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# MULTIDIMENSIONAL EXTENSION OF L.C. YOUNG'S INEQUALITY

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ABSTRACT. A classical inequality of L. C. Young is extended to higher dimensions, and using this extension sufficient conditions for the existence of integral $\int_{[0,1]^n} f dg$ are given, where both $f$ and $g$ are functions of finite higher variations.

*Key words and phrases:* Inequalities, Stieltjes Integral, $p$-variation.

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## 1. INTRODUCTION

In this paper we consider the existence of the integral $\int_{[0,1]^n} f dg$, where $f$ and $g$ are functions of bounded higher variations. In the sequel we explain the meaning of this integral and we will also define the higher variations of functions of several variables. Such integrals occur naturally in the study of stochastic differential equations. In 1935 a paper that appeared in Acta Mathematica [6], L. C. Young gave sufficient conditions for the existence of Riemann-Stieltjes integral $\int_0^1 f(x)dg(x)$, where $f$ is a function of bounded $p$-variation, $g$ is a functions of bounded $q$-variation, and $\frac{1}{p}+\frac{1}{q}>1$ (see Theorem 1.1). This result of L. C. Young has received considerable attention to understand the Ito map, and to develop a stochastic integration theory based on his techniques. Using Young's integral T. Lyon solved a differential equation drived by rough signals that are of bounded $p$-variation with $p<2$ [2, 3]. Since almost surely Brownian motion paths are not functions of bounded $p$-variation for $p<2$, it appears that stochastic differential equations driven by white noise may be well beyond the setting of Young's theory. However, it turns out that a certain set function associated with the Brownian motion process can be viewed as functions of bounded-$p$ variation in two variables [4]. Therefore, Young's ideas can still be used to construct stochastic integrals with respect to processes with rough sample paths such as the Brownian motion. In order to construct multiple stochastic integrals

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similar to the 1-dimensional construction described in [4], an exact $n$-dimensional analogue of L. C. Young's result is needed.

Although the motivation behind extended L. C. Young's inequality to higher dimension is to construct multiple stochastic integrals, the extension may be of independent interest. Interested reader may consult [4, 2] and [3] for application of L. C. Young's inequality in stochastic integration.

The key to Young's integration theorem is a discrete inequality. On the main we are interested in extending Young's discrete inequality to higher dimensions. Using the inequality one can establish an analogous Stieltjes type integration theorem. In this paper we do not strive to find the most general integration result, that is, we do not push the integration result to obtain Lebesgue-Stiletjes type integrals by removing conditions on continuity of the functions. Interested reader may consult Young's original work [6] – [8] for further developing or extending the integration theorems of this paper.

The main ingredients in the proof of $n$-dimensional result are still the techniques originally employed by L. C. Young to prove his one dimensional result. However, some modification of his techniques and a judicious choice of exponents which appear in the proof is required. To underscore this point, we should mention that, in his 1937 paper L. C. Young gave sufficient conditions for the existence of double Stieltjes integral $\int_0^1\int_0^1 f(x,y)dg(x,y)$ ([8, Theorem 6.3]). However, L. C. Young's 2-dimensional result is not the exact analogue of the one dimensional result, in the sense that, the conditions that $f$ and $g$ must satisfy in order for the double integral to exist (in Young-Stieltjes sense), are somewhat complicated. In the appendix of this paper we have stated a version of Young's theorem in this paper (see Theorem 3.1 in the Appendix). In particular, there is no obvious way of generalizing the two-dimensional version of L. C. Young's result to higher dimensions. Our main result is to prove an exact $n$-dimensional version of L. C. Young's one dimensional result. We also show that L. C. Young's 2-dimensional result follows from our $n$-dimensional result.

Functions of finite higher variations seem to have been considered for the first time by N. Wiener. His ideas were developed by L.C. Young and E. R. Love (for a complete detail see [1, 6, 7] and [8].

L.C. Young considered the $p$-th variation of a function $f(x)$, defined as

$$V_p(f,[a,b]) = V_p(f) = \left[\sup_{\tau}\left\{\sum_{j=1}^{n}|f(t_j)-f(t_{j-1})|^p\right\}\right]^{\frac{1}{p}}, \tag{1.1}$$

where $\tau$ denotes the partition $a = t_0 \le t_1 \le \cdots \le t_n = b$ of $[a,b]$. Existence proof of Riemann-Stieltjes integrals $\int_0^1 fdg$ where both $f$ and $g$ are functions of finite higher variations, was given by Young [6]:

**Theorem 1.1** (L.C. Young's Theorem/Inequality)**.** *If $V_p(f) < \infty$, $V_q(g) < \infty$, $\frac{1}{p}+\frac{1}{q} > 1$, and $f$ and $g$ have no common discontinuities, then the Riemann-Stieltjes integral $\int_0^1 fdg$ exists and*

$$\left|\int_0^1 f\,dg\right| \le \left(1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right)[|f(0)|+V_p(f)]V_q(g), \tag{1.2}$$

*where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.*

Multidimensional extension of Young's theorem is the main result of this paper. The multidimensional integral will be defined as limits of *Stieltjes* sums, and the integral will be referred to as the *Young-Stieltjes integral.*

1.1. **Young-Stieltjes integral of functions.** For the sake of clarity we define Young-Stieltjes integral of functions of two variables. Let $f$ and $g$ be functions defined on $[0,1]^2$ and $\pi =:$

$\{x_i\}_{i=0}^n \times \{y_j\}_{j=0}^m$ be a partition of $[0,1]^2$. That is, $\pi =: \{x_i\}_{i=0}^n \times \{y_j\}_{j=0}^m$ with $(x_i, y_j) \in [0,1]^2$. Let

$$L(f, g, \pi) = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\eta_i, \nu_j) \Delta_{i,j}^2 \pi(g), \tag{1.3}$$

where $(\eta_i, \nu_i) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and

$$\Delta_{i,j}^2 \pi(g) = g(x_i, y_j) - g(x_{i-1}, y_j) - g(x_i, y_{j-1}) + g(x_{i-1}, y_{j-1}).$$

Note that the above sum depends on the choice of intermediate values $(\eta_i, \nu_j)$. We say that the *Young-Stieltjes integral of f with respect to* $g$ exists, if there is a scalar $I(f, g)$ such that

$$\lim_{||\pi|| \to 0} |L(f, g, \pi) - I(f, g)| = 0. \tag{1.4}$$

Here $||\pi|| = \sup_{\{1 \le i \le n, 1 \le j \le n\}} \{\max\{|x_i - x_{i-1}|, |y_j - y_{j-1}|\}\}$. That is, the Young-Stieltjes integral exists if and only if there exists a scalar $I(f, g)$, such that $|L(f, g, \pi) - I(f, g)| < \epsilon$ for any given positive $\epsilon$, provided that the partition $\pi$ has norm $||\pi|| < \delta$, where $\delta$ depends only on $\epsilon$. If (1.4) holds, we say that $I(f, g)$ is the Young-Stieltjes integral of $f$ with respect to $g$.

