Journal of Inequalities in Pure and Applied Mathematics

EXISTENCE AND LOCAL UNIQUENESS FOR NONLINEAR LIDSTONE BOUNDARY VALUE PROBLEMS

JEFFREY EHME AND JOHNNY HENDERSON

Department of Mathematics
Box 214, Spelman College
Atlanta, Georgia 30314 USA.
EMail: jehme@spelman.edu

Department of Mathematics
Auburn University
Auburn, Alabama 36849-5310 USA.
EMail: hendej2@mail.auburn.edu

ISSN (electronic): 1443-5756
021-99

volume 1, issue 1, article 8,
2000.

*Received 12 January, 2000;
accepted 31 January, 2000.*

Communicated by: R.P. Agarwal

Abstract

Higher order upper and lower solutions are used to establish the existence and local uniqueness of solutions to $y^{(2n)} = f(t, y, y'', \ldots, y^{(2n-2)})$, satisfying boundary conditions of the form
$g_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) - y^{(2i-2)}(0) = 0,$
$h_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) - y^{(2i-2)}(0) = 0, 1 \leq i \leq n.$

2000 Mathematics Subject Classification: 34B15, 34A40
Key words: planar convex set, inequality, area, perimeter, diameter, width, inradius, circumradius

Contents

1. Introduction

In this paper we wish to consider the existence and local uniqueness to problems of the form

$$y^{(2n)} = f(t, y, y'', \dots, y^{(2n-2)}) \tag{1.1}$$

subject to boundary conditions of the form

$$\begin{aligned} g_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) - y^{(2i-2)}(0) = 0, \\ h_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) - y^{(2i-2)}(1) = 0, \end{aligned} \tag{1.2}$$

$1 \le i \le n$, where g_i and h_i are continuous functions. These conditions generalize the usual Lidstone boundary conditions, which have been of recent interest. See [1, 5].

The method of upper and lower solutions, sometimes referred to as differential inequalities, is generally used to obtain the existence of solutions within specified bounds determined by the upper and lower solutions. Important papers using these techniques include [2, 3, 4, 9, 11, 14, 15]. These techniques are also used in the more recent papers of Eloe and Henderson [8] and Thompson [17, 18]. This paper will consider problems described as fully nonlinear by Thompson in [17, 18].

The classic papers by Klassen [13] and Kelly [12] apply higher order upper and lower solutions methods. In addition, Šeda [16], Eloe and Grimm [7], and Hong and Hu [10] have also considered higher order methods involving upper and lower solutions.

In [6] Ehme, Eloe, and Henderson applied this method to $2n^{th}$ order problems in order to obtain the existence of solutions to problems with nonlinear boundary conditions. This paper extends those results to obtain a *unique* solution within the appropriate bounds.

2. Preliminaries

In this section we make some useful definitions and prove some elementary, yet key, lemmas. We will use the norm

$$||x|| = \max_{t\in[0,1]} \left\{|x(t)|, |x'(t)|, \dots, |x^{2n-2}(t)|\right\}$$

as our norm on $C^{2n-2}[0,1]$. We begin with the following representation lemma which converts our boundary value problem (1.1), (1.2) into an integral equation.

Lemma 2.1. *Suppose $x(t)$ is a solution to the integral equation*

$$x(t) = \sum_{i=1}^{n} g_i(x^{(2i-2)}(0), x^{(2i-2)}(1))p_i(t) + \sum_{i=1}^{n} h_i(x^{(2i-2)}(0), x^{(2i-2)}(1))q_i(t) \\ + \int_0^1 G(t,s) f\left(s, x(s), x''(s), \dots, x^{(2n-2)}(s)\right) ds$$

where $G(t,s)$ is the Green's function for $x^{(2n)} = 0, x^{(2i-2)}(0) = x^{(2i-2)}(1) = 0$, $1 \leq i \leq n$. Here the functions p_i and q_i satisfy

$$p_i^{(2j-2)}(0) = \delta_{ij}, p_i^{(2j-2)}(1) = 0,\ q_i^{(2j-2)}(0) = 0,\ q_i^{(2j-2)}(1) = \delta_{ij}, \quad 1 \leq i,j \leq n,$$

with p_i and q_i solutions to $x^{(2n)} = 0$. Then x is a solution to (1.1), (1.2). Conversely, if x is a solution to (1.1), (1.2), then x is a solution to the above integral equation.

