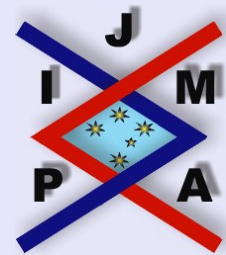


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## SOME PROPERTIES OF THE SERIES OF COMPOSED NUMBERS

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Abstract

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## Abstract

If  $c_n$  denotes the  $n$ -th composed number, one proves inequalities involving  $c_n, p_{c_n}, c_{p_n}$ , and one shows that the sequences  $(p_n)_{n \geq 1}$  and  $(c_{p_n})_{n \geq 1}$  are neither convex nor concave.

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# 1. Introduction

We are going to use the following notation

$\pi(x)$  the number of prime numbers  $\leq x$ ,

$C(x)$  the number of composed numbers  $\leq x$ ,

$p_n$  the  $n$ -th prime number,

$c_n$  the  $n$ -th composed number;  $c_1 = 4, c_2 = 6, \dots$ ,

$\log_2 n = \log(\log n)$ .

For  $x \geq 1$  we have the relation

$$(1.1) \quad \pi(x) + C(x) + 1 = [x].$$

Bojarincev proved (see [1], [4]) that

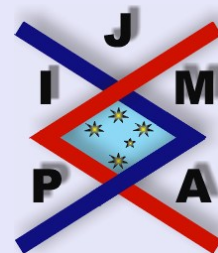
$$(1.2) \quad c_n = n \left( 1 + \frac{1}{\log n} + \frac{2}{\log^2 n} + \frac{4}{\log^3 n} + \frac{19}{2} \cdot \frac{1}{\log^4 n} + \frac{181}{6} \cdot \frac{1}{\log^5 n} + o\left(\frac{1}{\log^5 n}\right) \right).$$

Let us remark that

$$(1.3) \quad c_{k+1} - c_k = \begin{cases} 1 & \text{if } c_k + 1 \text{ is composed,} \\ 2 & \text{if } c_k + 1 \text{ is prime.} \end{cases}$$

In the proofs from the present paper, we shall need the following facts related to  $\pi(x)$  and  $p_n$ :

$$(1.4) \quad \text{for } x \geq 67, \quad \pi(x) > \frac{x}{\log x - 0.5}$$



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(see [7]);

$$(1.5) \quad \text{for } x \geq 3299, \quad \pi(x) > \frac{x}{\log x - \frac{28}{29}}$$

(see [6]);

$$(1.6) \quad \text{for } x \geq 4, \quad \pi(x) < \frac{x}{\log x - 1.12}$$

(see [6]);

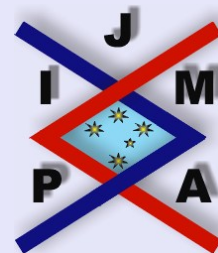
$$(1.7) \quad \text{for } n \geq 1, \quad \pi(x) = \frac{x}{\log x} \sum_{k=0}^n \frac{k!}{\log^k x} + O\left(\frac{x}{\log^{n+1} x}\right),$$

$$(1.8) \quad \text{for } n \geq 2, \quad p_n > n(\log n + \log_2 n - 1)$$

(see [2] and [3]);

$$(1.9) \quad \text{for } n \geq 6, \quad p_n < n(\log n + \log_2 n)$$

(see [7]).



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## 2. Inequalities Involving $c_n$

**Property 2.1.** *We have*

$$(2.1) \quad n \left( 1 + \frac{1}{\log n} + \frac{3}{\log^2 n} \right) > c_n > n \left( 1 + \frac{1}{\log n} + \frac{1}{\log^2 n} \right)$$

whenever  $n \geq 4$ .

*Proof.* If we take  $x = c_n$  in (1.1), then we get

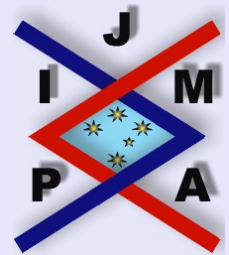
$$(2.2) \quad \pi(c_n) + n + 1 = c_n.$$

Now (1.4) implies that for  $n \geq 48$  we have

$$c_n > n + \pi(c_n) > n + \frac{n}{\log n}$$

and then

$$\begin{aligned} c_n &> n + \pi(c_n) > n + \pi \left( n \left( 1 + \frac{1}{\log n} \right) \right) \\ &> n + \frac{n \left( 1 + \frac{1}{\log n} \right)}{\log n + \log \left( 1 + \frac{1}{\log n} \right) - 0.5} \\ &> n + \frac{n \left( 1 + \frac{1}{\log n} \right)}{\log n} \\ &= n \left( 1 + \frac{1}{\log n} + \frac{1}{\log^2 n} \right). \end{aligned}$$



