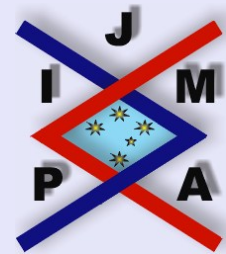


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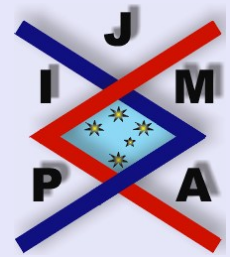


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Abstract

In the present article we study the asymptotic behavior of the sums $\sum_{n \leq x} \left| \frac{c_{n+1}}{p_{n+1}} - \frac{c_n}{p_n} \right|$ and $\sum_{n \leq x} \left| \frac{p_{n+1}}{c_{n+1}} - \frac{p_n}{c_n} \right|$, and of the series $\sum_{n=1}^{\infty} \left| \frac{c_{n+1}}{p_{n+1}} - \frac{c_n}{p_n} \right|^{\alpha}$, where p_n denotes the n -th prime number while c_n stands for the n -th composed number.

2000 Mathematics Subject Classification: 11A25, 11N05.

Key words: Sequences, Series, Prime Numbers, the Sequence of Composed Numbers, Asymptotic Formulae

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1. Introduction

We are going to use the following notation

$\pi(x)$ the number of prime numbers $\leq x$,

$C(x)$ the number of composed numbers $\leq x$,

p_n the n -th prime number,

c_n the n -th composed number; $c_1 = 4, c_2 = 6, \dots$,

$\log_2 n = \log(\log n)$.

The present work originates in a result due to Erdős and Prachar [2]: they proved that there exist $c', c'' > 0$ such that

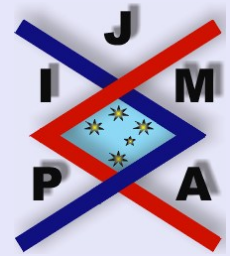
$$c' \log^2 x > \sum_{p_k \leq x} \left| \frac{p_{k+1}}{k+1} - \frac{p_k}{k} \right| > c'' \log^2 x \text{ for } x \geq 2,$$

that is

$$(1.1) \quad \sum_{p_k \leq x} \left| \frac{p_{k+1}}{k+1} - \frac{p_k}{k} \right| \asymp \log^2 x.$$

In a recent paper [3], Panaitopol proved that

$$(1.2) \quad \sum_{p_k \leq x} \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| \asymp \log \log x.$$



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The proofs of these results rely on the following result due to Schnirelmann: if for x positive and n a positive integer one denotes by $M(n, x)$ the number of the indices k such that $p_k \leq x$ and $p_{k+1} - p_k = n$, then

$$M(n, x) < c''' \frac{x}{\log^2 x} \sum_{d|n} \frac{1}{d},$$

where c''' is a positive constant.

In the present paper, several well known results will be used:

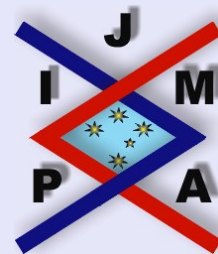
$$(1.3) \quad \pi(x) \sim \frac{x}{\log x};$$

$$(1.4) \quad p_n \sim n \log n;$$

$$(1.5) \quad \text{the series } \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^\alpha} \text{ is convergent if and only if } \alpha > 1;$$

$$(1.6) \quad \sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log \log x + O(1);$$

$$(1.7) \quad \sum_{2 \leq n \leq x} \frac{1}{n} = \log x + O(1);$$



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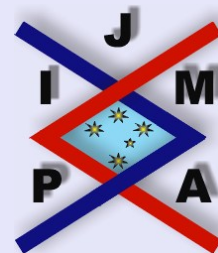
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$$(1.8) \quad \sum_{p \text{ prime} \leq x} \frac{\log p}{p} \sim \log x.$$

We also need the following result of Bojarincev [1]:

$$(1.9) \quad c_n = n + \frac{n}{\log n} \cdot u_n, \text{ where } \lim_{n \rightarrow \infty} u_n = 1.$$



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2. Properties of the Sequence $\left(\frac{n}{p_n}\right)_{n \geq 1}$

The series $\sum_{n=1}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|$ is divergent by (1.2). In connection with this fact we prove the following result.