To state the 2-dimensional version of our result, we need to introduce the notion of $p$-variation and mixed $p - q$ variation of functions of two variables.

Henceforth, whenever we deal with $p-$variation or mixed $p-q$-variations, we always assume that $p$'s and $q$'s are never smaller than 1. Let $p, q \ge 1$, then the $L(p-q)-$ *variation* of a function $f(x, y)$ on $[0,1]^2$ is defined to be

$$LV_{(p,q)}^{(2)}(f, [0,1]^2) = LV_{(p,q)}^{(2)}(f) = \sup_{\pi} \left\{ \left[ \sum_{i=1}^{n} \left[ \sum_{j=1}^{m} \left| \Delta_{i,j}^2 \pi(f) \right|^p \right]^{\left(\frac{q}{p}\right)} \right]^{\frac{1}{q}} \right\}, \tag{1.5}$$

where $\pi =: \{0 = x_0 \le x_1 \le \cdots \le x_n = 1\} \times \{0 = y_0 \le y_1 \le \cdots \le y_m = 1\}$ is a *partition of* $[0,1]^2$, and

$$\Delta_{i,j}^2 \pi(f) = f(x_i, y_j) - f(x_i, y_{j-1}) - f(x_{i-1}, y_j) + f(x_{i-1}, y_{j-1}).$$

Similarly $R(p-q)$-*variation* of a function $f(x, y)$ on $[0,1]^2$ is defined to be

$$RV_{(p,q)}^{(2)}(f, [0,1]^2) = RV_{(p,q)}^{(2)}(f) = \sup_{\pi} \left\{ \left[ \sum_{j=1}^{n} \left[ \sum_{i=1}^{m} \left| \Delta_{i,j}^2 \pi(f) \right|^p \right]^{\left(\frac{q}{p}\right)} \right]^{\frac{1}{q}} \right\}. \tag{1.6}$$

We define the left and right Wiener class-$p - q$ to be the space of functions defined as follows,

$$LW_{(p,q)}^{(2)} = \{f : [0,1]^2 \to \mathbf{C} : LV_{(p,q)}^{(2)}(f) + V_p(f(\cdot, 0), [0,1]) + V_q(f(0, \cdot), [0,1]) < \infty\},$$

where $V_p(f(\cdot, 0), [0,1])$ is the $p$-th variation of the function $x \to f(x, 0)$ as defined by (1.1). Similarly

$$RW_{(p,q)}^{(2)} = \left\{ f : [0,1]^2 \to \mathbf{C} : RV_{(p,q)}^{(2)}(f) + V_q(f(\cdot, 0), [0,1]) + V_p(f(0, \cdot), [0,1]) < \infty \right\}.$$

We define the left and right $p - q$-Wiener norm of $f \in LW_{(p,q)}^2$ or $f \in RW_{(p,q)}$ as follows:

$$\|f\|_{LW_{(p,q)}} = LV_{(p,q)}^{(2)}(f) + V_p(f(\cdot, 0), [0,1]) + V_q(f(0, \cdot), [0,1]) + |f(0,0)| \tag{1.7}$$

and

$$\|f\|_{RW_{(p,q)}} = RV_{(p,q)}^{(2)}(f) + V_q(f(\cdot, 0), [0,1]) + V_p(f(0, \cdot), [0,1]) + |f(0,0)|. \tag{1.8}$$

We also define the Wiener class-$p$ of functions of one variable, that is,

$$W_p[0,1] = \{f : [0,1] \to \mathbf{C} : V_p(f,[0,1])) < \infty\}. \tag{1.9}$$

When $p = q$ then $LV_{(p,p)} = RV_{(p,p)}$, consequently we write $W_p$, $V_p$ and $p$-variation instead of $LW_{(p,p)}$, $LV_{(p,p)}$ etc.

Before we can state our main result (Theorem 1.2), we need to define the notion of *jump point* of functions of several variables. We stay in a two-dimensional setting.

Let $f(x,y)$ be a function such that $V_p^{(2)}(f) < \infty$. For $\vec{x} = (x_1,x_2)$ and $\vec{y} = (y_1,y_2)$, we let

$$d(\vec{x},\vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}, \tag{1.10}$$

$$\Delta_{\vec{y}} f(\vec{x}) = f(x_1,x_2) - f(x_1,y_2) - f(y_1,x_2) + f(y_1,y_2). \tag{1.11}$$

For $\vec{x} \in [0,1]^2$, we let

$$J(f,\vec{x}) = \lim_{\delta \to 0} \sup\{\Delta_{\vec{y}} f(\vec{x}) : d(\vec{x},\vec{y}) < \delta\}. \tag{1.12}$$

We say that $f$ has a *jump at* $\vec{x}$ if $J(f,\vec{x}) > 0$. It can be shown that if $V_p^{(2)}(f) < \infty$ then $f$ has at most a countable number of jump points. If $f$ is continuous at $\vec{x}$ then $\vec{x}$ cannot be a jump point of $f$, but the converse is not true. Our main result is

**Theorem 1.2** (a). *Let $f \in W_p^{(2)}$, $V_q^{(2)}(g) < \infty$ and $\frac{1}{p} + \frac{1}{q} > 1$. If $f$ and $g$ do not have any common jump points then the Young-Stieltjes integral of $f$ with respect to $g$ exists, and*

$$\left| \int_0^1 \int_0^1 f(x,y) dg(x,y) \right| \le c(p,q) \, \|f\|_{W_p} V_q^{(2)}(g), \tag{1.13}$$

*where*

$$c(p,q) \le 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) + \inf\left\{(1+\zeta(\alpha))\left(1 + \zeta\left(\frac{1}{\alpha p} + \frac{1}{\alpha q}\right)\right)^{\alpha} : 1 < \alpha < \frac{1}{p} + \frac{1}{q}\right\}. \tag{1.14}$$

We also have the following result.