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By (1.6) and (2.2) it follows that

$$c_n \cdot \frac{\log c_n - 2.12}{\log c_n - 1.12} < n + 1.$$

Since  $c_n > n$ , it follows that  $\frac{\log c_n - 2.12}{\log c_n - 1.12} > \frac{\log n - 2.12}{\log n - 1.12}$  hence

$$(2.3) \quad n + 1 > c_n \cdot \frac{\log n - 2.12}{\log n - 1.12}.$$

Assume that there would exist  $n \geq 1747$  such that

$$c_n \geq n \left( 1 + \frac{1}{\log n} + \frac{3}{\log^2 n} \right).$$

Then a direct computation shows that (12) implies

$$\frac{1}{n} \geq \frac{0.88 \log n - 6.36}{\log^2 n (\log n - 1.12)}.$$

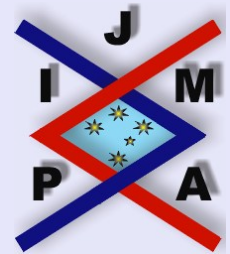
For  $n \geq 1747$ , one easily shows that  $\frac{0.88 \log n - 6.36}{\log n - 1.12} > \frac{1}{31}$ , hence  $\frac{1}{n} > \frac{1}{31 \log^2 n}$ . But this is impossible, since for  $n \geq 1724$  we have  $\frac{1}{n} < \frac{1}{31 \log^2 n}$ .

Consequently we have  $c_n < n \left( 1 + \frac{1}{\log n} + \frac{3}{\log^2 n} \right)$ . By checking the cases when  $n \leq 1746$ , one completely proves the stated inequalities.  $\square$

**Property 2.2.** *If  $n \geq 30,398$ , then the inequality*

$$p_n > c_n \log c_n$$

*holds.*



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*Proof.* We use (1.8), (2.1) and the inequalities

$$\log \left( 1 + \frac{1}{\log n} + \frac{3}{\log^2 n} \right) < \frac{1}{\log n} + \frac{3}{\log^2 n},$$

and

$$n(\log n + \log \log n - 1) > n \left( 1 + \frac{1}{\log n} + \frac{3}{\log^2 n} \right) \left( \log n + \frac{1}{\log n} + \frac{3}{\log^2 n} \right),$$

that is  $\log \log n > 2 + \frac{4}{\log n} + \frac{4}{\log^2 n} + \frac{6}{\log^3 n} + \frac{9}{\log^4 n}$ , which holds if  $n \geq 61,800$ . Now the proof can be completed by checking the remaining cases.  $\square$

**Proposition 2.1.** *We have*

$$\pi(n)p_n > c_n^2$$

whenever  $n \geq 19,421$ .

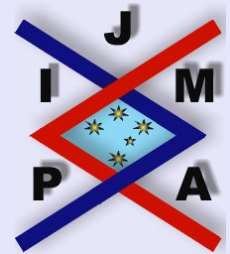
*Proof.* In view of the inequalities (1.5), (1.8) and (2.1), for  $n \geq 3299$  it remains to prove that  $\frac{\log n + \log_2 n - 1}{\log n - \frac{28}{29}} > \left( 1 + \frac{1}{\log n} + \frac{3}{\log^2 n} \right)^2$ , that is

$$\log \log n > \frac{59}{29} + \frac{5.069}{\log n} - \frac{0.758}{\log^2 n} + \frac{3.207}{\log^3 n} - \frac{8.68}{\log^4 n}.$$

It suffices to show that

$$\log \log n > \frac{59}{29} + \frac{5.069}{\log n}.$$

For  $n = 130,000$ , one gets  $2.466 \dots > 2.4649 \dots$ . The checking of the cases when  $n < 130,000$  completes the proof.  $\square$



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### 3. Inequalities Involving $c_{p_n}$ and $p_{c_n}$

**Proposition 3.1.** *We have*

$$(3.1) \quad p_n + n < c_{p_n} < p_n + n + \pi(n)$$

for  $n$  sufficiently large.

