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MATRIX AND OPERATOR INEQUALITIES

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ABSTRACT. In this paper we prove certain inequalities involving matrices and operators on Hilbert spaces. In particular inequalities involving the trace and the determinant of the product of certain positive definite matrices.

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1. INTRODUCTION

Inequalities have proved to be a powerful tool in mathematics , in particular in modeling error analysis for filtering and estimation problems, in adaptive stochastic control and for investigation of quantum mechanical Hamiltonians as it has been shown by Patel and Toda [10, 11, 12] and Lieb and Thirring [5].

It is the object of this paper to prove new interesting matrix and operator inequalities. We refer the reader to [4, 7, 8] for the basics of matrix and operator inequalities and for a survey of many other basic and important inequalities.

Through out the paper if A is an $n \times n$ matrix, we write trA to denote the trace of A and $\det A$ for the determinant of A. If A is positive definite we write $A > 0$. The adjoint of A (a matrix or operator) is denoted by A^*.

2. MATRIX INEQUALITIES

Through out this section, we work with square matrices on a finite dimensional Hilbert space.

Theorem 2.1. *If $A > 0$ and $B > 0$, then*

$$0 < tr\,(AB)^m < (tr\,(AB))^m \tag{2.1}$$

for any integer $m > 0$.

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Proof. The equality holds for $m = 1$. For $m > 1$, let $B = I$, and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A. Since $\sum_{i=1}^n \lambda_i^m < \left(\sum_{i=1}^n \lambda_i\right)^m$, then

$$0 < tr\left(A^m\right) < \left(trA\right)^m . \tag{2.2}$$

Since (2.2) is true for any $A > 0$, we let $D = B^{\frac{1}{2}}AB^{\frac{1}{2}}$. Then inequality (2.2) holds for D. Thus $0 < tr\left(D^m\right) < \left(trD\right)^m$, from which the result follows. □

Theorem 2.2. *Let A, B be positive definite matrices. Then*

$$0 < tr\left(AB\right)^m < \left[tr\left(AB\right)^s\right]^{\frac{m}{s}} , \tag{2.3}$$

provided that m and s are positive integers and $m > s$.

Proof. Clearly $tr\left(AB\right)^m = tr\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^m > 0$. Let $l_1, l_2, \dots, l_n$ be the eigenvalues of $A^{\frac{1}{2}}BA^{\frac{1}{2}}$. Then from Hardy's inequality [3] $\left(l_1^m + l_2^m + \dots + l_n^m\right)^{\frac{1}{m}} < \left(l_1^s + l_2^s + \dots + l_n^s\right)^{\frac{1}{s}}$ for $m > s > 0$, we get

$$\left[tr\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^m\right]^{\frac{1}{m}} < \left[tr\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^s\right]^{\frac{1}{s}} .$$

This implies (2.3). □

Theorem 2.3. *If $A_i > 0$ and $B_i > 0$ $(i = 1, 2, \dots, k)$, then*

$$\left(tr\sum_{i=1}^k A_iB_i\right)^2 \leq \left(tr\sum_{i=1}^k A_i^2\right) \cdot \left(tr\sum_{i=1}^k B_i^2\right) . \tag{2.4}$$

If $A_iB_i > 0$ $(i = 1, 2, \dots, k)$, then

$$\left(tr\sum_{i=1}^k A_iB_i\right)^2 < \left(tr\sum_{i=1}^k A_i^2\right) \cdot \left(tr\sum_{i=1}^k B_i^2\right) . \tag{2.5}$$

Proof. Since

$$0 \leq tr\sum_{i=1}^k \left(\theta A_i + B_i\right)^2 = \theta^2 tr\left(\sum_{i=1}^k A_i^2\right) + 2\theta tr\left(\sum_{i=1}^k A_iB_i\right) + tr\left(\sum_{i=1}^k B_i^2\right),$$

we conclude (2.4). To prove (2.5), it suffices to prove that

$$tr\left(\sum_{i=1}^k A_iB_i\right)^2 < \left(tr\sum_{i=1}^k \left(A_iB_i\right)\right)^2 . \tag{2.6}$$

Since $A_iB_i > 0$ for $i = 1, 2, \dots, k$, then $U = \sum_{i=1}^k A_iB_i > 0$. Therefore the inequality $tr\left(U\right)^2 < \left(trU\right)^2$ for positive definite U implies (2.6) and the proof is complete. □

Remark 2.4. The condition $A_iB_i > 0$ in (2.5) is essential as the following example shows.