**Theorem 1.2** (b). *Let $f \in RW^{(2)}_{(p_1,p_2)}$, $RV^{(2)}_{(q_1,q_2)}(g) < \infty$ and for $i = 1,2$, $\frac{1}{p_i} + \frac{1}{q_i} > 1$. If $f$ and $g$ do not have any common jump points then the Young-Stieltjes integral of $f$ with respect to $g$ exists, and*

$$\left| \int_0^1 \int_0^1 f(x,y) dg(x,y) \right| \le c \, \|f\|_{RW_{(p_1,p_2)}} RV^{(2)}_{(q_1,q_2)}(g), \tag{1.15}$$

*where*

$$\begin{aligned} c \le {} & \left(1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right) + \left(1 + \zeta\left(\frac{1}{p_2} + \frac{1}{q_2}\right)\right) \\ & + \min\left\{ \inf_{\{1 < \alpha < \frac{1}{p_2} + \frac{1}{q_2}\}} \left\{(1+\zeta(\alpha))\left(1 + \zeta\left(\frac{1}{\alpha p_1} + \frac{1}{\alpha q_1}\right)\right)^{\alpha} 1\right\} \right. \\ & \left. + \inf\left\{(1+\zeta(\alpha))(\left(1 + \zeta\left(\frac{1}{\alpha p_2} + \frac{1}{\alpha q_2}\right)\right)^{\alpha} : 1 < \alpha < \frac{1}{p_1} + \frac{1}{q_1}\right\}\right\}. \end{aligned} \tag{1.16}$$

*The theorem holds if we replace $RW$ and $RV$ with $LW$ and $LV$ throughout.*

Note that, when $p_1 = p_2$ and $q_1 = q_2$, 1.2(b) reduces to 1.2(a). And finally to state the $n$-dimensional version, we define the corresponding $W_p^n$ and $V_p^n$ classes of functions of $n$-variables.

Let $p \geq 1$ and $f$ be a function defined on $[0,1]^n$. Let

$$V_p^{(n)}(f,[0,1]^n) = \left( \sup_{\pi_1,\dots,\pi_n} \sum_{i_1,i_2,\cdots i_n} |\Delta_{i_1,\dots,i_n}^{\pi_1,\dots,\pi_n} f|^p \right)^{1/p}.$$

Here $\pi_i$ is a partition of $[0,1]$ and $\Delta_{i_1,\dots,i_n}^{\pi_1,\dots,\pi_n} f$ is the $n^{\text{th}}$-difference of $f$. The $n^{\text{th}}$-difference is a straightforward generalization of the 2nd-difference introduced prior to the statement of Theorem 1.2. Let $W_p^{(n)}([0,1]^n) = W_p^{(n)}$ denote the class of functions $f$ on $[0,1]^n$, such that, $V_p^{(n)}(f,[0,1]^n) < \infty$, and for each positive integer $k$ less than $n$; the function on $[0,1]^{n-k}$ obtained by keeping any $k$ coordinates of arguments of $f$ to the fixed value of 0, belongs to $W_p^{n-k}([0,1]^{n-k})$. For instance when $n=3$, $f \in W_p^{(3)}([0,1]^3)$ if and only if

$$\begin{aligned}\|f\|_{W_p^3} &= V_p^{(3)}(f,[0,1]^3) + V_p^{(2)}(f(0,\cdot,\cdot),[0,1]^2) + V_p^{(2)}(f(\cdot,0,\cdot),[0,1]^2) \\ &\quad + V_p^{(2)}(f(\cdot,\cdot,0),[0,1]^2) + V_p(f(\cdot,0,0),[0,1]) + V_p(f(0,\cdot,0),[0,1]) \\ &\quad + V_p(f(0,0,\cdot),[0,1]) + |f(0,0,0)|\end{aligned}$$

is finite. Stated below is the $n$-dimensional version of Theorem 1.2(a).

**Theorem 1.2** (c). *Let $f \in W_p^{(n)}$, $V_q^{(n)}(g) < \infty$ and $\frac{1}{p} + \frac{1}{q} > 1$. If $f$ and $g$ do not have any common jump points then the Young-Stieltjes integral of $f$ with respect to $g$ exists, and*

$$\left| \int_{[0,1]^n} f(x_1,\cdots,x_n)dg(x_1,\cdots,x_n) \right| \leq c(p,q)\, \|f\|_{W_p^n} V_q^{(n)}(g), \tag{1.17}$$

*where*

$$\begin{aligned} c(p,q) &\leq 2^{n-1}\left(1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right) \\ &\quad + 2^{n-2}\left[(1+\zeta(\alpha_1))\left(1+\zeta\left(\frac{1}{\alpha_1 p}+\frac{1}{\alpha_1 q}\right)\right)^{\alpha_1}\right] \\ &\quad + 2^{n-3}\left[(1+\zeta(\alpha_1))(1+\zeta(\alpha_2))^{\alpha_1}\left(1+\zeta\left(\frac{1}{\alpha_1\alpha_2 p}+\frac{1}{\alpha_1\alpha_2 q}\right)\right)^{\alpha_1\alpha_2}\right] \\ &\quad + \cdots \\ &\quad + \Bigg[(1+\zeta(\alpha_1))(1+\zeta(\alpha_2))^{\alpha_1}\cdots(1+\zeta(\alpha_{n-1}))^{\alpha_1\cdots\alpha_{n-2}} \\ &\quad \times \left(1+\zeta\left(\frac{1}{\alpha_1\alpha_2\cdots\alpha_{n-1}p}+\frac{1}{\alpha_1\alpha_2\cdots\alpha_{n-1}q}\right)\right)^{\alpha_1\alpha_2\cdots\alpha_{n-1}}\Bigg] \end{aligned} \tag{1.18}$$

*where for each* $1 \leq j \leq n-1,\ \ 1 < \alpha_j$, *and* $\alpha_1\alpha_2\cdots\alpha_{n-1} < \frac{1}{p} + \frac{1}{q}$.

## 2. HIGHER VARIATIONS OF SEQUENCES

In this section we will prove a discrete version of Theorem 1.2. We define the $p$-th variation of sequence of scalars.

Let $\theta =: \{k_i\}_{i=0}^n$ be a increasing sequence of positive integers. A *partition of* $\theta$ denoted by $\pi(\theta)$ is an increasing sequence of integers $\{j_i\}_{i=0}^m$ such that $\{j_i\}_{i=0}^m \subset \{k_i\}_{i=0}^n$, $j_0 = k_0$ and $j_m = k_n$. We note that if $\theta =: \{k_i\}_{i=0}^n$ is a increasing sequence of integers and $\pi(\theta)$ is partition of $\theta$, then any partition of $\pi(\theta)$ is also a partition of $\theta$. If $\theta =: \{0,1,2,...,n\}$, then we write

$\pi(n)$ instead of $\pi(\theta)$. That is, $\pi(n)$ denotes a partition of $\{0,1,2,...,n\}$. For a given sequence $a=\{a_i\}_{i=0}^n$ and a partition $\pi=:\{j_i\}_{i=1}^m$ of $\{0,1,2,...,n\}$, $\pi(a)$ denotes the sequence $\{a_{j_i}\}_{i=0}^m$.