Proof. Suppose x is a solution to the integral equation above. Then using the boundary conditions that the Green's function and the p_i and q_i satisfy at $t=0$, we obtain

$$x^{(2j-2)}(0) = g_j(x^{(2j-2)}(0), x^{(2j-2)}(1))p_j^{(2j-2)}(0).$$

But $p_j^{(2j-2)}(0) = 1$ implies

$$g_j(x^{(2j-2)}(0), x^{(2j-2)}(1)) - x^{(2j-2)}(0) = 0.$$

A similar argument at $t=1$ shows

$$h_i(x^{(2i-2)}(0), x^{(2i-2)}(1)) - x^{(2j-2)}(1) = 0.$$

This shows x satisfies the boundary conditions (1.2). The right hand side of the integral equation is $2n$ times differentiable. Differentiating the integral equation $2n$ times yields x satisfies (1.1) .

For the converse, suppose x satisfies (1.1), (1.2). Then

$$\frac{d^{2n}}{dt^{2n}}\left(x(t) - \int_0^1 G(t,s)f\big(s, x(s), \dots, x^{(2n-2)}(s)\big)ds\right) = 0.$$

Thus

$$x(t) - \int_0^1 G(t,s)f\big(s, x(s), \dots, x^{(2n-2)}(s)\big)ds = w(t)$$

where $w(t)$ is a $2n-1$ degree polynomial. The functions p_i, q_i, $1 \le i \le n$, form a basis for the $2n-1$ degree polynomials, hence there exists constants

$a_1, \dots, a_n, b_1, \dots, b_n$ such that

(2.1)
$$x(t) - \int_0^1 G(t,s) f\big(s, x(s), \dots, x^{(2n-2)}(s)\big) ds = \sum_{j=1}^{n} a_j p_j(t) + \sum_{j=1}^{n} b_j q_j(t).$$

Using the properties of the Green's function, we obtain for $1 \le i \le n$,

$$x^{(2i-2)}(0) = \sum_{j=1}^{n} a_j p_j^{(2i-2)}(0) + \sum_{j=1}^{n} b_j q_j^{(2i-2)}(0).$$

The properties of the p_i, q_i imply $x^{(2i-2)}(0) = a_i$. But x satisfies (1.2), hence

$$a_i = g_i(x^{(2i-2)}(0), x^{(2i-2)}(1)).$$

A similar argument shows

$$b_i = h_i(x^{(2i-2)}(0), x^{(2i-2)}(1)).$$

Equation (2.1) implies x satisfies the correct integral equation. $\square$

It is well known that for $0 \le i \le 2n-2$ the Green's function above satisfies

$$\sup\left\{ \int_0^1 \left| \frac{\partial^i G(t,s)}{\partial t^i} \right| ds \middle|\ t \in [0,1] \right\} \le M_{i+1}$$

for appropriate constants M_{i+1}. These constants will play a role in the statement of our main theorem.

The following key lemma will be indispensable in passing sign information from higher order derivatives to lower order derivatives.

