Journal of Inequalities in Pure and Applied Mathematics

SOME PROPERTIES OF THE SERIES OF COMPOSED NUMBERS

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volume 2, issue 3, article 38, 2001.

Received 27 March, 2001; accepted 11 June, 2001.

Communicated by: L. Toth

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ISSN (electronic): 1443-5756
027-01

Abstract

If c_n denotes the n-th composed number, one proves inequalities involving c_n, p_{c_n}, c_{p_n}, and one shows that the sequences $(p_n)_{n\geq 1}$ and $(c_{p_n})_{n\geq 1}$ are neither convex nor concave.

2000 Mathematics Subject Classification: 11A25, 11N05.
Key words: Prime Numbers, Composed Numbers, Asymptotic Behavior, Inequalities, Sums and Series.

The author gratefully acknowledges for partial support from Grant No. 7D/2000 awarded by the Consiliul Naţional al Cercetării Ştiinţifice din Învăţământul Superior, România.

Contents

1. Introduction

We are going to use the following notation

$$\begin{aligned} \pi(x) &\text{ the number of prime numbers } \le x,\\ C(x) &\text{ the number of composed numbers } \le x,\\ p_n &\text{ the } n\text{-th prime number},\\ c_n &\text{ the } n\text{-th composed number; } c_1 = 4, c_2 = 6, \dots,\\ \log_2 n &= \log(\log n). \end{aligned}$$

For $x \ge 1$ we have the relation

(1.1) $$\pi(x) + C(x) + 1 = [x].$$

Bojarincev proved (see [1], [4]) that

(1.2) $$c_n = n\left(1 + \frac{1}{\log n} + \frac{2}{\log^2 n} + \frac{4}{\log^3 n} + \frac{19}{2}\cdot\frac{1}{\log^4 n} + \frac{181}{6}\cdot\frac{1}{\log^5 n} + o\left(\frac{1}{\log^5 n}\right)\right).$$

Let us remark that

(1.3) $$c_{k+1} - c_k = \begin{cases} 1 & \text{if } c_k + 1 \text{ is composed,}\\ 2 & \text{if } c_k + 1 \text{ is prime.} \end{cases}$$

In the proofs from the present paper, we shall need the following facts related to $\pi(x)$ and p_n:

(1.4) $$\text{for } x \ge 67, \quad \pi(x) > \frac{x}{\log x - 0.5}$$

(see [7]);

$$\text{for } x \geq 3299, \quad \pi(x) > \frac{x}{\log x - \frac{28}{29}} \tag{1.5}$$

(see [6]);

$$\text{for } x \geq 4, \quad \pi(x) < \frac{x}{\log x - 1.12} \tag{1.6}$$

(see [6]);

$$\text{for } n \geq 1, \quad \pi(x) = \frac{x}{\log x} \sum_{k=0}^{n} \frac{k!}{\log^k x} + O\left(\frac{x}{\log^{n+1} x}\right), \tag{1.7}$$

$$\text{for } n \geq 2, \quad p_n > n(\log n + \log_2 n - 1) \tag{1.8}$$

(see [2] and [3]);

$$\text{for } n \geq 6, \quad p_n < n(\log n + \log_2 n) \tag{1.9}$$

(see [7]).

2. Inequalities Involving c_n

Property 2.1. *We have*

$$n\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right) > c_n > n\left(1+\frac{1}{\log n}+\frac{1}{\log^2 n}\right) \tag{2.1}$$

whenever $n \geq 4$.

Proof. If we take $x = c_n$ in (1.1), then we get

$$\pi(c_n) + n + 1 = c_n. \tag{2.2}$$

Now (1.4) implies that for $n \geq 48$ we have

$$c_n > n + \pi(c_n) > n + \frac{n}{\log n}$$

and then

$$\begin{aligned} c_n &> n + \pi(c_n) > n + \pi\left(n\left(1+\frac{1}{\log n}\right)\right) \\ &> n + \frac{n\left(1+\frac{1}{\log n}\right)}{\log n + \log\left(1+\frac{1}{\log n}\right) - 0.5} \\ &> n + \frac{n\left(1+\frac{1}{\log n}\right)}{\log n} \\ &= n\left(1+\frac{1}{\log n}+\frac{1}{\log^2 n}\right). \end{aligned}$$