2.1. $p-$**variation of Sequences.** Let $a=:\{a_i\}_{i=0}^n$ be a finite sequence of scalars. For any partition $\pi=\pi(n)=\{j_i\}_{i=0}^k$, where $\{j_i\}_{i=0}^m\subset\{0,1,2,...,n\}$, we define $\pi(a)$ to be the sequence $\{a_{j_i}\}_{i=0}^k$, and $\Delta_i(\pi(a))=a_{j_i}-a_{j_{i-1}}$. Let $\Delta\pi(a)$ denote the sequence $\{a_{j_i}-a_{j_{i-1}}\}_{i=1}^k$. Let $p>0$ and $V_p(a,\pi)=[\sum_i|\Delta_i(\pi(a))|^p]^{\frac{1}{p}}$. We define the $p$-variation of $\{a_i\}$ to be $V_p(a)=\sup_\pi V_p(a,\pi)$.

We now consider the variation of two-dimensional sequences.

**Definition 2.1.** Let $\theta=:\{k_j\}_{j=0}^m\times\{l_j\}_{j=0}^n$, where $\{k_j\}_{j=0}^m$ and $\{l_j\}_{j=0}^n$ are two increasing sequences of positive integers. A *partition of* $\theta$ denoted by $\pi(\theta)$ is a two-dimensional sequence $\{k'_j\}_{j=0}^{m'}\times\{l'_j\}_{j=0}^{n'}$ such that $\{k'_j\}_{j=0}^{m'}$ is a partition of $\{k_j\}_{j=0}^m$ as defined above in 2.1 and $\{l'_j\}_{j=0}^{n'}$ is a partition of $\{l_j\}_{j=0}^n$. If $\theta=\{0,1,...,n\}\times\{0,1,...,m\}$, then a partition of $\theta$ will be denoted by $\pi(n\times m)$.

2.2. **Variation of 2-Dimensional Sequences.** Let $a=\{a_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ be a two dimensional sequence of scalars and $\pi=:\{k_j\}_{j=0}^{m'}\times\{l_j\}_{j=0}^{n'}$ be a partition. Then $\pi(a)$ denotes the sequence $\{a_{k_i,l_j}\}_{i=0,j=0}^{i=m',j=n'}$. In particular, $\pi(a)_{i,j}=a_{k_i,l_j}$.

We define $\Delta_{1,i,j}\pi(a)=a_{k_i,l_j}-a_{k_{i-1},l_j},\quad\Delta_{2,i,j}\pi(a)=a_{k_i,l_j}-a_{k_i,l_{j-1}}$, and

$$\Delta_{i,j}^2\pi(a)=a_{k_i,l_j}-a_{k_{i-1},l_j}-a_{k_i,l_{j-1}}+a_{k_{i-1},l_{j-1}}.$$

Let $\Delta^2\pi(a)$ denote the sequence $\{\Delta_{i,j}^2\pi(a)\}_{i=0,j=0}^{i=m',j=n'}$, $\Delta_{1,j}\pi(a)$ denote the sequence $\{\Delta_{1,i,j}\pi(a)\}_{i=1}^{m'}$, and $\Delta_{2,i}\pi(a)$ denote the sequence $\{\Delta_{2,i,j}^2\pi(a)\}_{j=1}^{n'}$. For $p>0$, we define $V_p^{(2)}(a,\pi)=[\sum_{i,j}|\Delta_{i,j}^2(\pi(a))|^p]^{\frac{1}{p}}$.

We define the $p$-variation of $\{a_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ to be $V_p^{(2)}(a)=\sup_\pi V_p^{(2)}(a,\pi)$, and the $p$-*variation norm* of $\{a_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ to be

$$\|a\|_{W_p}=V_p^{(2)}(a)+V_p(\{a_{0,j}\}_j)+V_p(\{a_{i,0}\}_i)+|a_{0,0}|. \tag{2.1}$$

Given two partitions $\pi$ and $\theta$, we say $\theta$ refines $\pi$, if $\pi$ is a partition of $\theta$, and we write $\theta<\pi$. Let

$$V_{p,\theta(\pi)}^{(2)}(a)=\sup_{\theta<\tau<\pi}V_p^{(2)}(a,\tau). \tag{2.2}$$

Suppose $(a)=\{a_{i,j}\}_{i=0,j=0}^{i=n',j=m'}$ is a sequence of scalars and $\pi=\{k_i\}_{i=0}^n\times\{l_j\}_{j=0}^m$ a partition of $\{0,1,...,n\}\times\{0,1,2,...,m\}$. Let $\theta<\pi$, then every subdivision point of $\pi$ is also a subdivision point of $\theta$. Therefore, $\theta$ can be viewed as a product of two, two-dimensional sequences, that is,

$$\theta=:\{c_{i,j}\}_{i=0,j=0}^{i=n,j=r_i}\times\{d_{i,j}\}_{i=0,j=0}^{i=m,j=s_i},$$

where for each fixed $i\geq1$,

$$k_{i-1}=c_{i,0}\leq c_{i,1}\leq\cdots\leq c_{i,r_i}=k_i,$$
$$l_{i-1}=d_{i,0}\leq d_{i,1}\leq\cdots\leq d_{i,s_i}=l_i.$$

We now prove a discrete version of Theorem 1.2(a).

**Theorem 2.1.** *Let* $a=:\{a_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ *and* $b=:\{b_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ *be two sequences of scalars. Let* $p,q>0$, $\frac{1}{p}+\frac{1}{q}>1$. *Let*

$$L(a,b)=\sum_{i=1}^n\sum_{j=1}^m a_{i,j}\Delta_{i,j}^2b. \tag{2.3}$$

*Then*

$$|L(a,b) - a_{0,0}(b_{n,m} - b_{0,m} - b_{n,0} + b_{0,0})| \le c(p,q)\, \|a\|_{W_p} V_q^{(2)}(b), \tag{2.4}$$

*where* $c(p,q) \le \inf\left\{(1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)\right)^{\alpha} : 1 < \alpha < \frac{1}{p}+\frac{1}{q}\right\}$.