Lemma 2.2. *If* $x(t) \in C^2[0,1]$ *then*

$$x(t) = x(0)(1-t) + x(1)t + \int_0^1 H(t,s)x''(s)ds$$

where $H(t,s)$ *is the Green's function for*

$$x'' = 0, \qquad x(0) = x(1) = 0.$$

Proof. Let

$$u(t) = x(0)(1-t) + x(1)t + \int_0^1 H(t,s)x''(s)ds.$$

Then $u(0) = x(0), u(1) = x(1)$, and $u''(t) = x''(t)$. Hence by the uniqueness of solutions to

$$x'' = 0, \qquad x(0) = x(1) = 0,$$

it follows that $u(t) = x(t)$ for all t. □

Lemma 2.3. *Suppose* p_i *and* q_i *satisfy*

$$p_i^{(2j-2)}(0) = \delta_{ij},\ p_i^{(2j-2)}(1) = 0,\ q_i^{(2j-2)}(0) = 0,\ q_i^{(2j-2)}(1) = \delta_{ij}, \quad 1 \le i,j \le n,$$

with p_i *and* q_i *solutions to* $x^{(2n)} = 0$. *Then* $||p_i||, ||q_i|| \le 1$.

Proof. If $i = 1$ then $q_1(t) = t$ and the result clearly holds. Assume $i > 1$ and let $G_*(t,s)$ denote the Green's function for the $(2i-2)$ order Lidstone problem

$$x^{(2i-2)} = 0,\ x^{(2k)}(0) = 0,\ x^{(2l)}(1) = 0,\ x^{(2i-4)}(1) = 1,$$

where $0 \leq k \leq i-2$, and $0 \leq l \leq i-3$. It can easily be verified that

$$\left|\frac{\partial^r G_*}{\partial t^r}(t,s)\right| \leq 1 \quad \text{for all } \ t,s \in [0,1].$$

Set

$$v(t) = \int_0^1 G_*(t,s)s\,ds,$$

then $v^{(2i-2)}(t) = t$ and this yields

$$v^{(2i-2)}(0) = 0 \quad \text{and} \quad v^{(2i-2)}(1) = 1.$$

Obviously if $k \geq 2i$ then $v^{(k)}(0) = v^{(k)}(1) = 0$. If $k \leq 2i-4$, then the properties of the Green's function G_* imply $v^{(k)}(0) = 0, v^{(k)}(1) = 0$. By uniqueness, we see $v(t) = q_i(t)$. Thus for $1 \leq k \leq 2n-2$,

$$|q_i^{(k)}(t)| \leq \int_0^1 \left|\frac{\partial^r G_*}{\partial t^r}(t,s)s\right| ds \leq 1.$$

Hence $||q_i|| \leq 1$. The p_i are handled similarly. □

An *upper solution* for (1.1), (1.2) is a function $q(t) \in C^{(2n)}[0,1]$ satisfying

$$q^{(2n)} \leq f(t,q,q'',\ldots,q^{(2n-2)})$$

$$\begin{aligned}
g_i(q^{(2i-2)}(0),q^{(2i-2)}(1)) - q^{(2i-2)}(0) &\leq 0, \quad i = n-2k+2\\
h_i(q^{(2i-2)}(0),q^{(2i-2)}(1)) - q^{(2i-2)}(1) &\leq 0, \quad i = n-2k+2
\end{aligned}$$

$$\begin{aligned}
g_i(q^{(2i-2)}(0),q^{(2i-2)}(1)) - q^{(2i-2)}(0) &\geq 0, \quad i = n-2k+1\\
h_i(q^{(2i-2)}(0),q^{(2i-2)}(1)) - q^{(2i-2)}(1) &\geq 0, \quad i = n-2k+1
\end{aligned}$$

where $k \geq 1$.

A *lower solution* for (1.1), (1.2) is a function $p(t) \in C^{(2n)}[0,1]$ satisfying

$$p^{(2n)} \geq f(t, p, p'', \ldots, p^{(2n-2)})$$

$$\begin{aligned} g_i(p^{(2i-2)}(0), p^{(2i-2)}(1)) - p^{(2i-2)}(0) &\geq 0, \quad i = n-2k+2 \\ h_i(p^{(2i-2)}(0), p^{(2i-2)}(1)) - p^{(2i-2)}(1) &\geq 0, \quad i = n-2k+2 \end{aligned}$$

$$\begin{aligned} g_i(p^{(2i-2)}(0), p^{(2i-2)}(1)) - p^{(2i-2)}(0) &\leq 0, \quad i = n-2k+1 \\ h_i(p^{(2i-2)}(0), p^{(2i-2)}(1)) - p^{(2i-2)}(1) &\leq 0, \quad i = n-2k+1 \end{aligned}$$

where $k \geq 1$.