Example 2.1. Let

$$A = \begin{pmatrix} 4 & -3 \\ -3 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix},$$
$$C = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ -3 & 10 \end{pmatrix}.$$

It is clear that A, B, C, and D are positive definite matrices . Now

$$\left(AB + CD\right) = \begin{pmatrix} -10 & 10 \\ -9 & 66 \end{pmatrix}, \quad \left(AB + CD\right)^2 = \begin{pmatrix} 10 & 560 \\ -504 & 4266 \end{pmatrix}.$$

Thus

$$tr\,(AB+CD)^2 = 4276 > [tr\,(AB+CD)]^2 = 3136.$$

Remark 2.5. R. Bellman [1] proved that $tr\,(AB)^2 \leq tr\,(A^2B^2)$ (*) for positive definite matrices A and B. Further he asked: "Does the above inequality (*) hold for higher powers?". Such a question had been solved by E.H.Lieb and W.E. Thirring [5] ,where they proved

$$tr\,(AB)^m < tr\,(A^mB^m) \tag{2.7}$$

for any positive integer m, and for A, B positive definite matrices. In 1995, Changqin Xu [2] proved a particular case of (2.7): that is when A and B are 2×2 positive definite matrices. Notice that $(trAB)^m$ and $tr\,(A^mB^m)$ are upper bounds for $tr\,(AB)^m$ in (2.1) and (2.7) . One may ask what is max$\{tr\,(A^mB^m)\,, tr\,(AB)^m\}$. The following examples show that either $(trAB)^m$ or $tr\,(A^mB^m)$ can be the least.

Example 2.2. Let

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $tr\,(AB)^2 = 144 < 204 = tr\,(A^2B^2)$.

Example 2.3. Let

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then $tr\,(A^2B^2) = 25 < 36 = tr\,(AB)^2$.

Theorem 2.6. *If* $0 < A_1 \leq B_1$ *and* $0 < A_2 \leq B_2$, *then*

$$0 < tr\,(A_1A_2) \leq tr\,(B_1B_2)\,. \tag{2.8}$$

Proof. Since $0 < A_1 \leq B_1$ and $0 < A_2 \leq B_2$, it follows that

$$0 < A_2^{\frac{1}{2}}A_1A_2^{\frac{1}{2}} \leq A_2^{\frac{1}{2}}B_1A_2^{\frac{1}{2}}$$

and

$$0 < B_1^{\frac{1}{2}}A_2B_1^{\frac{1}{2}} \leq B_1^{\frac{1}{2}}B_2B_1^{\frac{1}{2}}. \tag{2.9}$$

Since trace is a monotone function on the definite matrices, we get

$$0 < tr\,(A_1A_2) \leq tr\,(B_1A_2)\,. \tag{2.10}$$

and

$$0 < tr\,(B_1A_2) \leq tr\,(B_1B_2)\,. \tag{2.11}$$

This implies (2.8). $\square$

Remark 2.7. The conditions $A_1 > 0$ and $A_2 > 0$ in Theorem 2.6. are essential even if A_1A_2 and B_1B_2 are symmetric as the following example shows.

Example 2.4. Let

$$A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is clear that $A_1 < B_1$ and $A_2 < B_2$. We have also

$$A_1A_2 = \begin{pmatrix} \frac{1}{2} & -\frac{7}{2} \\ -\frac{7}{2} & \frac{15}{2} \end{pmatrix}, \qquad B_1B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

and $tr\,(A_1A_2) = 8 > 7 = tr\,(B_1B_2)$.

Theorem 2.8. *If $A > 0$ and $B > 0$, then*

$$n\,(\det A \cdot \det B)^{\frac{m}{n}} \le tr\,(A^m B^m) \tag{2.12}$$

for any positive integer m.

Proof. Since A is diagonalizable, there exists an orthogonal matrix P and a diagonal matrix Λ such that $\Lambda = P^T AP$. So if the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\Lambda = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Let $b_{11}(m), b_{22}(m), \ldots, b_{nn}(m)$ denote the elements of $\left(PBP^T\right)^m$. Then

$$\begin{aligned} \frac{1}{n} tr\,(A^m B^m) &= \frac{1}{n} tr\left(P\Lambda^m P^T B^m\right) \\ &= \frac{1}{n} tr\left(\Lambda^m P^T B^m P\right) \\ &= \frac{1}{n} tr\left[\Lambda^m \left(P^T BP\right)^m\right] \\ &= \frac{1}{n}\left[\lambda_1^m b_{11}(m) + \lambda_2^m b_{22}(m) + \cdots + \lambda_n^m b_{nn}(m)\right]. \end{aligned} \tag{2.13}$$