*Proof.* By consecutive application of summation by parts we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}\Delta^2_{i,j} b &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i}\sum_{l=1}^{j}\Delta^2_{k,l} a \Delta^2_{i,j} b \\ &\quad + \sum_{i=1}^{n}\sum_{l=i}^{n}(a_{l,0}-a_{l-1,0})(b_{i,m}-b_{i,0}-b_{i-1,m}+b_{i-1,0}) \\ &\quad + \sum_{j=1}^{m}\sum_{l=j}^{m}(a_{0,l}-a_{0,l-1})(b_{n,j}-b_{0,j}-b_{n,j-1}+b_{0,j-1}) \\ &\quad + a_{0,0}(b_{n,m}-b_{0,m}-b_{n,0}+b_{0,0}) \\ &= I + II + III + IV.\end{aligned} \tag{2.5}$$

We now estimate $I$. For each $1 \le i \le n$, let

$$Q(0,i) = \sum_{j=1}^{m}\sum_{l=1}^{j}\Delta^2_{i+1,l}(a)\Delta^2_{i,j}(b), \tag{2.6}$$

$$S(0) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i}\sum_{l=1}^{j}\Delta^2_{k,l}(a)\Delta^2_{i,j}(b). \tag{2.7}$$

Choose $i_0$ with $1 \le i_0 \le n-1$ so that for each $i \le n-1$, the following holds:

$$|Q(0,i_0)| \le |Q(0,i)|. \tag{2.8}$$

For each $1 \le i \le n-1$, let

$$c^1_i = \begin{cases} i & \text{if } i < i_0 \\ i+1 & \text{if } i_0 \le i \le n-1. \end{cases} \tag{2.9}$$

Let $\pi_1 =: \{c^1_i\}_{i=0}^{n-1} \times \{j\}_{j=0}^{m}$ be a partition of $\{0,1,...,n\} \times \{0,1,2,...m\}$ and let

$$S(1) = \sum_{i=1}^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{i}\sum_{l=1}^{j}\Delta^2_{k,l}\pi_1(a)\Delta^2_{i,j}\pi_1(b). \tag{2.10}$$

The following equation is verified:

$$S(0) = S(1) - Q(0,i_0). \tag{2.11}$$

We now estimate $|Q(0,i_0)|$. Let $1 < \alpha < \frac{1}{p}+\frac{1}{q}$. By (2.8)

$$|Q(0,i_0)| \le \left(\prod_{i\ne i_0}|Q(0,i)|\right)^{\frac{1}{n-1}}.$$

An application of geometric-arithmetic mean inequality gives us

$$|Q(0,i_0)| \le \left(\frac{1}{n-1}\right)^{\alpha}\left(\sum_{i\ne i_0}|Q(0,i)|^{\frac{1}{\alpha}}\right)^{\alpha}. \tag{2.12}$$

For each $1 \leq j$, let $U(0,i,j) = \Delta^2_{i+1,j+1}\pi_1(a)\Delta^2_{i,j}\pi_1(b)$. For $1 \leq j \leq n-1$, let

$$W(a,p,j) = \left(\sum_{i=1}^{n-1} |\Delta^2_{i,j+1}\pi_1(a)|^p\right)^{\frac{1}{\alpha p}}, \quad W(b,q,j) = \left(\sum_{i=1}^{n-1} |\Delta^2_{i,j}\pi_1(b)|^q\right)^{\frac{1}{\alpha q}},$$

and

$$\tilde{U}(0,j) = W(b,q,j)W(a,p,j). \tag{2.13}$$

Choose $j_0$ with $1 \leq j_0 \leq m-1$ so that for each $j \leq m-1$, the following holds:

$$|\tilde{U}(0,j_0)| \leq |\tilde{U}(0,j)|. \tag{2.14}$$

For $0 \leq j \leq m-1$, let

$$d^1_j = \begin{cases} j & \text{if } j < j_0 \\ j+1 & \text{if } j_0 \leq j \leq m-1. \end{cases} \tag{2.15}$$

Now $\pi_2 =: \{c_i\}_{i=0}^n \times \{d^1_j\}_{j=1}^{m-1}$, is a partition which refines $\pi_1$. Let

$$Q(1,i) = \sum_{j=1}^{m-1}\sum_{l=1}^{j} \Delta^2_{i+1,l}\pi_1(a)\Delta^2_{i,j}\pi_1(b). \tag{2.16}$$

The following equation can be verified:

$$Q(1,i) = Q(0,i) - U(0,i,j_0). \tag{2.17}$$

Therefore, by Minkowski's inequality and the fact that $\alpha > 1$, we obtain

$$\sum_{i=1}^{n-1} |Q(1,i)|^{\frac{1}{\alpha}} \leq \sum_{i=1}^{n-1} |Q(1,i)|^{\frac{1}{\alpha}} + \sum_{i=1}^{n-1} |U(0,i,j_0,)|^{\frac{1}{\alpha}}. \tag{2.18}$$

We now estimate $\sum_{i=1}^{n-1} |U(0,i,j_0,)|^{\frac{1}{\alpha}}$. By (2.13) and Hölder's inequality with exponents $\alpha p$ and $\alpha q$, we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} |U(0,i,j_0,)|^{\frac{1}{\alpha}} &= \sum_{i=1}^{n-1} |\Delta^2_{i+1,j_0+1}\pi_1(a)\Delta^2_{i,j_0}\pi_1(b)|^{\frac{1}{\alpha}}. \\ &\leq \left[\sum_{i=1}^{n-1} |\Delta^2_{i+1,j_0+1}\pi_1(a)|^p\right]^{\frac{1}{\alpha p}} \left[\sum_{i=1}^{n-1} |\Delta^2_{i,j_0}\pi_2(b)|^q\right]^{\frac{1}{\alpha q}} \\ &= |\tilde{U}(0,j_0)|. \end{aligned}$$

Therefore, by (2.14)

$$\begin{aligned} \sum_{i=1}^{n-1} |U(0,i,j_0)|^{\frac{1}{\alpha}} &\leq \left(\prod_{j\neq j_0} \tilde{U}(0,j)\right)^{\frac{1}{m-1}} \\ &= \left(\prod_{j\neq j_0} W(b,q,j)\right)^{\frac{1}{m-1}} \left(\prod_{j\neq j_0} W(a,p,j)\right)^{\frac{1}{m-1}}. \end{aligned} \tag{2.19}$$