The function $f(t, x_1, \ldots, x_n)$ is said to be *Lip-qp* if there exist positive constants c_i such that for all $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ such that

$$(-1)^{i+1} p^{(2n-2i)}(t) \leq x_{n-i+1}, y_{n-i+1} \leq (-1)^{i+1} q^{(2n-2i)}(t), \quad 1 \leq i \leq n,$$

it follows that

$$|f(t, x_1, \ldots, x_n) - f(t, y_1, \ldots, y_n)| \leq \sum_{i=1}^{n} c_i |x_i - y_i|.$$

We note that if f is continuously differentiable on a suitable region, then f will be *Lip-qp*.

A boundary condition $g_i : R^2 \to R$ is said to be *increasing with respect to region-qp* if

$$(-1)^{i+1} p^{(2n-2i)}(0) \leq x \leq (-1)^{i+1} q^{(2n-2i)}(0),$$

and

$$(-1)^{i+1}p^{(2n-2i)}(1) \leq y \leq (-1)^{i+1}q^{(2n-2i)}(1)$$

imply

$$g_i(p^{(2n-2i)}(0), p^{(2n-2i)}(1)) \leq g_i(x,y) \leq g_i(q^{(2n-2i)}(0), q^{(2n-2i)}(1)) \text{ for } i \text{ odd},$$

and

$$g_i(q^{(2n-2i)}(0), q^{(2n-2i)}(1)) \leq g_i(x,y) \leq g_i(p^{(2n-2i)}(0), p^{(2n-2i)}(1)) \text{ for } i \text{ even}.$$

It should be noted that this condition is trivially satisfied if g_i is an increasing function of both of its arguments.

Throughout the rest of this paper, we shall assume our boundary conditions are Lipschitz. That is,

$$|g_i(x_1,x_2) - g_i(y_1,y_2)| \leq c_{1i}|x_1 - y_1| + c_{2i}|x_2 - y_2|$$

and

$$|h_i(x_1,x_2) - h_i(y_1,y_2)| \leq c_{3i}|x_1 - y_1| + c_{4i}|x_2 - y_2|,$$

for some constants $c_{\mu\nu}$.

3. Existence and Local Uniqueness

In this section, we present our main theorem, which establishes the existence and local uniqueness of a solution to (1.1), (1.2) that lies between an upper and lower solution.

Theorem 3.1. *Assume*

1. $f(t, x_1, \ldots, x_n) : [0,1] \times R^n \to R$ *is continuous;*
2. $f(t, x_1, \ldots, x_n)$ *is increasing in the* x_{n-2k+1} *variables for* $k \geq 1$*;*
3. $f(t, x_1, \ldots, x_n)$ *is decreasing in the* x_{n-2k} *variables for* $k \geq 1$.
   Assume, in addition, there exist q *and* p *such that*
   (a) q *and* p *are upper and lower solutions to (1.1), (1.2) respectively, so that* $(-1)^{i+1}p^{(2n-2i)}(t) \leq (-1)^{i+1}q^{(2n-2i)}(t)$ *for all* $t \in [0,1]$*;*
   (b) $f(t, x_1, \ldots, x_n)$ *is Lip-*qp*,*
   (c) *Each* g_i *and* h_i *is Lipschitz and increasing with respect to region-*qp.