Using the arithmetic-mean geometric- mean inequality [9], we get

$$\frac{1}{n} tr\,(A^m B^m) \ge \left[\lambda_1^m \lambda_2^m \cdots \lambda_n^m\right]^{\frac{1}{n}} \left[b_{11}(m)\, b_{22}(m) \cdots b_{nn}(m)\right]^{\frac{1}{n}}. \tag{2.14}$$

Since $\det A \le a_{11}a_{22}\cdots a_{nn}$ for any positive definite matrix A, [4] we conclude that

$$\det\left(P^T BP\right)^m \le b_{11}(m) \cdot b_{22}(m) \cdot \cdots \cdot b_{nn}(m) \tag{2.15}$$

and

$$\det \Lambda^m \le \lambda_1^m \lambda_2^m \cdots \lambda_n^m. \tag{2.16}$$

Therefore from (2.14) it follows that

$$\begin{aligned} \frac{1}{n} tr\,(A^m B^m) &\ge \left[\det(\Lambda^m)\right]^{\frac{1}{n}} \cdot \left[\det\left(P^T BP\right)^m\right]^{\frac{1}{n}} \\ &= \left[\det\left(P^T AP\right)\right]^{\frac{m}{n}} \cdot \left[\det\left(P^T BP\right)\right]^{\frac{m}{n}} \\ &= (\det A \cdot \det B)^{\frac{m}{n}}. \end{aligned}$$

Here we used the fact that $A > 0$ and $B > 0$. The proof is complete. □

Corollary 2.9. [6] *Let A and X be positive definite $n \times n$- matrices such that* $\det X = 1$. *Then*

$$n\,(\det A)^{\frac{1}{n}} \le tr\,(AX). \tag{2.17}$$

Proof. Take $B = X$ and $m = 1$ in Theorem 2.8. □

Theorem 2.10. *If $A \ge 0$, $B \ge 0$ and $AB = BA$, then*

$$2^{(m-1)n} \det\,(A^m + B^m) \ge \left[\det\,(A + B)\right]^m \tag{2.18}$$

and

$$2^{m-1} tr\,(A^m + B^m) \ge tr\,(A + B)^m \tag{2.19}$$

for any positive integer m.

Proof. To prove inequality (2.18), it is enough to prove

$$\frac{A^m + B^m}{2} \geq \left(\frac{A+B}{2}\right)^m \tag{2.20}$$

for any pair of commuting positive definite matrices A and B. We use induction to prove (2.20). Clearly (2.20) holds true for $m = 2$. Assume that (2.20) is true for $m = k$. We have to prove (2.20) for $m = k + 1$. Indeed, since

$$\frac{A^k + B^k}{2} \cdot \frac{A+B}{2} = \frac{A+B}{2} \cdot \frac{A^k + B^k}{2},$$

it follows that

$$\begin{aligned} \left(\frac{A+B}{2}\right)^{k+1} &\leq \frac{A+B}{2} \cdot \frac{A^k + B^k}{2} \\ &= \frac{A^{k+1} + B^{k+1}}{2} - \frac{A^{k+1} + B^{k+1}}{4} + \frac{BA^k + AB^k}{4} \\ &= \frac{A^{k+1} + B^{k+1}}{2} - \frac{A^{k+1} + B^{k+1} - BA^k - AB^k}{4} \\ &= \frac{A^{k+1} + B^{k+1}}{2} - \frac{\left(A^k - B^k\right)(A-B)}{4}. \end{aligned} \tag{2.21}$$

Now the equality

$$\left(A^k - B^k\right)(A-B) = \left(A^{k-1} + A^{k-2}B + \cdots + AB^{k-2} + B^{k-1}\right)(A-B)^2$$

for $A \geq 0$, $B \geq 0$ and $AB = BA$, implies $AB \geq 0$ [8]. Consequently

$$L = A^{k-1} + A^{k-2}B + \cdots + AB^{k-2} + B^{k-1} \geq 0.$$

Since $L \cdot (A-B)^2 = (A-B)^2 \cdot L$, then $\left(A^k - B^k\right)(A-B) \geq 0$. Therefore, from (2.21) we obtain

$$\left(\frac{A+B}{2}\right)^{k+1} \leq \frac{A^{k+1} + B^{k+1}}{2}.$$

The proof is complete. Inequality (2.19) follows directly from (2.20). □

Remark 2.11. The condition $AB = BA$ in inequality (2.20) is essential as the following example shows.

