)^2\right] \int_0^{\sigma} \exp\left[B(\Phi^{-1}(s) - \Phi^{-1}(\sigma))\right] f^*(s)\, ds d\sigma,$$

which, putting $\mu(t) = s$ and $s = \Phi(x_1)$, gives

$$\begin{aligned} u^\star(x) &= u^\star(x_1) \\ &\le \int_\lambda^{x_1} \exp\left(\frac{\tau^2}{2}\right)\left(\int_\tau^{+\infty} f^\star(\sigma)\exp\left(B(\sigma-\tau) - \frac{\sigma^2}{2}\right) d\sigma\right) d\tau, \end{aligned} \tag{3.9}$$

where λ is such that $\gamma_n(\Omega^\star) = \gamma_n(\Omega)$ with $\Omega^\star = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$. This completes the proof of (3.2) observing that the right-hand side of (3.9) is the solution of (1.3).

Let us prove now (3.3). Using the Hölder inequality and (3.7) we have

$$\begin{aligned} &-\frac{d}{dt}\int_{|u|>t} |\nabla u|^q \varphi(x)\, dx \\ &\qquad \le (-\mu'(t))^{1-\frac{q}{2}} \left(\int_0^{\mu(t)} \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(\mu(t)))\right] f^*(r)\, dr\right)^{\frac{q}{2}}. \end{aligned} \tag{3.10}$$

By (3.6) we have

$$(-\mu'(t))^{-\frac{q}{2}} (2\pi)^{-\frac{q}{2}} \exp\left[-\left(\frac{q}{2}\right)\Phi^{-1}(\mu(t))^2\right] \le \left(-\frac{d}{dt}\int_{|u|>t} |\nabla u|^2 \varphi(x)\, dx\right)^{\frac{q}{2}}. \tag{3.11}$$

Using (3.11), (3.7) and integrating between 0 and $+\infty$, (3.10) becomes

$$\begin{aligned} \int_\Omega |\nabla u|^q \varphi(x)\, dx &\le (2\pi)^{\frac{q}{2}} \int_0^{\gamma_n(\Omega)} \left(\int_0^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] f^*(r)\, dr\right)^q \\ &\qquad \times \exp\left[\left(\frac{q}{2}\right)\Phi^{-1}(s)^2\right] ds \\ &= (2\pi)^{\frac{1}{2}} \int_\lambda^{+\infty} \left(\int_\tau^{+\infty} f^\star(\sigma)\exp\left(B(\sigma-\tau) - \frac{\sigma^2}{2}\right) d\sigma\right)^q \\ &\qquad \times \exp\left[\frac{q}{2}\tau^2 - \frac{\tau^2}{2}\right] d\tau, \end{aligned}$$

that is, (3.3). □

In the next theorem we consider the case $c_0(x) \neq 0$ and we compare problem (1.1) with a "symmetrized" problem which also takes account of the influence of the term "$c(x)\, u(x)$".

Theorem 3.4. *Let Ω be an open set of $\mathbb{R}^n$ with $\gamma_n(\Omega) < 1$ and let $u \in H_0^1(\varphi, \Omega)$ be a solution of (1.1) with the assumptions (i), (ii) and (iii). Moreover, let*

$$c_0^+(x) = \max\{c_0(x), 0\}, c_0^-(x) = \max\{-c_0(x), 0\}, c_{0\star}^+(x) = c_{0*}^+(\Phi(x_1)).$$

We will assume that the problem

$$(3.12)\qquad \begin{cases} -\left(w_{x_1}\varphi\left(x\right)\right)_{x_1} - Bw_{x_1}\varphi\left(x\right) \\ \qquad\qquad + \left[c_{0\star}^{+}(x_1) - c_0^{-\star}(x_1)\right] w\varphi(x) = f^{\star}(x_1)\varphi\left(x\right) & \text{in } \Omega^{\star}, \\ w = 0 & \text{on } \partial\Omega^{\star}, \end{cases}$$

has a solution $w(x) = w^{\star}(x_1)$. *Then*

$$u^{*}\left(s\right) \leq w^{*}\left(s\right)$$

holds in $[0, s_1']$, *where* $s_1' = \inf\left\{s : c_{0*}^{+}\left(s\right) > 0\right\}$, *and*

$$\int_0^s u^{*}\left(r\right) dr \leq \int_0^s w^{*}\left(r\right) dr$$

holds in $]s_1', \gamma_n\left(\Omega\right)]$.

