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EXTENDING MEANS OF TWO VARIABLES TO SEVERAL VARIABLES

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Abstract

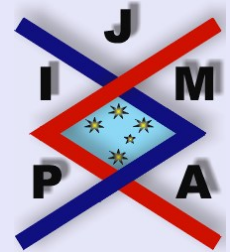
We present a method, based on series expansions and symmetric polynomials, by which a mean of two variables can be extended to several variables. We apply it mainly to the logarithmic mean.

2000 Mathematics Subject Classification: 26E60, 26D15.

Key words: Means, Logarithmic mean, Divided differences, Series expansions, Symmetric polynomials.

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1. Introduction

Throughout this paper, $n \geq 2$ is an integer and x_1, \dots, x_n are positive real numbers.

The logarithmic mean of x_1 and x_2 is defined by

$$(1.1) \quad L(x_1, x_2) = \frac{x_1 - x_2}{\ln x_1 - \ln x_2} \quad \text{if } x_1 \neq x_2,$$
$$L(x_1, x_1) = x_1.$$

There are several ways to extend this to n variables. Bullen ([1, p. 391]) writes that perhaps the most natural extension is due to Pittenger [13]. Based on an integral, it is

$$(1.2) \quad L(x_1, \dots, x_n) = \left[(n-1) \sum_{i=1}^n \frac{x_i^{n-2} \ln x_i}{\prod_{\substack{j=1 \\ j \neq i}}^n (\ln x_i - \ln x_j)} \right]^{-1}$$

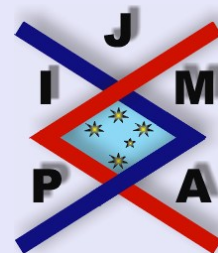
if all the x_i 's are unequal. Bullen ([1, p. 392]) also writes that another natural extension has been given by Neuman [9]. Based on the integral (6.3), it is

$$(1.3) \quad L(x_1, \dots, x_n) = (n-1)! \sum_{i=1}^n \frac{x_i}{\prod_{\substack{j=1 \\ j \neq i}}^n (\ln x_i - \ln x_j)}$$

if all the x_i 's are unequal. It is obviously different from (1.2).

If some of the x_i 's are equal, then (1.2) and (1.3) are defined by continuity.

Mustonen [6] gave (1.3) in 1976 but published it only recently [7] in the home page of his statistical data processing system, not in a journal. We will



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present his method. It is based on a series expansion and supports the notion that (1.3) is the most natural extension of (1.1).

In general, we call a continuous real function μ of two positive (or nonnegative) variables a mean if, for all $x_1, x_2, c > 0$ (or $x_1, x_2, c \geq 0$),

$$(i_1) \quad \mu(x_1, x_2) = \mu(x_2, x_1),$$

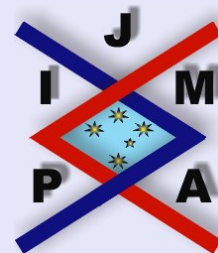
$$(i_2) \quad \mu(x_1, x_1) = x_1,$$

$$(i_3) \quad \mu(cx_1, cx_2) = c\mu(x_1, x_2),$$

$$(i_4) \quad x_1 \leq y_1, x_2 \leq y_2 \Rightarrow \mu(x_1, x_2) \leq \mu(y_1, y_2),$$

$$(i_5) \quad \min(x_1, x_2) \leq \mu(x_1, x_2) \leq \max(x_1, x_2).$$

Axiomatization of means is widely studied, see e.g. [1] and references therein.



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2. Polynomials Corresponding to a Mean

To extend the arithmetic and geometric means

$$A(x_1, x_2) = \frac{x_1 + x_2}{2}, \quad G(x_1, x_2) = (x_1 x_2)^{\frac{1}{2}}$$

to n variables is trivial, but to visualize our method, it may be instructive.