*Proof.* By (1.2) and (1.7) it follows that for  $n$  sufficiently large we have  $c_n = n + \pi(n) + \frac{n}{\log^2 n} + O\left(\frac{n}{\log^3 n}\right)$ , hence

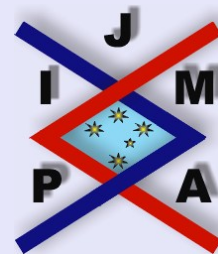
$$(3.2) \quad c_{p_n} = p_n + n + \frac{p_n}{\log^2 p_n} + O\left(\frac{n}{\log^2 n}\right).$$

Thus for  $n$  large enough we have  $c_{p_n} > p_n + n$ .

Since the function  $x \mapsto \frac{x}{\log^2 x}$  is increasing, one gets by (1.9)

$$\begin{aligned} \frac{p_n}{\log^2 p_n} &< \frac{n(\log n + \log_2 n)}{(\log n + \log(\log n + \log_2 n))^2} \\ &< \frac{n(\log n + \log_2 n)}{\log n(\log n + 2 \log_2 n)} \\ &< n \cdot \frac{\log n - \frac{1}{2} \log_2 n}{\log^2 n} \\ &= \pi(n) - \frac{1}{2} \cdot \frac{n \log_2 n}{\log^2 n} + O\left(\frac{n}{\log^2 n}\right). \end{aligned}$$

Both this inequality and (3.2) show that for  $n$  sufficiently large we have indeed  $c_{p_n} < p_n + n + \pi(n)$ . □



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**Proposition 3.2.** *If  $n$  is large enough, then the inequality*

$$p_{c_n} > c_{p_n}$$

*holds.*

*Proof.* By (2.1) it follows that

$$(3.3) \quad c_{p_n} = \pi(c_{p_n}) + p_n + 1.$$

Now (3.1) and (3.3) imply that for  $n$  sufficiently large we have  $\pi(c_{p_n}) < n + \pi(n)$ . But by (2.1) it follows that  $c_n > n + \pi(n)$ , hence  $c_n > \pi(c_{p_n})$ . If we assume that  $c_{p_n} > p_{c_n}$ , then we obtain the contradiction  $\pi(c_{p_n}) \geq \pi(p_{c_n}) = c_n$ . Consequently we must have  $c_{p_n} < p_{c_n}$ .  $\square$

It is easy to show that the sequence  $(c_n)_{n \geq 1}$  is neither convex nor concave. We are lead to the same conclusion by studying the sequences  $(c_{p_n})_{n \geq 1}$  and  $(p_{c_n})_{n \geq 1}$ . Let us say that a sequence  $(a_n)_{n \geq 1}$  has the property  $P$  when the inequality

$$a_{n+1} - 2a_n + a_{n-1} > 0$$

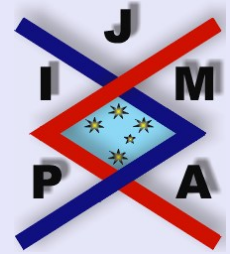
holds for infinitely many indices and the inequality

$$a_{n+1} - 2a_n + a_{n-1} < 0$$

holds also for infinitely many indices. Then we can prove the following fact.

**Proposition 3.3.** *Both sequences  $(c_{p_n})_{n \geq 1}$  and  $(p_{c_n})_{n \geq 1}$  have the property  $P$ .*

In order to prove it we need the following auxiliary result.



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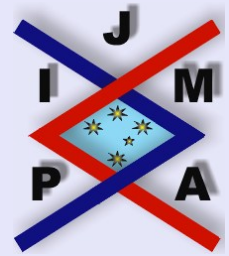


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**Lemma 3.4.** *If the sequence  $(a_n)_{n \geq n_1}$  is convex, then for  $m > n \geq n_1$  we have*

$$(3.4) \quad \frac{a_m - a_n}{m - n} \geq a_{n+1} - a_n.$$

*If the sequence  $(a_n)_{n \geq n_2}$  is concave, then for  $n > p \geq n_2$  we have*

$$(3.5) \quad \frac{a_n - a_p}{n - p} \geq a_{n+1} - a_n$$

whenever  $m > n \geq n_1$ .