Proof. Using (iii), the Schwartz inequality and the Hardy inequality we have,

$$(3.13)\qquad \begin{aligned} -\int_{|u|>t} c(x)\left|u\right| dx &\leq -\int_{|u|>t} c_0(x)\left|u\right|\varphi(x)dx \\ &\leq -\int_0^{\mu(t)} \left[c_{0*}^{+}(s) - c_0^{-*}(s)\right] u^{*}(s)ds, \end{aligned}$$

where $c_{0*}^{+}(s)$ is the increasing rearrangement of $c_0^{+}(x)$ and $\mu(t)$ is the distribution function of $u(x)$. Proceeding as in Theorem 3.1 and using (3.13) we obtain

$$(3.14)\qquad \begin{aligned} &-\frac{d}{dt}\int_{|u|>t} \left|\nabla u\right|^2 \varphi(x)dx \\ &\leq B\int_t^{\infty}\left(-\frac{d}{ds}\int_{|u|>s}\left|\nabla u\right|^2\varphi(x)dx\right)^{\frac{1}{2}}(-\mu'(s))^{\frac{1}{2}}ds \\ &\qquad - \int_0^{\mu(t)}\left[c_{0*}^{+}(s) - c_0^{-*}(s)\right] u^{*}(s)ds + \int_0^{\mu(t)} f^{*}(s)ds. \end{aligned}$$

By (3.6) we have

$$(3.15)\qquad \frac{1}{-\mu'\left(t\right)} \leq 2\pi\exp\left[\Phi^{-1}\left(s\right)^2\right]\left(-\frac{d}{dt}\int_{|u|>t}\left|\nabla u\right|^2\varphi(x)dx\right).$$

Following the same steps as in Theorem 3.1 and using (3.14), (3.15) becomes

$$(3.16)\qquad \begin{aligned} \left(-u^{*}\right)'\left(s\right) \leq 2\pi\exp\left[\Phi^{-1}\left(s\right)^2\right]\int_0^s \exp\left[B(\Phi^{-1}\left(r\right) - \Phi^{-1}\left(s\right))\right] \\ \times\left[f^{*}\left(r\right) + \left[c_0^{-*}\left(r\right) - c_{0*}^{+}\left(r\right)\right]u^{*}\left(r\right)\right]dr. \end{aligned}$$

Now if we consider the problem (3.12) we can proceed in the same way except that the inequalities should be replaced by equalities.

Then if we call $w(x) = w^{\star}(x)$ the solution of the problem (3.12), we obtain the following equality

$$(3.17)\qquad \begin{aligned} \left(-w^{*}\right)'\left(s\right) = 2\pi\exp\left[\Phi^{-1}\left(s\right)^2\right]\int_0^s \exp\left[B(\Phi^{-1}\left(r\right) - \Phi^{-1}\left(s\right))\right] \\ \times\left[f^{*}\left(r\right) + \left[c_0^{-*}\left(r\right) - c_{0*}^{+}\left(r\right)\right]w^{*}\left(r\right)\right]dr. \end{aligned}$$

To prove the comparison result, we put $v(s) = u^*(s) - w^*(s)$. By (3.16) and (3.17) we have

$$(3.18) \quad (-v)'(s) \le 2\pi \exp\left[\Phi^{-1}(s)^2\right] \int_0^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] \times [c_0^{-*}(r) - c_{0*}^{+}(r)] v(r)\, dr.$$

Let us suppose $c_0^{-*}(x)\,,\ c_{0*}^{+}(x) \neq 0$ and let us put $s_1' = \inf\left\{s : c_{0*}^{+}(s) > 0\right\}$ and $s_0' = \sup\left\{s : c_0^{-*}(s) > 0\right\}$. We write

$$V_1(s) = \int_0^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] c_0^{-*}(r)\, v(r) dr, \qquad s \in [0, s_0']$$
$$V_2(s) = \int_{s_1'}^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] c_{0*}^{+}(r)\, v(r) dr, \qquad s \in \left]s_1', \gamma_n(\Omega)\right].$$