Theorem 2.1. *The series*

$$\sum_{n=3}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right| \cdot \frac{1}{(\log \log n)^\alpha}$$

is convergent if and only if $\alpha > 1$.

Let us first prove the following auxiliary result.

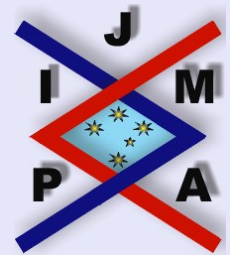
Lemma 2.2. *Consider the sequences $(a_n)_{n \geq 1}$, $(x_n)_{n \geq 1}$ and $(s_n)_{n \geq 1}$, where $s_n = \sum_{i=1}^n a_i$. If the sequence $(s_n x_n)_{n \geq 1}$ is convergent, then one of the series*

$$\sum_{n=1}^{\infty} s_n(x_{n+1} - x_n) \text{ and } \sum_{n=1}^{\infty} a_n x_n$$

is convergent if and only if the other one is convergent.

Proof. If $\sum_{n=1}^{\infty} s_n(x_{n+1} - x_n)$ is convergent, then $\lim_{n \rightarrow \infty} s_n(x_{n+1} - x_n) = 0$. But $\lim_{n \rightarrow \infty} s_n x_n = k$ for some $k \in \mathbb{R}$, hence $\lim_{n \rightarrow \infty} s_n x_{n+1} = k$ and $\lim_{n \rightarrow \infty} (s_{n+1} - s_n)x_{n+1} = 0$.

On the other hand, if $\sum_{n=1}^{\infty} a_n x_n$ is convergent, then $\lim_{n \rightarrow \infty} a_{n+1} x_{n+1} = 0$, hence $\lim_{n \rightarrow \infty} (s_{n+1} - s_n)x_{n+1} = 0$.



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Now let us denote $S_n = \sum_{i=1}^n a_i x_i$ and $\sigma_n = \sum_{i=1}^n s_i(x_{i+1} - x_i)$. Then for each p we have

$$(2.1) \quad S_{n+p} - S_n = \sigma_{n+p} - \sigma_n + s_{n+p}x_{n+p} - s_{n+1}x_{n+1} + (s_{n+1} - s_n)x_{n+1}.$$

Since we have just seen that in either case we have $\lim_{n \rightarrow \infty} (s_{n+1} - s_n)x_{n+1} = 0$, relation (2.1) implies that in either case one of the sequences $(S_n)_{n \geq 1}$ and $(\sigma_n)_{n \geq 1}$ is Cauchy if and only if the other one is Cauchy. Now, by Cauchy's criterion, one of the two series is convergent if and only if the other one is convergent. \square

Proof of Theorem 2.1. If $\alpha \leq 0$, then the series is divergent by (1.2). Next assume $\alpha > 0$ and choose $a_n = \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|$ and $x_n = \frac{1}{(\log \log n)^\alpha}$.

If we consider the function $f : [n, n+1] \rightarrow \mathbb{R}$, $f(x) = (\log \log x)^{-\alpha}$, then Lagrange's theorem implies

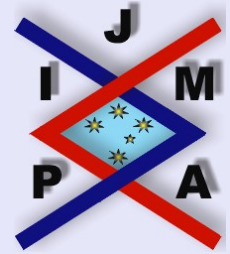
$$(\log \log(n+1))^{-\alpha} - (\log \log n)^{-\alpha} = -\frac{\alpha(\log \log \theta_n)^{-\alpha-1}}{\theta_n \log \theta_n},$$

where $n < \theta_n < n+1$. Since $\theta_n \sim n$, it follows that

$$(2.2) \quad x_{n+1} - x_n \sim \frac{-\alpha}{n \log n (\log \log n)^{\alpha+1}}.$$