Example 2.5. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}.$$

It is clear that $A > 0$, $B > 0$ and $AB \neq BA$. For $m = 3$ inequality (2.20) becomes

$$4\left(A^3 + B^3\right) \geq (A+B)^3. \tag{2.22}$$

Easily we find that

$$4\left(A^3 + B^3\right) = \begin{pmatrix} 256 & -216 \\ -216 & 196 \end{pmatrix}, \qquad (A+B)^3 = \begin{pmatrix} 84 & -45 \\ -45 & 24 \end{pmatrix}$$

and $\det C = -9$ which implies that $C < 0$.

3. OPERATOR INEQUALITIES

In this section we consider inequalities involving operators on separable Hilbert space X. We start with the following simple well-known inequality.

Theorem 3.1. *Let S and T be self-adjoint bounded linear operators on the Hilbert space H. Then*

$$\frac{ST+TS}{2} \leq \left(\frac{S+T}{2}\right)^2. \tag{3.1}$$

Proof. Since

$$\left(\frac{S+T}{2}\right)^2 - \frac{ST+TS}{2} = \left(\frac{S+T}{2}\right)^2$$

and since the square of the self-adjoint operator is a non-negative operator, we get $\left(\frac{S+T}{2}\right)^2 \geq 0$. The claim of the theorem now follows. □

Now we present a similar type result as Theorem 3.1 but for non-self-adjoint case . More precisely:

Theorem 3.2. *Let S and T be bounded linear operators on a Hilbert space X. Assume S to be self-adjoint . Then*

$$\frac{1}{4}K^2 + H_p \leq H_Q, \tag{3.2}$$

where

$$P = \frac{1}{2}(ST+TS), \qquad Q = \left(\frac{S+T}{2}\right)^2, \tag{3.3}$$

$$H_P = \frac{1}{2}(P+P^*), \qquad H_Q = \frac{1}{2}(Q+Q^*) \tag{3.4}$$

and $K = \frac{1}{2}(T+T^*)$.

Proof. For any bounded linear operator T we have

$$T = \frac{1}{2}\left[(T+T^*) + (T-T^*)\right] = H_T + K,$$

where $H_T = \frac{1}{2}(T+T^*)$. Inequality (3.1) can be applied to the self-adjoint operators S and H_T, so we get

$$\langle Ux, x\rangle \geq 0, \tag{3.5}$$

where $U = (S+H_T)^2 - 2(SH_T + H_TS)$. Now we have

$$\begin{aligned} U &= (S+H_T)^2 - S(T+T^*) - (T+T^*)S \\ &= (S+H_T)^2 - 4H_P \\ &= \frac{1}{2}\left[2S^2 + \frac{T^2+(T^*)^2}{2} + \frac{1}{2}(TT^*+T^*T) - 4H_P\right] \\ &= \frac{1}{2}\left(2S^2 + T^2 + (T^*)^2 - 4H_P - 2K^2\right) \\ &= \frac{1}{2}\left(8H_Q - 4H_P - 4H_P - 2K^2\right) \\ &= 4\left(H_Q - H_P - \frac{1}{4}K^2\right). \end{aligned} \tag{3.6}$$

Therefore the required inequality (3.2) follows from (3.6) and (3.5). □

Remark 3.3. When both S and T are not self-adjoint operators , Theorem 3.2 does not hold. The following example illustrates this fact.

Example 3.1. Let S and T be defined on $\mathbb{R}^2 \to \mathbb{R}^2$ by the following matrices

$$S = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 6 \\ -3 & 2 \end{pmatrix}.$$

By computation we find that

$$\begin{aligned} P &= \frac{1}{2}(ST+TS) = \begin{pmatrix} 7 & 12 \\ -6 & 14 \end{pmatrix}, \\ Q &= \left(\frac{S+T}{2}\right)^2 = \begin{pmatrix} -1 & 8 \\ -4 & 7 \end{pmatrix}, \qquad K^2 = \begin{pmatrix} -\frac{81}{4} & 0 \\ 0 & -\frac{81}{4} \end{pmatrix}, \\ H_P &= \begin{pmatrix} 7 & 3 \\ 3 & 14 \end{pmatrix}, \qquad H_Q = \begin{pmatrix} -1 & 2 \\ 2 & 7 \end{pmatrix}, \\ H_Q - H_P - \frac{1}{4}K^2 &= \begin{pmatrix} -\frac{47}{16} & -1 \\ -1 & -\frac{31}{16} \end{pmatrix} < 0. \end{aligned}$$

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