We assume initially that $c_0^{-*}(s)$ is continuous at s_0', we have to prove, now, that $V_1(s) \le 0$. Arguing as in [1] we observe that the existence of a solution $w(x) = w^\star(x)$ of (3.12), that implies (3.17), guarantees that the problem:

$$\begin{cases} -\left(c_0^{-*}(s)^{-1} Z'(s)\right)' = (2\pi) \exp\left[\Phi^{-1}(s)^2\right] Z(s) + 2\pi \exp\left[\Phi^{-1}(s)^2\right] \\ \qquad\qquad \times \int_0^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] f^*(r)\, dr, \\ Z(0) = Z'(s_0') = 0, \end{cases}$$

has the following positive solution

$$Z(s) = \int_0^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] c_0^{-*}(r)\, w^*(r) dr.$$

This allows us to state (see [5]) that the problem

$$(3.19) \quad \begin{cases} \left(c_0^{-*}(s)^{-1} \xi'(s)\right)' + \lambda (2\pi) \exp\left[\Phi^{-1}(s)^2\right] \xi(s) = 0, \\ \xi(0) = \xi'(s_0') = 0, \end{cases}$$

has the first eigenvalue $\lambda_1 > 1$, and consequently in the following problem

$$\begin{cases} \left(c_0^{-*}(s)^{-1} V_1'(s)\right)' + 2\pi \exp\left[\Phi^{-1}(s)^2\right] V_1(s) \ge 0, \\ V_1(0) = V_1'(s_0') = 0, \end{cases}$$

we have

$$V_1(s) \le 0 \quad \text{and} \quad V_1'(s) \le 0, \qquad s \in [0, s_0'],$$

that is,

$$(3.20) \quad u^*(s) \le w^*(s), \qquad s \in [0, s_0'].$$

On the other hand, from (3.18) and (3.20), we have

$$\begin{cases} \left(c_{0*}^{+}(s)^{-1} V_2'(s)\right)' - 2\pi \exp\left[\Phi^{-1}(s)^2\right] V_2(s) \ge 0, \\ V_2(s_1') = V_2'(\gamma_n(\Omega)) = 0. \end{cases}$$

Here we can use the maximum principle to obtain $V_2(s) \leq 0$. For Lemma 2.2 with $h(r) = c_{0*}^{+}(r)^{-1}$, we have for each $s \in \left]s_1', \gamma_n(\Omega)\right]$,

$$\int_{s_1'}^{s} \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] u^*(r)dr \leq \int_{s_1'}^{s} \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right] w^*(r)dr, \tag{3.21}$$

that is,

$$u^*(s_1') \leq w^*(s_1').$$

On the other hand, from (3.16), (3.17), (3.20) and the definition of $v(s)$ we have

$$-v'(s) \leq 0, \qquad s \in [s_0', s_1'].$$

Now since $u^*(s_1') \leq w^*(s_1')$, integrating between s and s_1' we obtain

$$u^*(s) \leq w^*(s), \qquad s \in [s_0', s_1']. \tag{3.22}$$

From (3.20) and (3.22) we have

$$u^*(s) \leq w^*(s), \qquad s \in [0, s_1'].$$

Moreover, for $s \in \left[s_{1,}'\gamma_n(\Omega)\right]$, (3.21) becomes

$$0 \geq \int_0^s \exp\left[B(\Phi^{-1}(r) - \Phi^{-1}(s))\right][u^*(r) - w^*(r)]dr,$$

that is,

$$\int_0^s u^*(r)dr \leq \int_0^s w^*(r)dr, \qquad s \in [s_1', \gamma_n(\Omega)].$$

Finally we can remove the hypothesis about the continuity of $c_0^{-*}(s)$ at s_0' proceeding by approximations.

If $c_{0*}^{+}(x) = 0$ or $c_0^{-*}(x) = 0$ then $s_1' = \gamma_n(\Omega)$ or $s_0' = 0$ and the result follows with the obvious modifications. □

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