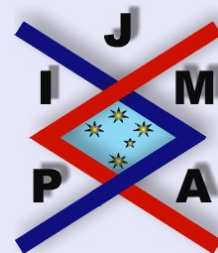
Substituting

$$(2.1) \quad x_1 = e^{u_1}, \quad x_2 = e^{u_2},$$

we have

$$(2.2) \quad \begin{aligned} A(x_1, x_2) &= \tilde{A}(u_1, u_2) \\ &= \frac{1}{2} (e^{u_1} + e^{u_2}) \\ &= \frac{1}{2} \left(1 + u_1 + \frac{u_1^2}{2!} + \cdots + 1 + u_2 + \frac{u_2^2}{2!} + \cdots \right) \\ &= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \cdot \frac{u_1^2 + u_2^2}{2} + \frac{1}{3!} \cdot \frac{u_1^3 + u_2^3}{2} + \cdots, \end{aligned}$$

$$(2.3) \quad \begin{aligned} G(x_1, x_2) &= \tilde{G}(u_1, u_2) \\ &= (e^{u_1} e^{u_2})^{\frac{1}{2}} = e^{\frac{u_1 + u_2}{2}} \\ &= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \left(\frac{u_1 + u_2}{2} \right)^2 + \cdots \\ &= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \cdot \frac{(u_1 + u_2)^2}{2^2} + \frac{1}{3!} \cdot \frac{(u_1 + u_2)^3}{2^3} + \cdots, \end{aligned}$$



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$$\begin{aligned}
(2.4) \quad L(x_1, x_2) &= \tilde{L}(u_1, u_2) \\
&= \frac{e^{u_1} - e^{u_2}}{u_1 - u_2} \\
&= \left(1 + u_1 + \frac{u_1^2}{2!} + \cdots - 1 - u_2 - \frac{u_2^2}{2!} - \cdots \right) (u_1 - u_2)^{-1} \\
&= \left(u_1 - u_2 + \frac{u_1^2 - u_2^2}{2!} + \frac{u_1^3 - u_2^3}{3!} + \cdots \right) (u_1 - u_2)^{-1} \\
&= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \cdot \frac{u_1^2 + u_1 u_2 + u_2^2}{3} \\
&\quad + \frac{1}{3!} \cdot \frac{u_1^3 + u_1^2 u_2 + u_1 u_2^2 + u_2^3}{4} + \cdots .
\end{aligned}$$

All these expansions are of the form

$$(2.5) \quad 1 + P_1(u_1, u_2) + \frac{1}{2!} P_2(u_1, u_2) + \frac{1}{3!} P_3(u_1, u_2) + \cdots ,$$

where the P_m 's are symmetric homogeneous polynomials of degree m . In all of them,

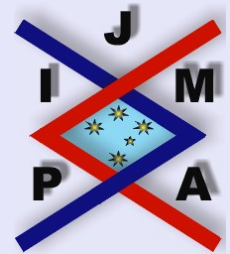
$$P_1(u_1, u_2) = \frac{u_1 + u_2}{2} = A(u_1, u_2).$$

The coefficients of

$$(2.6) \quad P_m(u_1, u_2) = b_0 u_1^m + b_1 u_1^{m-1} u_2 + \cdots + b_m u_2^m$$

are nonnegative numbers with sum 1. They are for A

$$b_0 = \frac{1}{2}, \quad b_1 = \cdots = b_{m-1} = 0, \quad b_m = \frac{1}{2},$$



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for G

$$b_k = \binom{m}{k} 2^{-m} \quad (0 \leq k \leq m),$$

and for L

$$b_0 = \dots = b_m = \frac{1}{m+1}.$$

Let μ be a mean of two variables. Assume that it has a valid expansion (2.5). Fix $m \geq 2$, and denote by $P_m[\mu]$ the polynomial (2.6). Its coefficients define a discrete random variable, denoted by $X_m[\mu]$, whose value is k ($0 \leq k \leq m$) with probability b_k . In particular, $X_m[A]$ is distributed uniformly over $\{0, m\}$, and $X_m[G]$ binomially and $X_m[L]$ uniformly over $\{0, \dots, m\}$. Their variances satisfy

$$\text{Var } X_m[G] \leq \text{Var } X_m[L] \leq \text{Var } X_m[A],$$

which is an interesting reminiscent of

$$(2.7) \quad G(x_1, x_2) \leq L(x_1, x_2) \leq A(x_1, x_2).$$

Let $u_1, u_2 \geq 0$, then (2.7) holds in fact termwise:

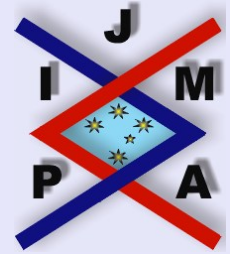
$$(2.8) \quad P_m[G](u_1, u_2) \leq P_m[L](u_1, u_2) \leq P_m[A](u_1, u_2)$$

for all $m \geq 1$. The functions

$$R_m[\mu](u_1, u_2) = (P_m[\mu](u_1, u_2))^{\frac{1}{m}}$$

are means. In particular, for A they are moment means

$$R_m[A](u_1, u_2) = \left(\frac{u_1^m + u_2^m}{2} \right)^{\frac{1}{m}} = M_m(u_1, u_2),$$



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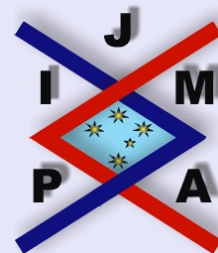
for G all of them are equal to the arithmetic mean

$$R_m[G](u_1, u_2) = \frac{u_1 + u_2}{2} = A(u_1, u_2),$$

and for L they are special cases of complete symmetric polynomial means and Stolarsky means (see e.g. [1, pp. 341, 393])

$$R_m[L](u_1, u_2) = \left[\frac{u_1^{m+1} - u_2^{m+1}}{(m+1)(u_1 - u_2)} \right]^{\frac{1}{m}} = \left(\frac{u_1^m + u_1^{m-1}u_2 + \cdots + u_2^m}{m+1} \right)^{\frac{1}{m}}.$$

Since the $P_m|\mu$'s are symmetric and homogeneous polynomials of two variables, they can be extended to n variables. Thus μ can also be likewise extended.



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3. Trivial Extensions: A and G

Consider first A . By (2.2),

$$P_m[A](u_1, u_2) = \frac{u_1^m + u_2^m}{2}.$$

To extend it to n variables is actually as trivial as to extend A directly. We obtain

$$P_m[A](u_1, \dots, u_n) = \frac{u_1^m + \dots + u_n^m}{n},$$

and so

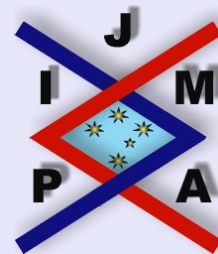
$$\begin{aligned} A(x_1, \dots, x_n) &= \sum_{m=0}^{\infty} \frac{1}{m!} P_m[A](u_1, \dots, u_n) \\ &= \frac{1}{n} \left(\sum_{m=0}^{\infty} \frac{u_1^m}{m!} + \dots + \sum_{m=0}^{\infty} \frac{u_n^m}{m!} \right) \\ &= \frac{1}{n} (e^{u_1} + \dots + e^{u_n}) = \frac{x_1 + \dots + x_n}{n}. \end{aligned}$$

Next, study G . By (2.3),

$$P_m[G](u_1, u_2) = \left(\frac{u_1 + u_2}{2} \right)^m,$$

which can be immediately extended to

$$P_m[G](u_1, \dots, u_n) = \left(\frac{u_1 + \dots + u_n}{n} \right)^m,$$



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and so

$$\begin{aligned}
 G(x_1, \dots, x_n) &= \sum_{m=0}^{\infty} \frac{1}{m!} P_m[G](u_1, \dots, u_n) \\
 &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{u_1 + \dots + u_n}{n} \right)^m \\
 &= e^{\frac{u_1 + \dots + u_n}{n}} = (e^{u_1} \dots e^{u_n})^{\frac{1}{n}} = (x_1 \dots x_n)^{\frac{1}{n}}.
 \end{aligned}$$

We present a “termwise” (cf. (2.8)) proof of the geometric-arithmetic mean inequality

$$(3.1) \quad G(x_1, \dots, x_n) \leq A(x_1, \dots, x_n).$$

We can assume that $u_1, \dots, u_n \geq 0$; if not, consider $cG \leq cA$ for a suitable $c > 0$. Let $m \geq 1$. Then

$$(3.2) \quad P_m[G](u_1, \dots, u_n) \leq P_m[A](u_1, \dots, u_n)$$

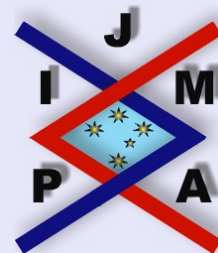
or equivalently

$$(3.3) \quad R_m[G](u_1, \dots, u_n) \leq R_m[A](u_1, \dots, u_n),$$

since

$$\frac{u_1 + \dots + u_n}{n} \leq \left(\frac{u_1^m + \dots + u_n^m}{n} \right)^{\frac{1}{m}}$$

by Schlömilch’s inequality (see e.g. [1, p. 203]). Therefore (3.1) follows.



