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AN EXTENDED HARDY-HILBERT INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. In this paper, it is shown that an extended Hardy-Hilbert's integral inequality with weights can be established by introducing a power-exponent function of the form $ax^{1+x}(a > 0, x \in [0, +\infty))$, and the coefficient $\frac{\pi}{(a)^{1/q}(b)^{1/p}\sin\pi/p}$ is shown to be the best possible constant in the inequality. In particular, for the case $p = 2$, some extensions on the classical Hilbert's integral inequality are obtained. As applications, generalizations of Hardy-Littlewood's integral inequality are given.

Key words and phrases: Power-exponent function, Weight function, Hardy-Hilbert's integral inequality, Hardy-Littlewood's integral inequality.

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1. INTRODUCTION

The famous Hardy-Hilbert's integral inequality is

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}} \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}}, \tag{1.1}$$

where $p > 1,\ q = p/(p-1)$ and the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is best possible (see [1]). In particular, when $p = q = 2$, the inequality (1.1) is reduced to the classical Hilbert integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \le \pi \left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}}, \tag{1.2}$$

where the coefficient π is best possible.

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Recently, the following result was given by introducing the power function in [2]:

$$\int_a^b \int_a^b \frac{f(x)\,g(y)}{x^t + y^t}dxdy \leq \left\{\omega(t,p,q)\int_a^b x^{1-t} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\omega(t,q,p)\int_a^b x^{1-t} g^q(x)\,dx\right\}^{\frac{1}{q}}, \tag{1.3}$$

where t is a parameter which is independent of x and y, $\omega(t,p,q) = \frac{\pi}{t\sin\frac{\pi}{pt}} - \varphi(q)$ and here the function φ is defined by

$$\varphi(r) = \int_0^{a/b} \frac{u^{t-2+1/r}}{1+u^t}du, \quad r = p,q.$$

However, in [2] the best constant for (1.3) was not determined.

Afterwards, various extensions on the inequalities (1.1) and (1.2) have appeared in some papers (such as [3, 4] etc.). The purpose of the present paper is to show that if the denominator $x + y$ of the function on the left-hand side of (1.1) is replaced by the power-exponent function $ax^{1+x} + by^{1+y}$, then we can obtain a new inequality and show that the coefficient $\frac{\pi}{(a)^{1/q}(b)^{1/p}\sin\pi/p}$ is the best constant in the new inequality. In particular if $p = 2$ then several extensions of (1.2) follow. As its applications, it is shown that extensions on the Hardy-Littlewood integral inequality can be established.

Throughout this paper we stipulate that $a > 0$ and $b > 0$.

For convenience, we give the following lemma which will be used later.

Lemma 1.1. *Let* $h(x) = \frac{x}{1+x+x\ln x}$, $x \in (0,+\infty)$, *then there exists a function* $\varphi(x)$ $\left(0 \leq \varphi(x) < \frac{1}{2}\right)$, *such that* $h(x) = \frac{1}{2} - \varphi(x)$.

Proof. Consider the function defined by

$$s(x) = \frac{1+x}{x} + \ln x, \quad x \in (0,+\infty).$$

It is easy to see that the minimum of $s(x)$ is 2. Hence $s(x) \geq 2$, and $h(x) = s^{-1}(x) \leq \frac{1}{2}$. Obviously $h(x) = \frac{1}{s(x)} > 0$. We can define a nonnegative function φ by

$$\varphi(x) = \frac{1-x+x\ln x}{2(1+x+x\ln x)} \quad x \in (0,+\infty). \tag{1.4}$$

Hence we have $h(x) = \frac{1}{2} - \varphi(x)$. The lemma follows. □

2. MAIN RESULTS

Define a function by

$$\omega(r,x) = x^{(1+x)(1-r)}\left(\frac{1}{2} - \varphi(x)\right)^{r-1} \quad x \in (0,+\infty), \tag{2.1}$$

where $r > 1$ and $\varphi(x)$ is defined by (1.4).