Applying geometric-arithmetic mean inequality to right side of the previous inequality, we obtain

$$\sum_{i=1}^{n-1} |U(0,i,j_0)|^{\frac{1}{\alpha}} \le \left(\frac{1}{m-1}\right)^{\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)} \left[\sum_{j\neq j_0} (W(b,q,j))^{\alpha q}\right]^{\frac{1}{\alpha q}} \left[\sum_{j\neq j_0}^{m-1} (W(a,p,j))^{\alpha p}\right]^{\frac{1}{\alpha p}}.$$

Now

$$\left[\sum_{j\neq j_0} (W(b,q,j))^{\alpha q}\right]^{\frac{1}{\alpha q}} \le \left[\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} |\Delta^2_{i,j}\pi_2(b)|^q\right]^{\frac{1}{\alpha q}} \le \left(V_q^{(2)}(b)\right)^{\frac{1}{\alpha}}.$$

Similarly

$$\left[\sum_{j\neq j_0} (W(a,p,j))^{\alpha p}\right]^{\frac{1}{\alpha p}} \le \left(V_p^{(2)}(a)\right)^{\frac{1}{\alpha}}.$$

Combining (2.19) and the last three inequalities, we obtain

$$\sum_{i=1}^{n-1} |U(0,i,j_0)|^{\frac{1}{\alpha}} \le \left(\frac{1}{m-1}\right)^{\frac{1}{\alpha p}+\frac{1}{\alpha q}} \left(V_q^{(2)}(b)\right)^{\frac{1}{\alpha}} \left(V_p^{(2)}(a)\right)^{\frac{1}{\alpha}}. \tag{2.20}$$

Combining inequalities (2.18) and (2.20), we obtain

$$\sum_{i=1}^{n-1} |Q(0,i)|^{\frac{1}{\alpha}} \le \sum_{i=1}^{n-1} |Q(1,i)|^{\frac{1}{\alpha}} + \left(\frac{1}{m-1}\right)^{\frac{1}{\alpha p}+\frac{1}{\alpha q}} \left[V_q^{(2)}(b) V_p^{(2)}(a)\right]^{\frac{1}{\alpha}}. \tag{2.21}$$

By a similar argument we break up $Q(1,i)$ as the difference of two quantities (compare with the equation following (2.17)), that is

$$Q(2,i) = Q(1,i) - U(1,i,j_1), \tag{2.22}$$

where for each $1 \le j \le n-2$,

$$U(1,i,j) = \Delta^2_{i+1,j+1}\pi_2(a)\Delta^2_{i,j}\pi_2(b),$$

and $j_1$ is chosen so that for each $j \le m-2$,

$$\left(\sum_{i=1}^{n-1} |\Delta^2_{i,j_1+1}\pi_2(a)|^p\right)^{\frac{1}{\alpha p}} \left(\sum_{i=1}^{n-1} |\Delta^2_{i,j_1}\pi_2(b)|^q\right)^{\frac{1}{\alpha q}}$$
$$\le \left(\sum_{i=1}^{n-1} |\Delta^2_{i,j+1}\pi_2(a)|^p\right)^{\frac{1}{\alpha p}} \left(\sum_{i=1}^{n-1} |\Delta^2_{i,j}\pi_2(b)|^q\right)^{\frac{1}{\alpha q}}.$$

(This last inequality is to be compared with (2.13) and (2.14)). By Minkowski's inequality

$$\sum_{i=1}^{n-1} |Q(1,i)|^{\frac{1}{\alpha}} \le \sum_{i=1}^{n-1} |Q(2,i)|^{\frac{1}{\alpha}} + \sum_{i=1}^{n-1} |U(1,i,j_1|^{\frac{1}{\alpha}}. \tag{2.23}$$

The quantity $\sum_{i=1}^{n-1} |U(1,i,j_1|^{\frac{1}{\alpha}}$ is estimated in exactly the same manner as we estimated $\sum_{i=1}^{n-1} |U(0,i,j_0|^{\frac{1}{\alpha}}$. We obtain

$$\sum_{i=1}^{n-1} |U(1,i,j_1|^{\frac{1}{\alpha}} \le \left(\frac{1}{m-2}\right)^{\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)} \left[V_q^{(2)}(b) V_p^{(2)}(a)\right]^{\frac{1}{\alpha}}. \tag{2.24}$$

Combining (2.21), (2.22), (2.23) and (2.24) we obtain that,

$$\sum_{i=1}^{n-1} |Q(0,i)|^{\frac{1}{\alpha}} \leq \sum_{i=1}^{n-1} |Q(2,i)|^{\frac{1}{\alpha}} + \left(\frac{1}{m-1}\right)^{\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)} \left[V_q^{(2)}(b)V_p^{(2)}(a)\right]^{\frac{1}{\alpha}} + \left(\frac{1}{m-2}\right)^{\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)} \left[V_q^{(2)}(b)V_p^{(2)}(a)\right]^{\frac{1}{\alpha}}. \tag{2.25}$$

Continuing this process by breaking up the expression $Q(2,i)$ and so on, we obtain

$$\sum_{i=1}^{n-1} |Q(0,i)|^{\frac{1}{\alpha}} \leq \zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right) \left[V_q^{(2)}(b)V_p^{(2)}(a)\right]^{\frac{1}{\alpha}}. \tag{2.26}$$

Consequently by (2.11), (2.12) and (2.26), we obtain

$$|S(0)| \leq |S(1)| + \left(\frac{1}{n-1}\right)^{\alpha} \zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)^{\alpha} V_q^{(2)}(b)V_p^{(2)}(a). \tag{2.27}$$

Now expression $S(1)$ is similar to $S(0)$, thus it can be estimated in the same manner, i.e., we can write

$$S(1) = S(2) - Q(1,i_1), \tag{2.28}$$

where $S(2)$ and $Q(1,i_1)$ are obtained in the same manner as $S(1)$ and $Q(0,i_0)$ were obtained from $S(0)$. Furthermore each $i \leq n-2$, $Q(1,i_1)$ satisfies the following inequality (compare with (2.8)),

$$|Q(1,i_1)| \leq |Q(1,i)|. \tag{2.29}$$

Estimating $|Q(1,i_1)|$ the way we estimated $|Q(0,i_0)|$, we obtain

$$|Q(1,i_1)| \leq \left(\frac{1}{n-2}\right)^{\alpha} \zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)^{\alpha} V_q^{(2)}(b)V_p^{(2)}(a). \tag{2.30}$$

Consequently by (2.27), (2.28) and (2.30), we obtain

$$|S(0)| \leq |S(2)| + \left(\frac{1}{n-2}\right)^{\alpha} \zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)^{\alpha} V_q^{(2)}(b)V_p^{(2)}(a) + \left(\frac{1}{n-2}\right)^{\alpha} \zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)^{\alpha} V_q^{(2)}(b)V_p^{(2)}(a). \tag{2.31}$$