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4. Extending L

Let $1 \leq m \leq n$. The m th complete symmetric polynomial of $u_1, \dots, u_n \geq 0$ (see e.g. [1, p. 341]) is defined by

$$C_m(u_1, \dots, u_n) = \sum_{i_1 + \dots + i_n = m} u_1^{i_1} \dots u_n^{i_n}.$$

(Here $i_1, \dots, i_n \geq 0$, and we define $0^0 = 1$.)

Let us now study L . Denote $Q_m = P_m[L]$. By (2.4),

$$Q_m(u_1, u_2) = \frac{u_1^m + u_1^{m-1}u_2 + \dots + u_2^m}{m+1}.$$

This can be easily extended to

$$(4.1) \quad Q_m(u_1, \dots, u_n) = \binom{n+m-1}{m}^{-1} C_m(u_1, \dots, u_n).$$

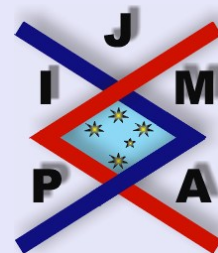
The corresponding mean,

$$R_m[L](u_1, \dots, u_n) = Q_m(u_1, \dots, u_n)^{\frac{1}{m}},$$

is called [1] the m th complete symmetric polynomial mean of u_1, \dots, u_n .

Thus we extend

$$(4.2) \quad L(x_1, \dots, x_n) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} Q_m(u_1, \dots, u_n).$$



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We compute this explicitly. Fix $m \geq 2$. Assume that $u_1, \dots, u_n \geq 0$ are all unequal. We claim that if $2 \leq n \leq m + 1$, then $C_{m-n+1}(u_1, \dots, u_n)$ is the $(n - 1)$ th divided difference of the function $f(u) = u^m$ with arguments u_1, \dots, u_n . In other words,

$$(4.3) \quad C_{m-n+1}(u_1, \dots, u_n) = \frac{C_{m-n+2}(u_2, \dots, u_n) - C_{m-n+2}(u_1, \dots, u_{n-1})}{u_n - u_1}.$$

(For $n = 2$, we have simply $C_{m-1}(u_1, u_2) = \frac{u_2^m - u_1^m}{u_2 - u_1}$.)

To prove this, note that for $k \geq 1$

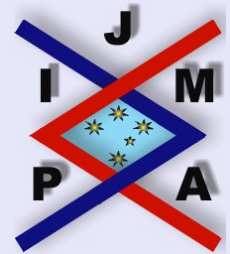
$$(4.4) \quad C_k(u_1, \dots, u_n) = u_n^k + u_n^{k-1}C_1(u_1, \dots, u_{n-1}) \\ + \dots + u_n C_{k-1}(u_1, \dots, u_{n-1}) + C_k(u_1, \dots, u_{n-1})$$

and

$$C_k(u_1, \dots, u_n) = C_k(u_1, u_n) + C_{k-1}(u_1, u_n)C_1(u_2, \dots, u_{n-1}) \\ + \dots + C_1(u_1, u_n)C_{k-1}(u_2, \dots, u_{n-1}) + C_k(u_2, \dots, u_{n-1}).$$

Hence,

$$C_{m-n+2}(u_2, \dots, u_n) - C_{m-n+2}(u_1, \dots, u_{n-1}) \\ = C_{m-n+2}(u_2, \dots, u_n) - C_{m-n+2}(u_2, \dots, u_{n-1}, u_1) \\ = u_n^{m-n+2} + u_n^{m-n+1}C_1(u_2, \dots, u_{n-1}) + \dots + C_{m-n+2}(u_2, \dots, u_{n-1}) \\ - u_1^{m-n+2} - u_1^{m-n+1}C_1(u_2, \dots, u_{n-1}) - \dots - C_{m-n+2}(u_2, \dots, u_{n-1}) \\ = (u_n^{m-n+2} - u_1^{m-n+2}) + (u_n^{m-n+1} - u_1^{m-n+1})C_1(u_2, \dots, u_{n-1}) + \dots \\ + (u_n - u_1)C_{m-n+1}(u_2, \dots, u_{n-1})$$



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$$\begin{aligned}
&= (u_n - u_1) \left[C_{m-n+1}(u_1, u_n) + C_{m-n}(u_1, u_n)C_1(u_2, \dots, u_{n-1}) + \dots \right. \\
&\quad \left. + C_{m-n+1}(u_2, \dots, u_{n-1}) \right] \\
&= (u_n - u_1)C_{m-n+1}(u_1, \dots, u_n),
\end{aligned}$$

and (4.3) follows.