Then, if

$$\max\left\{\sum_{i=1}^{n}(c_{1i}+c_{2i}+c_{3i}+c_{4i}) + M_{j+1}\sum_{i=1}^{n} c_i \,\Big|\, j = 0, \ldots, n-2\right\} < 1,$$

there exists a unique solution $x(t)$ *to (1.1), (1.2) such that*

$$(-1)^{i+1}p^{(2n-2i)}(t) \leq (-1)^{i+1}x^{(2n-2i)}t) \leq (-1)^{i+1}q^{(2n-2i)}(t) \;\; \textit{for all} \;\; t \in [0,1]$$

and $i = 1, 2, \ldots, n$.

Proof. For $1 \le j \le n$, define

$$\alpha_{2n-2j}(y^{(2n-2j)}(t)) = \begin{cases} \max\{p^{(2n-2j)}(t), \min\{y^{(2n-2j)}(t), q^{(2n-2j)}(t)\}\}, & \text{if } j \text{ is odd,} \\ \max\{q^{(2n-2j)}(t), \min\{y^{(2n-2j)}(t), p^{(2n-2j)}(t)\}\}, & \text{if } j \text{ is even,} \end{cases}$$

where y is a function defined on $[0,1]$. If $y^{(2n-2j)}$ is continuous, then α_{2n-2j} is continuous. Moreover,

$$(-1)^{i+1}p^{(2n-2i)}(t) \le (-1)^{i+1}\alpha_{2n-2i}(y^{(2n-2i)}(t)) \le (-1)^{i+1}q^{(2n-2i)}(t) \quad \text{for all } \ t \in [0,1]$$

and $i = 1, 2, \dots, n$. Define $F_1 : [0,1] \times C^{2n-2}[0,1] \to R$ by

$$F_1(t, y, y'', \dots, y^{(2n-2)}) = f(t, \alpha_0(y(t)), \dots, \alpha_{2n-2}(y^{(2n-2)}(t))).$$

A tedious, but straight forward, computation shows each α_{2n-2i} is a non-expansive function. Thus

$$\left|F_1(t, y, y'', \dots, y^{(2n-2)}) - F_1(t, z, z'', \dots, z^{(2n-2)})\right| \le \sum_{i=1}^{n} c_i |y^{(2i-2)}(t) - z^{(2i-2)}(t)|.$$

F_1 is also continuous. Choose $c_0 > 0$ such that

$$\max\left\{\sum_{i=1}^{n}(c_{1i} + c_{2i} + c_{3i} + c_{4i}) + M_{j+1}\sum_{i=1}^{n} c_i \,|\, j = 0, \dots, n-2\right\} + c_0 < 1.$$

Now define $F_2 : [0,1] \times C^{2n-2}[0,1] \to R$ by

$$F_2(t, y, y'', \dots, y^{(2n-2)}) = \begin{cases} F_1(t, y, y'', \dots, y^{(2n-2)}) + c_0(y^{(2n-2)}(t) - q^{(2n-2)}(t)), \\ \qquad \text{if } y^{(2n-2)}(t) > q^{(2n-2)}(t) \\ F_1(t, y, y'', \dots, y^{(2n-2)}), \\ \qquad \text{if } p^{(2n-2)}(t) \le y^{(2n-2)}(t) \le q^{(2n-2)}(t) \\ F_1(t, y, y'', \dots, y^{(2n-2)}) - c_0(p^{(2n-2)}(t) - y^{(2n-2)}(t)), \\ \qquad \text{if } y^{(2n-2)}(t) < p^{(2n-2)}(t) \end{cases}$$

Then F_2 is continuous. By considering various cases, it can be shown that F_2 satisfies

$$\begin{aligned} &\left|F_2(t, y, y'', \dots, y^{(2n-2)}) - F_2(t, z, z'', \dots, z^{(2n-2)})\right| \\ &\qquad \le \sum_{i=1}^{n-1} c_i |y^{(2i-2)} - z^{(2i-2)}| + (c_n + c_0)|y^{(2n-2)} - z^{(2n-2)}|. \end{aligned}$$

This shows F_2 is also Lipschitz.