*Proof.* In the first case, for  $i \geq n$  we have  $a_{i+1} - a_i \geq a_{n+1} - a_n$ , hence  $\sum_{i=n}^{m-1} (a_{i+1} - a_i) \geq (m - n)(a_{n+1} - a_n)$ , that is (3.4). The inequality (3.5) can be proved similarly.  $\square$

*Proof of Proposition 3.3.* Erdős proved in [3] that, with  $d_n = p_{n+1} - p_n$ , we have

$\limsup_{n \rightarrow \infty} \frac{\min(d_n, d_{n+1})}{\log n} = \infty$ . In particular, the set  $M = \{n \mid \min(d_n, d_{n+1}) > 2 \log n\}$  is infinite.

For every  $n$ , at least one of the numbers  $n$  and  $n + 1$  is composed, that is, either  $n = c_m$  or  $n + 1 = c_m$  for some  $m$ . Consequently, there exist infinitely many indices  $m$  such that  $p_{c_{m+1}} - p_{c_m} > 2 \log c_m$ . Since  $c_{m+1} \geq c_m + 1$  and  $c_m > m$ , we get infinitely many values of  $m$  such that

$$(3.6) \quad p_{c_{m+1}} - p_{c_m} > 2 \log m.$$

Let  $M'$  be the set of these numbers  $m$ .

If we assume that the sequence  $(p_{c_n})_{n \geq n_1}$  is convex, then (3.4) implies that for  $m \in M'$  we have

$$\frac{p_{c_{2m}} - p_{c_m}}{m} \geq p_{c_{m+1}} - p_{c_m} > 2 \log m,$$

hence  $p_{c_{2m}} > 2m \log m + p_{c_m}$ . But this is a contradiction because  $c_n \sim n$  and  $p_n \sim n \log n$ , that is  $p_{c_{2m}} \sim 2m \log 2m$  and  $p_{c_m} \sim m \log m$ .

On the other hand, if we assume that the sequence  $(p_{c_n})_{n \geq n_2}$  is concave, then (3.5) implies that for  $x \in M'$  we have

$$\frac{p_{c_m} - p_{c[\frac{m}{2}]}}{m - [\frac{m}{2}]} \geq p_{c_{m+1}} - p_{c_m} > 2 \log m,$$

that is

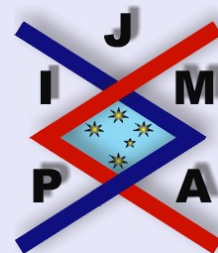
$$1 > \frac{2 \left( m - \left[ \frac{m}{2} \right] \right) \log m + p_{c[\frac{m}{2}]}}{p_{c_m}}.$$

For  $m \rightarrow \infty$ ,  $m \in M'$ , the last inequality implies the contradiction  $1 \geq 1 + \frac{1}{2}$ . Consequently the sequence  $(p_{c_n})_{n \geq 1}$  has the property  $P$ .

Now let us assume that the sequence  $(c_{p_n})_{n \geq n_1}$  is convex. Then for  $n \in M$ ,  $n \geq n_1$ , we get by (3.4)

$$\frac{c_{p_{2n}} - c_{p_n}}{n} \geq c_{p_{n+1}} - c_{p_n} \geq p_{n+1} - p_n > 2 \log n.$$

If we take  $n \rightarrow \infty$ ,  $n \in M$ , in the inequality  $1 > (2n \log n + c_{p_n})/c_{p_{2n}}$ , then we obtain the contradiction  $1 \geq \frac{3}{2}$ .



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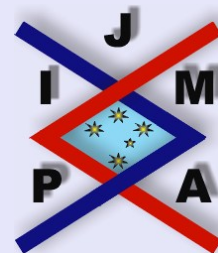
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Finally, if we assume that the sequence  $(c_{p_n})_{n \geq n_2}$  is concave, then (3.5) implies that for  $n \in M$ ,  $n \geq n_2$ , we have

$$\frac{c_{p_n} - c_{p_{\lfloor n/2 \rfloor}}}{n - \lfloor \frac{n}{2} \rfloor} \geq c_{p_{n+1}} - c_{p_n} \geq p_{n+1} - p_n > 2 \log n,$$

which is again a contradiction. □




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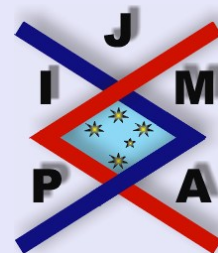
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