Theorem 2.1. *Let*

$$0 < \int_0^\infty \omega(p,x)\,f^p(x)\,dx < +\infty, \quad 0 < \int_0^\infty \omega(q,x)\,g^q(x)\,dx < +\infty,$$

the weight function $\omega(r,x)$ *is defined by (2.1),* $\frac{1}{p}+\frac{1}{q}=1$, *and* $p\geq q>1$. *Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{ax^{1+x}+by^{1+y}}dxdy \leq \frac{\mu\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty \omega(p,x)f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega(q,x)g^q(x)\,dx\right\}^{\frac{1}{q}}, \tag{2.2}$$

where $\mu=(1/a)^{1/q}(1/b)^{1/p}$*and the constant factor* $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ *is best possible.*

Proof. Let $f(x)=F(x)\left\{(ax^{1+x})'\right\}^{\frac{1}{q}}$ and $g(y)=G(y)\left\{(by^{1+y})'\right\}^{\frac{1}{p}}$. Define two functions by

$$\alpha=\frac{F(x)\left\{(by^{1+y})'\right\}^{\frac{1}{p}}}{(ax^{1+x}+by^{1+y})^{\frac{1}{p}}}\left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{pq}} \text{ and} \tag{2.3}$$

$$\beta=\frac{G(y)\left\{(ax^{1+x})'\right\}^{\frac{1}{q}}}{(ax^{1+x}+by^{1+y})^{\frac{1}{q}}}\left(\frac{by^{1+y}}{ax^{1+x}}\right)^{\frac{1}{pq}}.$$

Let us apply Hölder's inequality to estimate the right hand side of (2.2) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{ax^{1+x}+by^{1+y}}dxdy=\int_0^\infty\int_0^\infty \alpha\beta dxdy \leq \left\{\int_0^\infty\int_0^\infty \alpha^p dxdy\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty \beta^q dxdy\right\}^{\frac{1}{q}}. \tag{2.4}$$

It is easy to deduce that

$$\int_0^\infty\int_0^\infty \alpha^p dxdy=\int_0^\infty\int_0^\infty \frac{(by^{1+y})'}{ax^{1+x}+by^{1+y}}\left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}}F^p(x)\,dxdy = \int_0^\infty \omega_q F^p(x)\,dx.$$

We compute the weight function ω_q as follows:

$$\omega_q=\int_0^\infty \frac{(by^{1+y})'}{ax^{1+x}+by^{1+y}}\left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}}dy = \int_0^\infty \frac{1}{ax^{1+x}+by^{1+y}}\left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}}d\left(by^{1+y}\right).$$

Let $t=by^{1+y}/ax^{1+x}$. Then we have

$$\omega_q=\int_0^\infty \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{q}}dt=\frac{\pi}{\sin\frac{\pi}{q}}=\frac{\pi}{\sin\frac{\pi}{p}}.$$

Notice that $F(x)=\left\{(ax^{1+x})'\right\}^{-1/q}f(x)$. Hence we have

$$\int_0^\infty\int_0^\infty \alpha^p dxdy=\frac{\pi}{\sin\frac{\pi}{p}}\int_0^\infty\left(\left(ax^{1+x}\right)'\right)^{1-p}f^p(x)\,dx, \tag{2.5}$$

and ,similarly,

$$\int_0^\infty\int_0^\infty \beta^q dxdy=\frac{\pi}{\sin\frac{\pi}{p}}\int_0^\infty\left(\left(by^{1+y}\right)'\right)^{1-q}g^q(y)\,dy. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.3), we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{ax^{1+x}+by^{1+y}}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty \left(\left(ax^{1+x}\right)'\right)^{1-p} f^p(x)\,dx\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty \left(\left(by^{1+y}\right)'\right)^{1-q} g^q(y)\,dy\right\}^{\frac{1}{q}}. \tag{2.7}$$

We need to show that the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ contained in (2.7) is best possible.