For $1 \le i \le n$, define the bounded functions $\hat{g}_i$ and $\hat{h}_i$ by

$$\begin{aligned} \hat{g}_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) &= g_i(\alpha_{2i-2}(y^{(2i-2)}(0)), \alpha_{2i-2}(y^{(2i-2)}(1))) \\ \hat{h}_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) &= h_i(\alpha_{2i-2}(y^{(2i-2)}(0)), \alpha_{2i-2}(y^{(2i-2)}(1))). \end{aligned}$$

It can be shown that the fact that g_i and h_i are *Lip-qp* implies $\hat{g}_i$ and $\hat{h}_i$ are *Lip-qp*

for the constants $c_{1i}, c_{2i}, c_{3i}, c_{4i}$. Define $T : C^{2n-2}[0,1] \to C^{2n-2}[0,1]$ by

$$Tx(t) = \sum_{i=1}^{n} \hat{g}_i(x^{(2i-2)}(0), x^{(2i-2)}(1))p_i(t) + \sum_{i=1}^{n} \hat{h}_i(x^{(2i-2)}(0), x^{(2i-2)}(1))q_i(t) + \int_0^1 G(t,s)F_2(s, x(s), x''(s), \dots, x^{(2n-2)}(s))ds.$$

For $0 \le k \le 2n-2$, and $x, y \in C^{2n-2}[0,1]$, it follows that

$$\begin{aligned}
&\left|(Tx)^{(k)}(t) - (Ty)^{(k)}(t)\right| \\
&\quad \le \Bigg| \sum_{i=1}^{n} \hat{g}_i(x^{(2i-2)}(0), x^{(2i-2)}(1))p_i^{(k)}(t) - \sum_{i=1}^{n} \hat{g}_i(y^{(2i-2)}(0), y^{(2i-2)}(1))p_i^{(k)}(t) \\
&\qquad + \sum_{i=1}^{n} \hat{h}_i(x^{(2i-2)}(0), x^{(2i-2)}(1))q_i^{(k)}(t) - \sum_{i=1}^{n} \hat{h}_i(y^{(2i-2)}(0), y^{(2i-2)}(1))q_i^{(k)}(t) \\
&\qquad + \int_0^1 \frac{\partial^k G}{\partial t^k}(t,s) F_2(s, x(s), x''(s), \dots, x^{(2n-2)}(s)) \\
&\qquad\quad - F_2(s, y(s), y''(s), \dots, y^{(2n-2)}(s))\, ds \Bigg| \\
&\quad \le \sum_{i=1}^{n} (c_{1i}|x^{(2i-2)}(0) - y(^{(2i-2)}(0)| + c_{2i}|x^{(2i-2)}(1) - y(^{(2i-2)}(1)|) \cdot ||p_i|| \\
&\qquad + \sum_{i=1}^{n} (c_{3i}|x^{(2i-2)}(0) - y(^{(2i-2)}(0)| + c_{4i}|x^{(2i-2)}(1) - y(^{(2i-2)}(1)|) \cdot ||q_i|| \\
&\qquad + M_{k+1}(\sum_{i=1}^{n-1} c_i||x-y|| + (c_n + c_0)||x-y||)
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{i=1}^{n}(c_{1i}||x-y|| + c_{2i}||x-y||) + \sum_{i=1}^{n}(c_{3i}||x-y|| + c_{4i}||x-y||) \\
&\quad + M_{k+1}\left(\sum_{i=1}^{n-1} c_i||x-y|| + (c_n + c_0)||x-y||\right) \\
&\quad \left(\sum_{i=1}^{n}(c_{1i} + c_{2i} + c_{3i} + c_{4i}) + M_{k+1}\left(\sum_{i=1}^{n-1} c_i + (c_n + c_0)\right)\right) ||x-y||.
\end{aligned}
$$

The choice of c_0 guarantees the above growth constant is strictly less than one. As this is true for each k, it follows T is a contraction and hence has a unique fixed point x.