By (1.6) and (2.2) it follows that

$$c_n \cdot \frac{\log c_n - 2.12}{\log c_n - 1.12} < n + 1.$$

Since $c_n > n$, it follows that $\frac{\log c_n - 2.12}{\log c_n - 1.12} > \frac{\log n - 2.12}{\log n - 1.12}$ hence

$$n + 1 > c_n \cdot \frac{\log n - 2.12}{\log n - 1.12}. \tag{2.3}$$

Assume that there would exist $n \geq 1747$ such that

$$c_n \geq n \left(1 + \frac{1}{\log n} + \frac{3}{\log^2 n}\right).$$

Then a direct computation shows that (12) implies

$$\frac{1}{n} \geq \frac{0.88 \log n - 6.36}{\log^2 n(\log n - 1.12)}.$$

For $n \geq 1747$, one easily shows that $\frac{0.88 \log n - 6.36}{\log n - 1.12} > \frac{1}{31}$, hence $\frac{1}{n} > \frac{1}{31 \log^2 n}$. But this is impossible, since for $n \geq 1724$ we have $\frac{1}{n} < \frac{1}{31 \log^2 n}$.

Consequently we have $c_n < n \left(1 + \frac{1}{\log n} + \frac{3}{\log^2 n}\right)$. By checking the cases when $n \leq 1746$, one completely proves the stated inequalities. □

Property 2.2. *If $n \geq 30,398$, then the inequality*

$$p_n > c_n \log c_n$$

holds.

Proof. We use (1.8), (2.1) and the inequalities

$$\log\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right) < \frac{1}{\log n}+\frac{3}{\log^2 n},$$

and

$$n(\log n+\log\log n-1) > n\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right)\left(\log n+\frac{1}{\log n}+\frac{3}{\log^2 n}\right),$$

that is $\log\log n > 2+\frac{4}{\log n}+\frac{4}{\log^2 n}+\frac{6}{\log^3 n}+\frac{9}{\log^4 n}$, which holds if $n \geq 61,800$. Now the proof can be completed by checking the remaining cases. □

Proposition 2.1. *We have*

$$\pi(n)p_n > c_n^2$$

whenever $n \geq 19,421$.

Proof. In view of the inequalities (1.5), (1.8) and (2.1), for $n \geq 3299$ it remains to prove that $\frac{\log n+\log_2 n-1}{\log n-\frac{28}{29}} > \left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right)^2$, that is

$$\log\log n > \frac{59}{29}+\frac{5.069}{\log n}-\frac{0.758}{\log^2 n}+\frac{3.207}{\log^3 n}-\frac{8.68}{\log^4 n}.$$

It suffices to show that

$$\log\log n > \frac{59}{29}+\frac{5.069}{\log n}.$$

For $n = 130,000$, one gets $2.466\cdots > 2.4649\ldots$. The checking of the cases when $n < 130,000$ completes the proof. □

3. Inequalities Involving c_{p_n} and p_{c_n}

Proposition 3.1. *We have*

$$p_n + n < c_{p_n} < p_n + n + \pi(n) \tag{3.1}$$

for n sufficiently large.

Proof. By (1.2) and (1.7) it follows that for n sufficiently large we have $c_n = n + \pi(n) + \frac{n}{\log^2 n} + O\left(\frac{n}{\log^3 n}\right)$, hence

$$c_{p_n} = p_n + n + \frac{p_n}{\log^2 p_n} + O\left(\frac{n}{\log^2 n}\right). \tag{3.2}$$

Thus for n large enough we have $c_{p_n} > p_n + n$.

Since the function $x \mapsto \frac{x}{\log^2 x}$ is increasing, one gets by (1.9)

$$\begin{aligned}\frac{p_n}{\log^2 p_n} &< \frac{n(\log n + \log_2 n)}{(\log n + \log(\log n + \log_2 n))^2} \\ &< \frac{n(\log n + \log_2 n)}{\log n(\log n + 2\log_2 n)} \\ &< n \cdot \frac{\log n - \frac{1}{2}\log_2 n}{\log^2 n} \\ &= \pi(n) - \frac{1}{2} \cdot \frac{n \log_2 n}{\log^2 n} + O\left(\frac{n}{\log^2 n}\right).\end{aligned}$$

Both this inequality and (3.2) show that for n sufficiently large we have indeed $c_{p_n} < p_n + n + \pi(n)$. □

Proposition 3.2. *If n is large enough, then the inequality*

$$p_{c_n} > c_{p_n}$$

holds.

Proof. By (2.1) it follows that

$$c_{p_n} = \pi(c_{p_n}) + p_n + 1. \tag{3.3}$$

Now (3.1) and (3.3) imply that for n sufficiently large we have $\pi(c_{p_n}) < n + \pi(n)$. But by (2.1) it follows that $c_n > n + \pi(n)$, hence $c_n > \pi(c_{p_n})$. If we assume that $c_{p_n} > p_{c_n}$, then we obtain the contradiction $\pi(c_{p_n}) \geq \pi(p_{c_n}) = c_n$. Consequently we must have $c_{p_n} < p_{c_n}$. □

It is easy to show that the sequence $(c_n)_{n\geq 1}$ is neither convex nor concave. We are lead to the same conclusion by studying the sequences $(c_{p_n})_{n\geq 1}$ and $(p_{c_n})_{n\geq 1}$. Let us say that a sequence $(a_n)_{n\geq 1}$ *has the property P* when the inequality

$$a_{n+1} - 2a_n + a_{n-1} > 0$$

holds for infinitely many indices and the inequality

$$a_{n+1} - 2a_n + a_{n-1} < 0$$

holds also for infinitely many indices. Then we can prove the following fact.