Define two functions by

$$\tilde{f}(x) = \begin{cases} 0, & x \in (0,1) \\ \left(ax^{1+x}\right)^{-(1+\varepsilon)/p}\left(ax^{1+x}\right)', & x \in [1,+\infty) \end{cases}$$

and

$$\tilde{g}(y) = \begin{cases} 0, & y \in (0,1) \\ \left(by^{1+y}\right)^{-(1+\varepsilon)/q}\left(by^{1+y}\right)', & y \in [1,+\infty) \end{cases}.$$

Assume that $0 < \varepsilon < \frac{q}{2p} \quad (p \ge q > 1)$. Then

$$\int_0^{+\infty} \left(\left(ax^{1+x}\right)'\right)^{1-p} \tilde{f}^p(x)\,dx = \int_1^{+\infty} \left(ax^{1+x}\right)^{-1-\varepsilon} d\left(ax^{1+x}\right) = \frac{1}{\varepsilon}.$$

Similarly, we have

$$\int_0^\infty \left(\left(by^{1+y}\right)'\right)^{1-q} \tilde{g}^q(y)\,dy = \frac{1}{\varepsilon}.$$

If $\frac{\pi}{\sin\frac{\pi}{p}}$ is not best possible, then there exists $k>0$, $k<\frac{\pi}{\sin\frac{\pi}{p}}$ such that

$$\int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\,\tilde{g}(y)}{ax^{1+x}+by^{1+y}}dxdy < k\left(\int_0^\infty \left(\left(ax^{1+x}\right)'\right)^{1-p} \tilde{f}^p(x)\,dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty \left(\left(by^{1+y}\right)'\right)^{1-q} \tilde{g}^q(y)\,dy\right)^{\frac{1}{q}} = \frac{k}{\varepsilon}. \tag{2.8}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\,\tilde{g}(y)}{ax^{1+x}+by^{1+y}}dxdy \\
&\quad= \int_1^\infty \int_1^\infty \frac{\left\{\left(ax^{1+x}\right)^{-\frac{1+\varepsilon}{p}}\left(ax^{1+x}\right)'\right\}\left\{\left(by^{1+y}\right)^{-\frac{1+\varepsilon}{q}}\left(by^{1+y}\right)'\right\}}{ax^{1+x}+by^{1+y}}dxdy \\
&\quad= \int_1^\infty \left\{\int_1^\infty \frac{\left(by^{1+y}\right)^{-\frac{1+\varepsilon}{q}}}{ax^{1+x}+by^{1+y}}d\left(by^{1+y}\right)\right\}\left\{\left(ax^{1+x}\right)^{-\frac{1+\varepsilon}{p}}\left(ax^{1+x}\right)'\right\}dx \\
&\quad= \int_1^\infty \left\{\int_{b/ax^{1+x}}^\infty \frac{1}{1+t}\left(\frac{1}{t}\right)^{-\frac{1+\varepsilon}{q}}dt\right\}\left(ax^{1+x}\right)^{-1-\varepsilon}d\left(ax^{1+x}\right) \\
&\quad= \frac{1}{\varepsilon}\int_{b/ax^{1+x}}^\infty \frac{1}{1+t}\left(\frac{1}{t}\right)^{-\frac{1+\varepsilon}{q}}dt.
\end{aligned}$$

If the lower limit b/ax^{1+x} of this integral is replaced by zero, then the resulting error is smaller than $\frac{\left(b/ax^{1+x}\right)^{\alpha}}{\alpha}$, where α is positive and independent of ε. In fact, we have

$$\int_0^{b/ax^{1+x}} \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1+\varepsilon}{q}} dt < \int_0^{b/ax^{1+x}} t^{-(1+\varepsilon)/q} dt = \frac{\left(b/ax^{1+x}\right)^{\beta}}{\beta}$$

where $\beta = 1-(1+\varepsilon)/q$. If $0<\varepsilon<\frac{q}{2p}$, then we may take α such that

$$\alpha = 1 - \frac{1+q/2p}{q} = \frac{1}{2p}.$$

Consequently, we get

$$\int_0^{\infty}\int_0^{\infty} \frac{\tilde{f}(x)\,\tilde{g}(y)}{ax^{1+x}+by^{1+y}}dxdy > \frac{1}{\varepsilon}\left\{\frac{\pi}{\sin\frac{\pi}{p}} + o(1)\right\} \quad (\varepsilon \to 0). \tag{2.9}$$

Clearly, when ε is small enough, the inequality (2.7) is in contradiction with (2.9). Therefore, $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible value for which the inequality (2.7) is valid.