By a well-known formula of divided differences (see e.g. [4, p. 148]), we now have

$$C_{m-n+1}(u_1, \dots, u_n) = \sum_{i=1}^n \frac{u_i^m}{U_i},$$

where

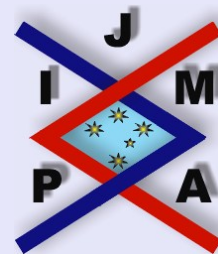
$$U_i = \prod_{\substack{j=1 \\ j \neq i}}^n (u_i - u_j).$$

Therefore, since

$$\frac{1}{(m-n+1)!} \binom{n+(m-n+1)-1}{m-n+1}^{-1} = \frac{(n-1)!}{m!},$$

we obtain

$$\begin{aligned}
\frac{1}{(m-n+1)!} Q_{m-n+1}(u_1, \dots, u_n) &= \frac{(n-1)!}{m!} C_{m-n+1}(u_1, \dots, u_n) \\
&= \frac{(n-1)!}{m!} \sum_{i=1}^n \frac{u_i^m}{U_i}.
\end{aligned}$$



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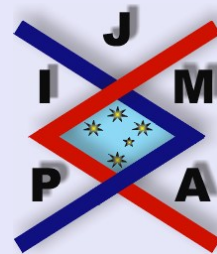
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Hence, and because the m th divided difference of the function $f(u) = u^m$ is 1 if $m = n - 1$ and 0 if $m \leq n - 2$, we have

$$\begin{aligned}
 L(x_1, \dots, x_n) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} Q_k(u_1, \dots, u_n) \\
 &= 1 + \sum_{m=n}^{\infty} \frac{1}{(m-n+1)!} Q_{m-n+1}(u_1, \dots, u_n) \\
 &= 1 + (n-1)! \sum_{m=n}^{\infty} \frac{1}{m!} \sum_{i=1}^n \frac{u_i^m}{U_i} \\
 &= (n-1)! \sum_{m=n-1}^{\infty} \frac{1}{m!} \sum_{i=1}^n \frac{u_i^m}{U_i} \\
 &= (n-1)! \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i=1}^n \frac{u_i^m}{U_i} \\
 &= (n-1)! \sum_{i=1}^n \frac{1}{U_i} \sum_{m=0}^{\infty} \frac{u_i^m}{m!} = (n-1)! \sum_{i=1}^n \frac{e^{u_i}}{U_i} \\
 &= (n-1)! \sum_{i=1}^n \frac{e^{u_i}}{\prod_{\substack{j=1 \\ j \neq i}}^n (u_i - u_j)} \\
 &= (n-1)! \sum_{i=1}^n \frac{x_i}{\prod_{\substack{j=1 \\ j \neq i}}^n (\ln x_i - \ln x_j)}.
 \end{aligned}$$

Thus (1.3) is found.



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5. Numerical Computation of L

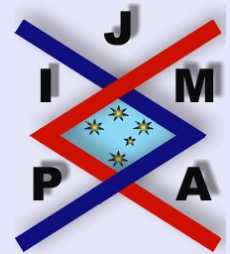
Mustonen [7] noted that, in computing L numerically, the explicit formula (1.3) is very unstable. He programmed a fast and stable algorithm based on (4.1), (4.2), and (4.4). His experiments lead to a conjecture that, denoting $G_n = G(1, \dots, n)$ and $L_n = L(1, \dots, n)$, we have

$$\lim_{n \rightarrow \infty} (G_{n+1} - G_n) = \lim_{n \rightarrow \infty} (L_{n+1} - L_n) = \frac{1}{e}$$

and

$$\lim_{n \rightarrow \infty} \frac{G_n}{n} = \lim_{n \rightarrow \infty} \frac{L_n}{n} = \frac{1}{e}.$$

For G_n , these limit conjectures can be proved by using Stirling's formula. For L_n , they remain open.