By (1.2) and (1.4) it follows that $s_n \asymp \log \log n$ and (2.2) implies

$$(2.3) \quad s_n(x_{n+1} - x_n) \asymp -\frac{1}{n \log n (\log \log n)^\alpha}.$$



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If $\alpha > 1$, then we have

$$\lim_{n \rightarrow \infty} s_n x_n = \lim_{n \rightarrow \infty} \frac{\log \log n}{(\log \log n)^\alpha} = 0,$$

while for $\alpha = 1$ we get $\lim_{n \rightarrow \infty} s_n x_n = 1$. Thus, for $\alpha \geq 1$, the above lemma implies that one of the series $\sum_{n=3}^{\infty} s_n(x_{n+1} - x_n)$ and $\sum_{n=3}^{\infty} a_n x_n$ is convergent if and only if the other one is convergent. In view of (1.5) and (2.3), the series $\sum_{n=3}^{\infty} a_n x_n$ is convergent for $\alpha > 1$ and divergent for $\alpha = 1$.

Finally, if $0 < \alpha < 1$, then

$$x_n \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right| > \frac{1}{\log \log n} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|$$

and the desired conclusion follows. □

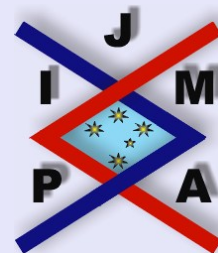
Consequence 1. *If $\alpha > 1$, then the series*

$$\sum_{n=1}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^\alpha$$

is convergent.

Proof. We have

$$\left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^{\alpha-1} \leq \max \left(\frac{n+1}{p_{n+1}}, \frac{n}{p_n} \right)^{\alpha-1}.$$



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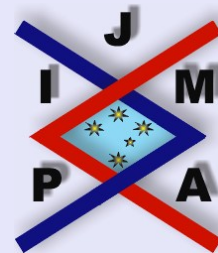
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For $K > 0$ and $n \geq 3$, we have $\frac{1}{(\log n)^{\alpha-1}} < \frac{K}{(\log \log n)^\alpha}$ and (1.4) implies that $\frac{n+1}{p_{n+1}} \sim \frac{n}{p_n} \sim \frac{1}{\log n}$. There exists K' such that

$$\left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^\alpha < K' \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right| \frac{1}{(\log \log n)^\alpha}$$

and the convergence of the series $\sum_{n=1}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^\alpha$ follows by Theorem 2.1. \square



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3. Properties of the Sequence $\left(\frac{c_n}{p_n}\right)_{n \geq 1}$

Since $c_n \sim n$ (see (1.9)), for the sequence $\left(\frac{c_n}{p_n}\right)_{n \geq 1}$ we obtain properties which are similar to those of the sequence $\left(\frac{n}{p_n}\right)_{n \geq 1}$.

In connection with (1.2) we have the following fact.

Theorem 3.1. *We have*

$$\sum_{p_k \leq x} \left| \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} \right| \asymp \log \log x$$

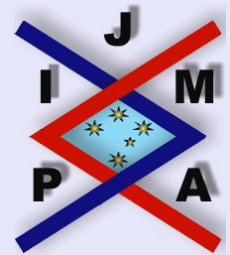
for every $x > e$.

Proof. If we denote $\alpha_k = c_{k+1} - c_k - 1$, then it follows that $\alpha_k = 0$ if $c_k + 1$ is a composed number, and $\alpha_k = 1$ if $c_k + 1$ is prime. In the last case $c_k + 1 = p_m$. Setting $k = k(m)$, we deduce by (1.9) that $c_k - k \sim \frac{k}{\log k}$ and $\log k \sim \log c_k \sim \log p_m \sim \log m$. It then follows that $k(m) - p_m \sim -\frac{p_m}{\log m}$ and (1.4) implies that

$$(3.1) \quad k(m) = p_m - my_m \text{ with } \lim_{m \rightarrow \infty} y_m = 1.$$