Let $u = ax^{1+x}$ and $v = by^{1+y}$. Then

$$u' = ax^{1+x}\left(\frac{1+x}{x} + \ln x\right) = ax^{1+x}h^{-1}(x).$$

Similarly, we have $v' = by^{1+y}h^{-1}(y)$. Substituting them into (2.7) and then using Lemma 1.1, the inequality (2.2) yields after simplifications. The constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ is best possible, where $\mu = (1/a)^{1/q}(1/b)^{1/p}$. Thus the proof of the theorem is completed. □

It is known from (2.1) that

$$\omega(r,x) = x^{(1+x)(1-r)}\left(\frac{1}{2}-\varphi(x)\right)^{r-1} = \left(\frac{1}{2}\right)^{r-1} x^{(1+x)(1-r)}\left(1-2\varphi(x)\right)^{r-1}.$$

The following result is equivalent to Theorem 2.1.

Theorem 2.2. *Let $\varphi(x)$ be a function defined by (1.4), $\frac{1}{p}+\frac{1}{q}=1$ and $p \geq q > 1$. If*

$$0 < \int_0^{\infty} x^{(1+x)(1-p)}\left(1-2\varphi(x)\right)^{p-1} f^p(x)\,dx < +\infty \quad \text{and}$$
$$0 < \int_0^{\infty} y^{(1+y)(1-q)}\left(1-2\varphi(y)\right)^{q-1} g^q(y)\,dy < +\infty,$$

then

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty} &\frac{f(x)\,g(y)}{ax^{1+x}+by^{1+y}}dxdy \\ &\leq \frac{\mu\pi}{2\sin\frac{\pi}{p}}\left\{\int_0^{\infty} x^{(1+x)(1-p)}\left(1-2\varphi(x)\right)^{p-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \\ &\qquad \times \left\{\int_0^{\infty} y^{(1+y)(1-q)}\left(1-2\varphi(y)\right)^{q-1} g^q(y)\,dy\right\}^{\frac{1}{q}},\end{aligned} \tag{2.10}$$

where $\mu = (1/a)^{1/q}(1/b)^{1/p}$and the constant factor $\frac{\mu\pi}{2\sin\frac{\pi}{p}}$ is best possible.

In particular, for case $p=2$, some extensions on (1.2) are obtained. According to Theorem 2.1, we get the following results.

Corollary 2.3. *If*

$$0 < \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x)\right) f^2(x)\, dx < +\infty \quad \textit{and}$$
$$0 < \int_0^\infty y^{-(1+y)} \left(\frac{1}{2} - \varphi(y)\right) g^2(y)\, dy < +\infty,$$

where $\varphi(x)$ *is a function defined by (1.4), then*

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{\sqrt{ab}} \left\{\int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x)\right) f^2(x)\, dx\right\}^{\frac{1}{2}} \times \left\{\int_0^\infty y^{-(1-y)} \left(\frac{1}{2} - \varphi(y)\right) g^2(y)\, dy\right\}^{\frac{1}{2}}, \tag{2.11}$$

where the constant factor $\frac{\pi}{\sqrt{ab}}$ *is best possible.*

Corollary 2.4. *Let* $\varphi(x)$ *be a function defined by (1.4). If*

$$0 < \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x)\right) f^2(x)\, dx < +\infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,f(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{\sqrt{ab}} \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x)\right) f^2(x)\, dx, \tag{2.12}$$

where the constant factor $\frac{\pi}{\sqrt{ab}}$ *is best possible.*

A equivalent proposition of Corollary 2.3 is:

Corollary 2.5. *Let* $\varphi(x)$ *be a function defined by (1.4),*

$$0 < \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx < +\infty \quad \textit{and}$$
$$0 < \int_0^\infty y^{-(1+y)} (1 - 2\varphi(y)) g^2(y)\, dy < +\infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{2\sqrt{ab}} \left\{\int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx\right\}^{\frac{1}{2}} \times \left\{\int_0^\infty y^{-(1+y)} (1 - 2\varphi(y)) g^2(y)\, dy\right\}^{\frac{1}{2}}, \tag{2.13}$$

where the constant factor $\frac{\pi}{2\sqrt{ab}}$ *is best possible.*

Similarly, an equivalent proposition to Corollary 2.4 is:

Corollary 2.6. *Let* $\varphi(x)$ *be a function defined by (1.4). If*

$$0 < \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx + \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,f(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{2\sqrt{ab}} \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx, \tag{2.14}$$

where the constant factor $\frac{\pi}{2\sqrt{ab}}$ *is best possible.*

3. APPLICATION

In this section, we will give various extensions of Hardy-Littlewood's integral inequality.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)\,dx, \quad n = 0, 1, 2, \ldots$$

then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)\,dx, \tag{3.1}$$

where π is the best constant that keeps (3.1) valid. In our previous paper [5], the inequality (3.1) was extended and the following inequality established:

$$\int_0^{\infty} f^2(x)\,dx < \pi \int_0^1 h^2(x)\,dx, \tag{3.2}$$

where $f(x) = \int_0^1 t^x h(x)\,dx$, $x \in [0, +\infty)$.

Afterwards the inequality (3.2) was refined into the form in the paper [6]:

$$\int_0^{\infty} f^2(x)\,dx \le \pi \int_0^1 t h^2(t)\,dt. \tag{3.3}$$

We will further extend the inequality (3.3), some new results can be obtained by further extending inequality (3.3).

Theorem 3.1. *Let $h(t) \in L^2(0,1)$, $h(t) \neq 0$. Define a function by*

$$f(x) = \int_0^1 t^{u(x)} |h(t)|\,dt, \quad x \in [0, +\infty),$$

where $u(x) = x^{1+x}$. Also, let $\varphi(x)$ be a weight function defined by (1.4), $(r = p, q)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p \ge q > 1$. If

$$0 < \int_0^{\infty} x^{(1+x)(1-r)} \left(\frac{1}{2} - \varphi(x)\right)^{r-1} f^r(x)\,dx < +\infty,$$

then

$$\left(\int_0^{\infty} f^2(x)\,dx\right)^2 < \frac{\mu\pi}{\sin\frac{\pi}{p}} \left(\int_0^{\infty} x^{(1+x)(1-p)} \left(\frac{1}{2} - \varphi(x)\right)^{p-1} f^p(x)\,dx\right)^{\frac{1}{p}} \times \left(\int_0^{\infty} y^{(1+y)(1-q)} \left(\frac{1}{2} - \varphi(y)\right) f^q(y)\,dy\right)^{\frac{1}{q}} \int_0^1 t h^2(t)\,dt, \tag{3.4}$$

where the constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ in (3.4) is best possible, and $\mu = (1/a)^{1/q}(1/b)^{1/p}$.

Proof. Let us write $f^2(x)$ in the form:

$$f^2(x) = \int_0^1 f(x)\, t^{u(x)} |h(t)|\,dt.$$

We apply, in turn, Schwarz's inequality and Theorem 2.1 to obtain

$$\begin{aligned}\left(\int_0^\infty f^2(x)\,dx\right)^2 &= \left\{\int_0^\infty\left(\int_0^1 f(x)\,t^{u(x)}\,|h(t)|\,dt\right)dx\right\}^2\\
&= \left\{\int_0^1\left(\int_0^\infty f(x)\,t^{u(x)-1/2}dx\right)t^{1/2}\,|h(t)|\,dt\right\}^2\\
&\le \int_0^1\left(\int_0^\infty f(x)\,t^{u(x)-1/2}dx\right)^2 dt\int_0^1 th^2(t)\,dt\\
&= \int_0^1\left(\int_0^\infty f(x)\,t^{u(x)-1/2}dx\right)\left(\int_0^\infty f(y)\,t^{u(y)-1/2}dy\right)dt\int_0^1 th^2(t)\,dt\\
&= \int_0^1\left(\int_0^\infty\int_0^\infty f(x)\,f(y)\,t^{u(x)+u(y)-1}dxdy\right)dt\int_0^1 th^2(t)\,dt\\
&= \left(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{u(x)+u(y)}dxdy\right)\int_0^1 th^2(t)\,dt\\
&\le \frac{\mu\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty x^{(1+x)(1-p)}\left(\frac{1}{2}-\varphi(x)\right)^{p-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\int_0^\infty y^{(1+y)(1-q)}\left(\frac{1}{2}-\varphi(y)\right)^{q-1}f^q(y)\,dy\right\}^{\frac{1}{q}}\int_0^1 th^2(t)\,dt.\end{aligned}\tag{3.5}$$