Continuing the above process by breaking up $S(2)$, we obtain

$$|S(0)| \leq \zeta(\alpha)\zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)^{\alpha} V_q^{(2)}(b)V_p^{(2)}(a). \tag{2.32}$$

This gives the estimate on $I$. To estimate $II$ and $III$, we note that $II$ and $III$ are one dimensional version of $I$. It can be shown that (see e.g. [6]),

$$II \leq \zeta\left(\frac{1}{p}+\frac{1}{q}\right) V_p^{(1)}(\{a_{i,0}\}_{i=1}^{n})V_q^{(1)}(\{b_{i,m}-b_{i,0}\}_{i=1}^{n}), \tag{2.33}$$

$$III \leq \zeta\left(\frac{1}{p}+\frac{1}{q}\right) V_p^{(1)}(\{a_{0,j}\}_{j=1}^{m})V_q^{(1)}(\{b_{n,j}-b_{0,j}\}_{j=1}^{m}). \tag{2.34}$$

It is easy to see that

$$\begin{aligned} V_q^{(1)}(\{b_{n,j}-b_{0,j}\}_{j=1}^{m}) &\leq V_q^{(2)}(b), \\ V_q^{(1)}(\{b_{i,m}-b_{i,0}\}_{i=1}^{n}) &\leq V_q^{(2)}(b). \end{aligned}$$

Consequently $I + II + III \leq c(p,q)\,\|a\|_{W_p} V_q^2(b)$. This completes the proof of the Theorem 2.1. $\square$

To prove Theorem 1.2(a), a more general version of Theorem 2.1 must be proved, the proof of which parallels the proof of Theorem 2.1. This theorem is needed to show that the Young-Stieltjes sums approximating the integral of $f$ with respect to $g$ form a Cauchy net.

**Theorem 2.2.** *Let $a =: \{a_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ and $b =: \{b_{i,j}\}_{i=0,j=0}^{i=n,j=m}$ be two sequences of scalars. Let $\pi =: \{e_i\}_{i=0}^{n_1} \times \{f_j\}_{j=0}^{m_1}$ be a partition of*

$$\{0,1,...,n\} \times \{0,1,2,...m\}.$$

*This means $\pi =: \{0 = e_0 < e_1 < \cdots < e_{n_1} = n\} \times \{0 = f_0 < f_1 < \cdots < f_{m_1} = m\}$, where $e_i$'s and $f_j$'s are integers. Let $L(a,b) = \sum_{i=1}^{n_1}\sum_{j=1}^{m_1} a_{i,j}\Delta_{i,j}(b)$, and*

$$L(a,b,\pi) = \sum_i \sum_j \pi_{i,j}(a)\Delta_{i,j}(\pi(b)).$$

*(Recall $\Delta_{i,j}(\pi(b)) = b_{e_i,f_j} - b_{e_i,f_{j-1}} - b_{e_{i-1},f_j} + b_{e_{i-1},f_{j-1}}$ and $\pi_{i,j}(a) = a_{e_i,f_j}$).*

*If $\frac{1}{p} + \frac{1}{q} > 1$, then*

$$\begin{aligned}
(2.35)\quad |L(a,b) - L(a,b,\pi)| &\leq c(p,q)V_{p,\pi}^{(2)}(a)V_{q,\pi}^{(2)}(b) \\
&\quad + \left|\sum_{i=1}^{n_1}\sum_{j=1}^{m} a_{e_i,j}(b_{e_i,j} - b_{e_{(i-1)},j} - b_{e_i,j-1} + b_{e_{(i-1)},j-1})\right| \\
&\quad + \left|\sum_{j=1}^{m_1}\sum_{i=1}^{n} a_{i,f_j}(b_{i,f_j} - b_{i-1,f_j} - b_{i,f_{(j-1)}} + b_{i-1,f_{(j-1)}})\right| \\
&= I + II + III,
\end{aligned}$$

*where $c(p,q) \leq \inf\left\{(1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\alpha q}\right)\right)^{\alpha} : 1 < \alpha < \frac{1}{p}+\frac{1}{q}\right\}$.*

Using Theorems 2.1 and 2.2, Theorems 1.2(a) through 1.2(c) can be proved following closely the proof of L. C. Young's original result.

## 3. APPENDIX

As it was pointed out, in [8] Young considered the higher variations of functions of two variables defined on $[0,1]^2$ and gave existence proof of the double Young -Stieltjes integral $\int_0^1\int_0^1 f dg$. In this appendix we show that Theorem 1.2 (by Theorem 1.2 we mean Theorems 1.2(a) and 1.2(b).).

In his paper, Young considered the more general type of variation in terms of Orlicz functions rather than $p$ or $p-q$ variation and he uses the concept of $p-$ and $q-$bivariations. However, Young's generalization of Theorem 1.1, is not the exact analogue of Theorem 1.1. In particular, the condition $1/p + 1/q > 1$ in the statement of Theorems 1.1 and 1.2 are replaced by a stronger condition, roughly given by $1/p + 1/2q \geq 1$. For the precise statement of Young's two dimensional extension we refer the reader to Theorem 6.3 in [8]. Below we state a special case of Young's 2-dimensional result, so the reader can compare the result with Theorem 1.2. Young's result can be obtained from 1.2. We first define the concept of $p$ and $q$-bivariation of a function of two variables. We say that $f(x,y)$ is function of bounded $p$ and $q-$ bivariation if there exists a pair of constants $P$ and $Q$ such that, for each fixed pair $y_1, y_2 \in [0,1]$, the total $p-$variation of the function of one variable $f(\cdot, y_1) - f(\cdot, y_2)$ is less than $P$ and for each fixed pair $x_1, x_2 \in [0,1]$, the total $q$-variation of the function $f(x_1,\cdot) - f(x_2,\cdot)$ is less than $Q$.