We have by (1.9)

$$\begin{aligned} \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} &= \frac{c_k + 1 + \alpha_k}{p_{k+1}} - \frac{c_k}{p_k} = \frac{\alpha_k + 1}{p_{k+1}} - \frac{c_k(p_{k+1} - p_k)}{p_k p_{k+1}} \\ &= \frac{\alpha_k + 1}{p_{k+1}} - \frac{k(p_{k+1} - p_k)}{p_k p_{k+1}} - \frac{ku_k}{(\log k)p_k p_{k+1}} \end{aligned}$$



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$$= \left(\frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right) + \frac{\alpha_k}{p_{k+1}} - \frac{ku_k(p_{k+1} - p_k)}{(\log k)p_k p_{k+1}}.$$

We have the inequality

$$(3.2) \quad \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| - \left| \frac{\alpha_k}{p_{k+1}} \right| - \left| \frac{ku_k(p_{k+1} - p_k)}{(\log k)p_k p_{k+1}} \right| \\ \leq \left| \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} \right| \leq \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| + \left| \frac{\alpha_k}{p_{k+1}} \right| + \left| \frac{ku_k(p_{k+1} - p_k)}{(\log k)p_k p_{k+1}} \right|.$$

We have

$$\frac{ku_k(p_{k+1} - p_k)}{(\log k)p_k p_{k+1}} \sim \frac{k(p_{k+1} - p_k)}{k^2 \log^3 k} = \frac{p_{k+1} - p_k}{k \log^3 k}.$$

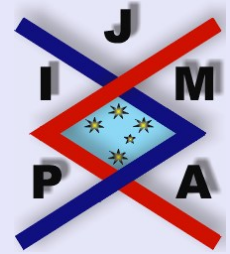
Panaitopol proves in [4] that for $\beta > 2$ the series $\sum_{n=2}^{\infty} \frac{p_{n+1} - p_n}{n \log^{\beta} n}$ is convergent, hence the series $\sum_{k=2}^{\infty} \frac{ku_k(p_{k+1} - p_k)}{(\log k)p_k p_{k+1}}$ is also convergent.

We have furthermore

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{p_{k+1}} = \sum' \frac{1}{p_{k+1}},$$

where \sum' extends over the values of k such that $\alpha_k = 1$, that is, $c_k + 1 = p_m$. Then by (2.2) and (1.4) we get

$$(3.3) \quad p_{k+1} \sim p_k \sim p_{p_m} \sim m \log^2 m.$$



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Since the series $\sum_{m=2}^{\infty} \frac{1}{m \log^2 m}$ is convergent, it then follows that the series $\sum_{k=1}^{\infty} \frac{\alpha_k}{p_{k+1}}$ is also convergent. Now (3.2) implies that

$$\sum_{k=1}^n \left| \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} \right| = \sum_{k=1}^n \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| + O(1)$$

and the desired conclusion follows. \square

Analogously to the Erdős-Prachar theorem, we shall prove the following fact.

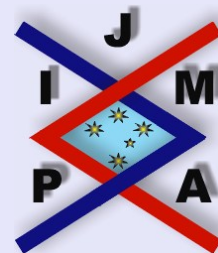
Theorem 3.2. *We have*

$$\sum_{p_k \leq x} \left| \frac{p_{k+1}}{c_{k+1}} - \frac{p_k}{c_k} \right| \asymp \log^2 x$$

for every $x > 1$.

Proof. We have

$$(3.4) \quad \frac{p_{k+1}}{c_{k+1}} - \frac{p_k}{c_k} = \left(\frac{p_{k+1}}{k+1} - \frac{p_k}{k} \right) + p_k \left(\frac{1}{k(k+1)} - \frac{c_{k+1} - c_k}{c_{k+1}c_k} \right) + \frac{(p_{k+1} - p_k)(k+1 - c_{k+1})}{(k+1)c_{k+1}}.$$