We now demonstrate $\alpha_{2i-2}(x^{(2i-2)}(t)) = x^{(2i-2)}(t)$, for $1 \le i \le n$. Suppose there exists t_0 such that $x^{(2n-2)}(t_0) > q^{(2n-2)}(t_0)$. Without loss of generality, assume $x^{(2n-2)}(t_0) - q^{(2n-2)}(t_0)$ is maximized. If $t_0 = 0$ then Lemma 2.1 implies

$$
\begin{aligned}
x^{(2n-2)}(0) &= \hat{g}_n(x^{(2n-2)}(0), x^{(2n-2)}(1)) \\
&\le \hat{g}_n(q^{(2n-2)}(0), q^{(2n-2)}(1)) \\
&= g_n(q^{(2n-2)}(0), q^{(2n-2)}(1)) \\
&\le q^{(2n-2)}(0)
\end{aligned}
$$

which is a contradiction. A similar argument applies if $t_0 = 1$. Hence $t_0 \in (0, 1)$.

Thus

$$\begin{aligned}
0 &\geq x^{(2n)}(t_0) - q^{(2n)}(t_0) \\
&\geq F_2(t_0, x(t_0), \dots, x^{(2n-2)}(t_0)) - f(t_0, q(t_0), \dots, q^{(2n-2)}(t_0)) \\
&\geq F_1(t_0, x(t_0), \dots, x^{(2n-2)}(t_0)) + c_0|x^{(2n-2)}(t_0) - q^{(2n-2)}(t_0)| \\
&\quad - f(t_0, q(t_0), \dots, q^{(2n-2)}(t_0)) \\
&\geq f(t_0, q(t_0), \dots, q^{(2n-2)}(t_0)) + c_0|x^{(2n-2)}(t_0) - q^{(2n-2)}(t_0)| \\
&\quad - f(t_0, q(t_0), \dots, q^{(2n-2)}(t_0)) \\
&> 0
\end{aligned}$$

where use was made of the increasing/decreasing properties of F_1 and f. This contradiction shows $x^{(2n-2)}(t) \leq q^{(2n-2)}(t)$ for all t. A similar argument establishes $p^{(2n-2)}(t) \leq x^{(2n-2)}(t)$. Now suppose $x^{(2n-4)}(t_0) < q^{(2n-4)}(t_0)$. The same argument using the boundary conditions can be used to establish $t_0 \neq 0, 1$. Thus using Lemma 2.2,

$$\begin{aligned}
x^{(2n-4)}(t) - q^{(2n-4)}(t) &= (x^{(2n-4)}(0) - q^{(2n-4)}(0))(1-t) + (x^{(2n-4)}(1) \\
&\quad - q^{(2n-4)}(1))t \\
&\quad + \int_0^1 H(t,s)(x^{(2n-2)}(s) - q^{(2n-2)}(s))ds \\
&\geq 0
\end{aligned}$$

for $t_0 \in (0,1)$. Thus $x^{(2n-4)}(t) \geq q^{(2n-4)}(t)$ for all $t_0 \in [0,1]$. A similar argument establishes $p^{(2n-4)}(t) \geq x^{(2n-4)}(t)$ for all $t_0 \in [0,1]$. Continuing in this manner, we obtain

$$(-1)^{i+1}p^{(2n-2i)}(t) \leq (-1)^{i+1}x^{(2n-2i)}t) \leq (-1)^{i+1}q^{(2n-2i)}(t) \text{ for all } t \in [0,1]$$

and $i = 1, 2, \ldots, n$, which is equivalent to $\alpha_{2i-2}(x^{(2i-2)}(t)) = x^{(2i-2)}(t)$, for $1 \leq i \leq n$. But this in turn implies

$$\begin{aligned}
F_2(t, x, x'', \ldots, x^{(2n-2)}) &= f(t, x, x'', \ldots, x^{(2n-2)}), \\
\hat{g}_i(x^{(2i-2)}(0), x^{(2i-2)}(1)) &= g_i(\alpha_{2i-2}(x^{(2i-2)}(0)), \alpha_{2i-2}(x^{(2i-2)}(1))), \\
\hat{h}_i(x^{(2i-2)}(0), x^{(2i-2)}(1)) &= h_i(\alpha_{2i-2}(x^{(2i-2)}(0)), \alpha_{2i-2}(x^{(2i-2)}(1))).
\end{aligned}$$

This implies x is a solution to (1.1), (1.2) satisfying the appropriate bounds.