**Theorem 3.1** (Special version of Theorem 6.3 in [8])**.** *Let $f$ be a function of bounded $p_1-$ and $p_2-$bivariation such that for each $x$ and $y$ in $[0,1]$ $f(x,0)=f(0,y)=0$. And for fixed $x_1, x_2, y_1, y_2$,*

$$|g(x_1,y_1)-g(x_1,y_2)-g(x_2,y_1)+g(x_2,y_2)| \le |x_1-x_2|^{\frac{1}{q_1}}|y_2-y_1|^{\frac{1}{q_2}}. \tag{A1}$$

*Then the Young-Stieltjes integral of $f$ with respect to $g$ exists, provided that there exist positive strictly increasing functions $h$ and $k$, such that*

$$h(x)k(x)=x \text{ and } \sum_n h\left(\frac{1}{n^{\frac{1}{p_1}}}\right)\left(\frac{1}{n^{\frac{1}{q_1}}}\right)+\sum_n k\left(\frac{1}{n^{\frac{1}{p_2}}}\right)\left(\frac{1}{n^{\frac{1}{q_2}}}\right)<\infty. \tag{*}$$

*To show that Theorem 1.2 implies Theorem 3.1, we must relate the concept of $p$- and $q$-bivariation to the concept of $p-q$ variation as defined by equations (1.5) and (1.6). Following theorem is the consequence of the results proven in* [5] *(see Theorem 1.4 and Corollary 3.1 in* [5]*).*

**Theorem 3.2.** [5]. *If $f$ is a function of $p_1$ and $p_2$-bivariation, then*

$$LV_{(2,p_1)}(f)+RV_{(2,p_2)}(f)<\infty. \tag{A2}$$

*Further more if $p_1\le 2$ then $RV_{(p_1,2)(f)}$ is finite. If $p_1>2$ then $V_{p_1}(f)$ is finite. Similarly if $p_2\le 2$ then $LV_{(p_2,2)(f)}$ is finite. If $p_2>2$ then $V_{p_2}(f)$ is finite. If $p_1=p_2=p\le 2$ then $V_{(\frac{4p}{2+p})}(f)$ is finite. If $p_1=p_2=p>2$ then $V_p(f)$ is finite.*

W now examine the conditions given in Theorem 3.1. Condition on $g$, that is,

$$|g(x_1,y_1)-g(x_1,y_2)-g(x_2,y_1)+g(x_2,y_2)| \le |x_1-x_2|^{\frac{1}{q_1}}|y_2-y_1|^{\frac{1}{q_2}}$$

implies that

$$LV_{(q_1,q_2)}(g)+RV_{(q_1,q_2)}(g)<\infty.$$

The fact that $f$ vanishes on each axis implies that

$$\|f\|_{LW_{(\cdot,\cdot)}}=LV_{(\cdot,\cdot)}(f), \|f\|_{RW_{(\cdot,\cdot)}}=RV_{(\cdot,\cdot)}(f),$$

and $\|f\|_{W_{(\cdot)}}=V_{(\cdot)}(f)$. Since $h$ and $k$ are decreasing functions, (*) implies that

$$h\left(n^{-\frac{1}{p_2}}\right)\le c\left(\frac{1}{n}\right)^{\left(\frac{p_1}{p_2}-\frac{p_1}{q_1p_2}\right)}, \tag{A3}$$

where $c$ is a fixed universal constant. We also have,

$$k\left(n^{-\frac{1}{p_1}}\right)\le c\left(\frac{1}{n}\right)^{\left(\frac{p_2}{p_1}-\frac{p_2}{q_2p_1}\right)}. \tag{A4}$$

Since $h(x)k(x)=x$, (*) and the previous set of inequalities imply that,

$$\sum_{n=1}\left(\frac{1}{n}\right)^{\left(\frac{1}{p_1}+\frac{1}{q_1}+\frac{p_2}{p_1q_2}-\frac{p_2}{p_1}\right)}+\sum_{n=1}\left(\frac{1}{n}\right)^{\left(\frac{1}{p_2}+\frac{1}{q_2}+\frac{p_1}{p_2q_1}-\frac{p_1}{p_2}\right)}<\infty. \tag{A5}$$

On the other hand theorem (A2) implies that

$$LV_{(2,p_1)}(f)+RV_{(2,p_2)}(f)<\infty. \tag{A6}$$

Consequently, if we want to use Theorem 1.2 to establish the existence of the Young-Stieltjes integral of $f$ with respect to $g$, we must show that either

$$\frac{1}{p_1}+\frac{1}{q_1}>1 \quad\text{and}\quad \frac{1}{2}+\frac{1}{q_2}>1; \tag{A7}$$

or

$$\frac{1}{p_2} + \frac{1}{q_2} > 1 \quad \text{and} \quad \frac{1}{2} + \frac{1}{q_1} > 1. \tag{A8}$$

Since $q_1 \geq$ and $q_2 \geq 1$, (A5) implies that $\frac{1}{p_i} + \frac{1}{q_i} > 1$ for $i = 1, 2$. If $p_1 \geq 2$ then (A8) holds and if $p_2 \geq 2$ then (A7) holds. Also if $\frac{1}{2} + \frac{1}{q_1} > 1$, then (A8) holds. Suppose that $p_i < 2$ for $i = 1, 2$ and $\frac{1}{2} + \frac{1}{q_1} \leq 1$. Now (A5) implies that

$$\frac{1}{p_1} + \frac{1}{q_1} + \frac{p_2}{p_1 q_2} - \frac{p_2}{p_1} > 1. \tag{A9}$$

This last inequality and the assumptions on $p_1$, $p_2$ and $q_1$ (i.e., $1 \leq p_i \leq 2$ and $\frac{1}{2} + \frac{1}{q_1} \leq 1$), imply that $\frac{1}{2} + \frac{1}{q_2} > 1$. Therefore (A7) holds. This shows that Theorem 1.2 implies Theorem 3.1.

## REFERENCES


[1] E. LOVE AND L.C. YOUNG, Sur une classe de fonctionnelles linéaires, *Fundamenta Mathematicae*, T. XXVIII, 243–257.

[2] T.J. LYONS, Differential equations driven by rough signals (I): an extension of an inequality of L. C. Young, *Mathematics Research Letters,* **1** (1994), 451–464.

[3] T.J. LYONS, Differential equations driven by rough signals, Preprint, (1995).

[4] N. TOWGHI, Stochastic integration of process of finite generalized variations I, *Annals of Probability*, **23**(2) (1995), 629–667.

[5] N. TOWGHI, Littlewood's Inequality for $p-$Bimeasures, *J. Ineq. Pure & Appl. Math.*, **3**(2) (2002), Article 19. (`http://jipam.vu.edu.au/v3n2/031_01.html`).

[6] L.C. YOUNG, An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.*, **67** (1936), 251–282.

[7] L.C. YOUNG, Inequalities connected with bounded $p$-th power variation in the Wiener sense, *Proc. London Math. Soc.*, **43**(2) (1937), 449–467.

[8] L.C. YOUNG, General inequalities of Stieltjes integrals and the convergence of Fourier series, *Math. Ann.*, **115** (1938), 581–612.