Since $h(t)\neq 0,\ f^2(x)\neq 0$. It is impossible to take equality in (3.5). We therefore complete the proof of the theorem. □

An equivalent proposition to Theorem 3.1 is:

Theorem 3.2. *Let the functions $h(t)$, $f(x)$ and $u(x)$ satisfy the assumptions of Theorem 3.1, and assume that*

$$0<\int_0^\infty x^{(1+x)(1-r)}\left(1-2\varphi(x)\right)^{r-1}f^r(x)\,dx<+\infty\quad(r=p,q).$$

Then

$$\left(\int_0^\infty f^2(x)\,dx\right)^2<\frac{\mu\pi}{2\sin\frac{\pi}{p}}\left(\int_0^\infty x^{(1+x)(1-p)}\left(1-2\varphi(x)\right)^{p-1}f^p(x)\,dx\right)^{\frac{1}{p}}\times\left(\int_0^\infty y^{(1+y)(1-q)}\left(1-2\varphi(y)\right)^{q-1}f^q(y)\,dy\right)^{\frac{1}{q}}\int_0^1 th^2(t)\,dt,\tag{3.6}$$

and the constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ in (3.6) is best possible, where $\mu=(1/a)^{1/q}(1/b)^{1/p}$.

In particular, when $p=q=2$, we have the following result.

Corollary 3.3. *Let the functions $h(t)$, $f(x)$ and $u(x)$ satisfy the assumptions of Theorem 3.1, and assume that*

$$0<\int_0^\infty x^{-(1+x)}\left(\frac{1}{2}-\varphi(x)\right)f^2(x)\,dx<+\infty,$$

where $\varphi(x)$ is a function defined by (1.4). Then

$$\left(\int_0^\infty f^2(x)\,dx\right)^2<\frac{\pi}{\sqrt{ab}}\left(\int_0^\infty x^{-(1+x)}\left(\frac{1}{2}-\varphi(x)\right)f^2(x)\,dx\right)\int_0^1 th^2(t)\,dt,\tag{3.7}$$

and the constant factor $\frac{\pi}{\sqrt{ab}}$ *in (3.7) is best possible.*

A result equivalent to Corollary 3.3 is:

Corollary 3.4. *Let the functions* $h(t)$, $f(x)$ *and* $u(x)$ *satisfy the assumptions of Theorem 3.1, and assume that*

$$0 < \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) \, dx < +\infty,$$

where $\varphi(x)$ *is a function defined by (1.4). Then*

$$\left(\int_0^\infty f^2(x) \, dx \right)^2 < \frac{\pi}{2\sqrt{ab}} \left(\int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) \, dx \right) \int_0^1 t h^2(t) \, dt, \tag{3.8}$$

and the constant factor $\frac{\pi}{2\sqrt{ab}}$ *in (3.8) is best possible.*

The inequalities (3.4), (3.6), (3.7) and (3.8) are extensions of (3.3).

REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge 1952.

[2] JICHANG KUANG, On new extensions of Hilbert's integral inequality, *J. Math. Anal. Appl.*, **235**(2) (1999), 608–614.

[3] BICHENG YANG AND L. DEBNATH, On the extended Hardy-Hilbert's inequality, *J. Math. Anal. Appl.*, **272**(1) (2002), 187–199.

[4] BICHENG YANG, On a general Hardy-Hilbert's inequality with a best value, *Chinese Ann. Math. (Ser. A)*, **21**(4) (2000), 401–408.

[5] MINGZHE GAO, On Hilbert's inequality and its applications, *J. Math. Anal. Appl.*, **212**(1) (1997), 316–323.

[6] MINGZHE GAO, LI TAN AND L. DEBNATH, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, **229**(2) (1999), 682–689.