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By (1.9) we get

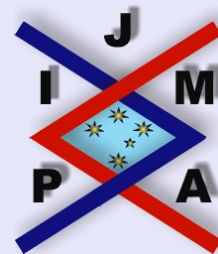
$$\begin{aligned}
 & \sum_{p_k \leq x} \left| \frac{(p_{k+1} - p_k)(k + 1 - c_{k+1})}{(k + 1)c_{k+1}} \right| \\
 &= - \sum_{p_k \leq x} \frac{(p_{k+1} - p_k)u_{k+1}}{c_{k+1} \log(k + 1)} \\
 &= O \left(\sum_{k=2}^{\pi(x)} \frac{p_{k+1} - p_k}{k \log k} \right) \\
 &= O \left(\frac{x}{\pi(x) \log \pi(x)} + \sum_{k=3}^{\pi(x)} p_k \left(\frac{1}{(k-1) \log(k-1)} - \frac{1}{k \log k} \right) \right).
 \end{aligned}$$

By (1.3) we have $\pi(x) \log \pi(x) \sim x$,

$$p_k \left(\frac{1}{(k-1) \log(k-1)} - \frac{1}{k \log k} \right) \sim \frac{p_k \log k}{k^2 \log^2 k} \sim \frac{1}{k}$$

and (1.7) implies

$$(3.5) \quad \sum_{p_k \leq x} \frac{(p_{k+1} - p_k)(k + 1 - c_{k+1})}{(k + 1)c_{k+1}} = O(\log x).$$



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We have also

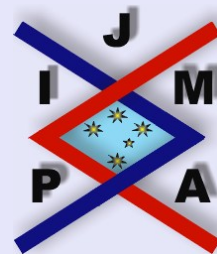
$$\begin{aligned} \sum_{p_k \leq x} \left| p_k \left(\frac{1}{k(k+1)} - \frac{c_{k+1} - c_k}{c_{k+1}c_k} \right) \right| \\ \leq \sum_{p_k \leq x}'' \left| p_k \left(\frac{1}{k(k+1)} - \frac{1}{c_k(c_k+1)} \right) \right| + 2 \sum_{p_k \leq x}' \left| \frac{p_k}{c_k(c_k+2)} \right|, \end{aligned}$$

where \sum' extends over the values of k such that $c_k + 1$ is a prime number, while \sum'' extends over the values of k such that $c_k + 1 = p_m$ is composed. By (1.9) we deduce

$$\begin{aligned} \sum_{p_k \leq x}'' \left| \frac{p_k(c_k - k)(c_k + k + 1)}{k(k+1)c_k(c_k+1)} \right| &= O \left(\sum_{p_k \leq x} \frac{k \log k}{k^4} \cdot k^2 \log k \right) \\ &= O \left(\sum_{p_k \leq x} \frac{1}{k} \right) = O(\log x). \end{aligned}$$

By (1.4) and (1.8) it follows that

$$\begin{aligned} \sum_{p_k \leq x}' \left| \frac{p_k}{c_k(c_k+2)} \right| &\sim \sum_{k \leq \pi(x)}' \frac{c_k \log c_k}{c_k(c_k+2)} \\ &\sim \sum_{c_k \leq x}' \frac{\log(c_k+1)}{c_k+1} \\ &= \sum_{p_m \leq x} \frac{\log p_m}{p_m} = O(\log x). \end{aligned}$$



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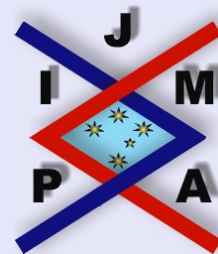
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Thus

$$(3.6) \quad \sum_{p_k \leq x} p_k \left| \frac{1}{k(k+1)} - \frac{c_{k+1} - c_k}{c_{k+1}c_k} \right| = O(\log x).$$

Now by (1.1), (3.4), (3.5) and (3.6) it follows that $\sum_{p_k \leq x} \left| \frac{p_{k+1}}{c_{k+1}} - \frac{p_k}{c_k} \right| \asymp \log^2 x$ and the proof is completed. \square



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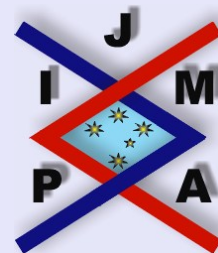
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