Proposition 3.3. *Both sequences $(c_{p_n})_{n\geq 1}$ and $(p_{c_n})_{n\geq 1}$ have the property P.*

In order to prove it we need the following auxiliary result.

Lemma 3.4. *If the sequence* $(a_n)_{n\geq n_1}$ *is convex, then for* $m > n \geq n_1$ *we have*

$$\frac{a_m - a_n}{m-n} \geq a_{n+1} - a_n. \tag{3.4}$$

If the sequence $(a_n)_{n\geq n_2}$ *is concave, then for* $n > p \geq n_2$ *we have*

$$\frac{a_n - a_p}{n-p} \geq a_{n+1} - a_n \tag{3.5}$$

whenever $m > n \geq n_1$.

Proof. In the first case, for $i \geq n$ we have $a_{i+1} - a_i \geq a_{n+1} - a_n$, hence $\sum_{i=n}^{m-1}(a_{i+1} - a_i) \geq (m-n)(a_{n+1} - a_n)$, that is, (3.4). The inequality (3.5) can be proved similarly. □

Proof of Proposition 3.3. Erdös proved in [3] that, with $d_n = p_{n+1} - p_n$, we have $\limsup_{n\to\infty} \frac{\min(d_n,d_{n+1})}{\log n} = \infty$. In particular, the set $M = \{n \mid \min(d_n, d_{n+1}) > 2\log n\}$ is infinite.

For every n, at least one of the numbers n and $n+1$ is composed, that is, either $n = c_m$ or $n+1 = c_m$ for some m. Consequently, there exist infinitely many indices m such that $p_{c_{m+1}} - p_{c_m} > 2\log c_m$. Since $c_{m+1} \geq c_m + 1$ and $c_m > m$, we get infinitely many values of m such that

$$p_{c_{m+1}} - p_{c_m} > 2\log m. \tag{3.6}$$

Let M' be the set of these numbers m.

If we assume that the sequence $(p_{c_n})_{n\geq n_1}$ is convex, then (3.4) implies that for $m \in M'$ we have

$$\frac{p_{c_{2m}} - p_{c_m}}{m} \geq p_{c_{m+1}} - p_{c_m} > 2\log m,$$

hence $p_{c_{2m}} > 2m\log m + p_{c_m}$. But this is a contradiction because $c_n \sim n$ and $p_n \sim n\log n$, that is $p_{c_{2m}} \sim 2m\log 2m$ and $p_{c_m} \sim m\log m$.

On the other hand, if we assume that the sequence $(p_{c_n})_{n\geq n_2}$ is concave, then (3.5) implies that for $x \in M'$ we have

$$\frac{p_{c_m} - p_{c[m/2]}}{m - \left[\frac{m}{2}\right]} \geq p_{c_{m+1}} - p_{c_m} > 2\log m,$$

that is

$$1 > \frac{2\left(m - \left[\frac{m}{2}\right]\right)\log m + p_{c[m/2]}}{p_{c_m}}.$$

For $m \to \infty$, $m \in M'$, the last inequality implies the contradiction $1 \geq 1 + \frac{1}{2}$. Consequently the sequence $(p_{c_n})_{n\geq 1}$ has the property P.

Now let us assume that the sequence $(c_{p_n})_{n\geq n_1}$ is convex. Then for $n \in M$, $n \geq n_1$, we get by (3.4)

$$\frac{c_{p_{2n}} - c_{p_n}}{n} \geq c_{p_{n+1}} - c_{p_n} \geq p_{n+1} - p_n > 2\log n.$$

If we take $n \to \infty$, $n \in M$, in the inequality $1 > (2n\log n + c_{p_n})/c_{p_{2n}}$, then we obtain the contradiction $1 \geq \frac{3}{2}$.

Finally, if we assume that the sequence $(c_{p_n})_{n \geq n_2}$ is concave, then (3.5) implies that for $n \in M$, $n \geq n_2$, we have

$$\frac{c_{p_n} - c_{p_{[n/2]}}}{n - \left[\frac{n}{2}\right]} \geq c_{p_{n+1}} - c_{p_n} \geq p_{n+1} - p_n > 2\log n,$$

which is again a contradiction. □

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