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6. Inequality $G \leq L \leq A$

It is natural to ask, whether

$$(6.1) \quad G(x_1, \dots, x_n) \leq L(x_1, \dots, x_n) \leq A(x_1, \dots, x_n)$$

is generally valid.

For $n = 2$, this inequality is old (see e.g. [1, pp. 168-169]). Carlson [2] (see also [1, p. 388]) sharpened the first part and Lin [5] (see also [1, p. 389]) the second:

$$(6.2) \quad (G(x_1, x_2)M_{1/2}(x_1, x_2))^{\frac{1}{2}} \leq L(x_1, x_2) \leq M_{1/3}(x_1, x_2).$$

Neuman [9] defined (as a special case of [9, Eq. (2.3)])

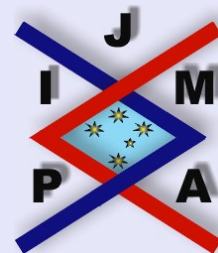
$$(6.3) \quad L(x_1, \dots, x_n) = \int_{E_{n-1}} \left(\exp \sum_{i=1}^n u_i \ln x_i \right) du,$$

where $u_1 + \dots + u_n = 1$,

$$E_{n-1} = \{(u_1, \dots, u_{n-1}) \mid u_1, \dots, u_{n-1} \geq 0, u_1 + \dots + u_{n-1} \leq 1\},$$

and $du = du_1 \cdots du_{n-1}$. He ([9], Theorem 1 and the last formula) proved (6.1) and reduced (6.3) into (1.3).

Pečarić and Šimić [12] tied Neuman's approach to a wider context. As a special case ([12, Remark 5.4]), they obtained (1.3).



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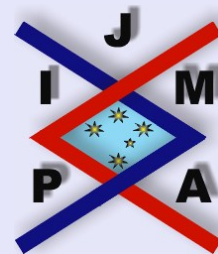
Let V denote the Vandermonde determinant and let V_i denote its subdeterminant obtained by omitting its last row and i th column. Xiao and Zhang [14] (unaware of [9]) defined

$$L(x_1, \dots, x_n) = \frac{(n-1)!}{V(\ln x_1, \dots, \ln x_n)} \sum_{i=1}^n (-1)^{n+i} x_i V_i(\ln x_1, \dots, \ln x_n),$$

which in fact equals to (1.3). Also they proved (6.1).

We conjecture that (6.2) can be extended to

$$(G(x_1, \dots, x_n) M_{1/2}(x_1, \dots, x_n))^{\frac{1}{2}} \leq L(x_1, \dots, x_n) \leq M_{1/3}(x_1, \dots, x_n).$$



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7. Inequalities $P_m[G] \leq P_m[L] \leq P_m[A]$

In view of (3.2) and (3.3), it is now natural to ask, whether (6.1) can be strengthened to hold termwise. In other words: Do we have

$$P_m[G] \leq P_m[L] \leq P_m[A]$$

or equivalently

$$R_m[G] \leq R_m[L] \leq R_m[A],$$

that is

$$(7.1) \quad \frac{u_1 + \cdots + u_n}{n} \leq Q_m(u_1, \dots, u_n)^{\frac{1}{m}} \leq \left(\frac{u_1^m + \cdots + u_n^m}{n} \right)^{\frac{1}{m}}$$

for all $u_1, \dots, u_n \geq 0$, $m \geq 1$?

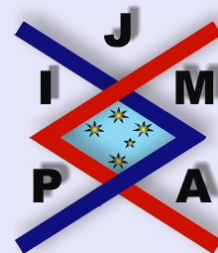
Fix u_1, \dots, u_n and denote $q_m = Q_m(u_1, \dots, u_n)^{\frac{1}{m}}$. Neuman ([8, Corollary 3.2]; see also [1, pp. 342-343]) proved that

$$(7.2) \quad k \leq m \Rightarrow q_k \leq q_m.$$

The first part of (7.1), $q_1 \leq q_m$, is therefore true. We conjecture that the second part is also true.

DeTemple and Robertson [3] gave an elementary proof of (7.2) for $n = 2$, but Neuman's proof for general n is advanced, applying B -splines.

Mustonen [7] gave an elementary proof of (7.1) for $n = 2$.