Suppose z is another solution to (1.1), (1.2) satisfying the appropriate bounds. Then, it must be the case that $\alpha_{2i-2}(z^{(2i-2)}(t)) = z^{(2i-2)}(t)$, for $1 \leq i \leq n$. Lemma 2.1 coupled with the definition of F_2, $\hat{g}_i$, and $\hat{h}_i$ imply $Tz = z$. But T has a unique fixed point, hence $x = z$. $\square$

References

[1] R. AGARWAL AND P.J.Y. WONG, Lidstone polynomials and boundary value problems, *Comput. Math. Appl.*, **17** (1989), 1397-1421.

[2] K. AKO, Subfunctions for ordinary differential equations I, *J. Fac. Sci. Univ. Tokyo*, **9** (1965), 17-43.

[3] K. AKO, Subfunctions for ordinary differential equations II, *Funckialaj Ekvacioj*, **10** (1967), 145-162.

[4] K. AKO, Subfunctions for ordinary differential equations III, *Funckialaj Ekvacioj*, **11** (1968), 111-129.

[5] J. DAVIS AND J. HENDERSON, Uniqueness implies existence for fourth order Lidstone boundary value problems, *PanAmer. Math. J.*, **8** (1998), 23-35.

[6] J. EHME, P. ELOE AND J. HENDERSON, Existence of solutions for $2n^{th}$ order nonlinear generalized Sturm-Liouville boundary value problems, preprint.

[7] P. ELOE AND L. GRIMM, Monotone iteration and Green's functions for boundary value problems, *Proc. Amer. Math. Soc.*, **78** (1980), 533-538.

[8] P. ELOE AND J. HENDERSON, A boundary value problem for a system of ordinary differential equations with impulse effects, *Rocky Mtn. J. Math.*, **27** (1997), 785-799.

[9] R. GAINES, *A priori* bounds and upper and lower solutions for nonlinear second-order boundary value problems, *J. Differential Equations*, **12** (1972), 291-312.

[10] S. HONG AND S. HU, A monotone iterative method for higher-order boundary value problems, *Math. Appl.*, **12** (1999), 14-18.

[11] L. JACKSON, Subfunctions and second-order differential equations, *Advances Math.*, **2** (1968), 307-363.

[12] W. KELLY, Some existence theorems for nth-order boundary value problems, *J. Differential Equations*, **18** (1975), 158-169.

[13] G. KLASSEN, Differential inequalities and existence theorems for second and third order boundary value problems, *J. Differential Equations*, **10** (1971), 529-537.

[14] J. MAWHIN, Topological Degree Methods in Nonlinear Boundary Value Problems, *Regional Conference Series in Math.*, No. 40, American Mathematical Society, Providence, 1979.

[15] M. NAGUMO, Über die Differentialgleichungen $y'' = f(x, y, y')$, *Proc. Phys.-Math. Soc. Japan*, **19** (1937), 861-866.

[16] V. ŠEDA, Two remarks on boundary value problems for ordinary differential equations, *J. Differential Equations*, **26** (1977), 278-290.

[17] H. THOMPSON, Second order ordinary differential equations with fully nonlinear two point boundary conditions I, *Pacific J. Math.*, **172** (1996), 255-276.

[18] H. THOMPSON, Second order ordinary differential equations with fully nonlinear two point boundary conditions II, *Pacific J. Math.*, **172** (1996), 279-297.