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8. Other Means

Pečarić and Šimić [12] (see also [1, p. 393]) studied a very large class of means, called *Stolarsky-Tobey means*, which includes all the ordinary means as special cases. They first defined these means for two variables and then, applying certain integrals, extended them to n variables. It might be of interest to apply our method to all these extensions, but we take only a small step towards this direction.

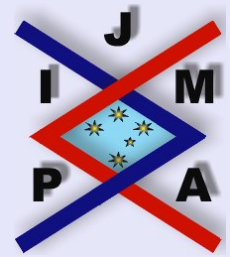
Let r and s be unequal nonzero real numbers. (Actually [12] allows $s = 0$ and [1] allows $r = 0$, both of which are obviously incorrect.) Consider ([12, Eq. (6)]) the mean

$$(8.1) \quad E_{r,s}(x_1, x_2) = \left(\frac{r}{s} \cdot \frac{x_1^s - x_2^s}{x_1^r - x_2^r} \right)^{\frac{1}{s-r}},$$

where $x_1 \neq x_2$. Assuming that $s \neq -r, -2r, \dots, -(n-2)r$, this can be extended ([12, Theorem 5.2(i)]) to

$$(8.2) \quad E_{r,s}(x_1, \dots, x_n) = \left[\frac{(n-1)! r^{n-1}}{s(s+r) \cdots (s+(n-2)r)} \sum_{i=1}^n \frac{x_i^{s+(n-2)r}}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_i^r - x_j^r)} \right]^{\frac{1}{s-r}},$$

where all the x_i 's are unequal.



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To extend (8.1) by our method, we simply note that

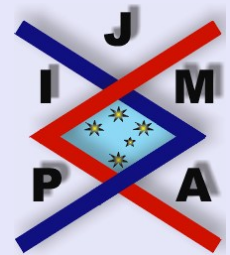
$$\begin{aligned} E_{r,s}(x_1, x_2) &= \left[\frac{x_1^s - x_2^s}{s(\ln x_1 - \ln x_2)} \bigg/ \frac{x_1^r - x_2^r}{r(\ln x_1 - \ln x_2)} \right]^{\frac{1}{s-r}} \\ &= \left(\frac{L(x_1^s, x_2^s)}{L(x_1^r, x_2^r)} \right)^{\frac{1}{s-r}}, \end{aligned}$$

which can be immediately extended to

$$\begin{aligned} (8.3) \quad E_{r,s}(x_1, \dots, x_n) &= \left(\frac{L(x_1^s, \dots, x_n^s)}{L(x_1^r, \dots, x_n^r)} \right)^{\frac{1}{s-r}} \\ &= \left\{ \sum_{i=1}^n \frac{x_i^s}{\prod_{j \neq i}^n [s(\ln x_i - \ln x_j)]} \bigg/ \sum_{i=1}^n \frac{x_i^r}{\prod_{j \neq i}^n [r(\ln x_i - \ln x_j)]} \right\}^{\frac{1}{s-r}} \\ &= \left[\left(\frac{r}{s} \right)^{n-1} \sum_{i=1}^n \frac{x_i^s}{\prod_{j \neq i}^n (\ln x_i - \ln x_j)} \bigg/ \sum_{i=1}^n \frac{x_i^r}{\prod_{j \neq i}^n (\ln x_i - \ln x_j)} \right]^{\frac{1}{s-r}}. \end{aligned}$$

This is obviously different from (8.2).

Unfortunately the problem of whether (8.3) indeed is a mean, i.e., whether it lies between the smallest and largest x_i , remains open.



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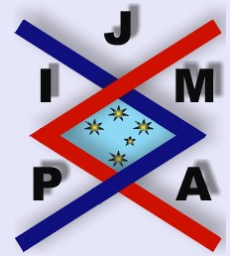
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Addendum

Neuman ([10, Theorem 6.2]) proved the second part of (7.1) and [11] showed that (8.3) is a mean.



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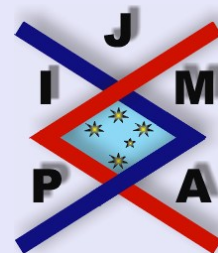
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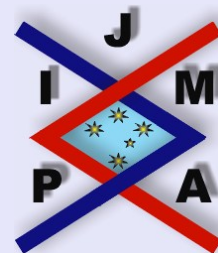
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