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AN INEQUALITY OF OSTROWSKI TYPE VIA POMPEIU'S MEAN VALUE THEOREM

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ABSTRACT. An inequality providing some bounds for the integral mean via Pompeiu's mean value theorem and applications for quadrature rules and special means are given.

Key words and phrases: Ostrowski's inequality, Pompeiu mean value theorem, quadrature rules, Special means.

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1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\,M, \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In [2], the author has proved the following Ostrowski type inequality.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ with $a > 0$ and differentiable on (a,b). Let $p \in \mathbb{R} \backslash \{0\}$ and assume that*

$$K_p(f') := \sup_{u \in (a,b)} \left\{ u^{1-p} |f'(u)| \right\} < \infty.$$

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Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{K_p(f')}{|p|(b-a)} \tag{1.2}$$
$$\times \begin{cases} 2x^p(x-A) + (b-x)L_p^p(b,x) - (x-a)L_p^p(x,a) & \text{if } p \in (0,\infty); \\ (x-a)L_p^p(x,a) - (b-x)L_p^p(b,x) - 2x^p(x-A) & \text{if } p \in (-\infty,-1) \cup (-1,0) \\ (x-a)L^{-1}(x,a) - (b-x)L^{-1}(b,x) - \frac{2}{x}(x-A) & \text{if } p = -1, \end{cases}$$

for any $x \in (a,b)$, *where for* $a \neq b$,

$$A = A(a,b) := \frac{a+b}{2}, \quad \textit{is the arithmetic mean,}$$

$$L_p = L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \textit{ is the } p - \textit{logarithmic mean } p \in \mathbb{R}\backslash\{-1,0\},$$

and

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a} \quad \textit{is the logarithmic mean.}$$

Another result of this type obtained in the same paper is:

Theorem 1.3. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *(with* $a > 0$*) and differentiable on* (a,b). *If*

$$P(f') := \sup_{u \in (a,b)} |uf'(x)| < \infty,$$

then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{P(f')}{b-a} \left[\ln\left[\frac{[I(x,b)]^{b-x}}{[I(a,x)]^{x-a}}\right] + 2(x-A)\ln x \right] \tag{1.3}$$

for any $x \in (a,b)$, *where for* $a \neq b$

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad \textit{is the identric mean.}$$

If some local information around the point $x \in (a,b)$ is available, then we may state the following result as well [2].

Theorem 1.4. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *Let* $p \in (0,\infty)$ *and assume, for a given* $x \in (a,b)$, *we have that*

$$M_p(x) := \sup_{u \in (a,b)} \left\{ |x-u|^{1-p} |f'(u)| \right\} < \infty.$$

Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{p(p+1)(b-a)} \left[(x-a)^{p+1} + (b-x)^{p+1}\right] M_p(x). \tag{1.4}$$

For recent results in connection to Ostrowski's inequality see the papers [3],[4] and the monograph [5].

The main aim of this paper is to provide some complementary results, where instead of using Cauchy mean value theorem, we use Pompeiu mean Value Theorem to evaluate the integral mean of an absolutely continuous function. Applications for quadrature rules and particular instances of functions are given as well.

2. POMPEIU'S MEAN VALUE THEOREM

In 1946, Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [7, p. 83]).

Theorem 2.1. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ in (x_1,x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{2.1}$$

Proof. Define a real valued function F on the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$ by

$$F(t) = tf\left(\frac{1}{t}\right). \tag{2.2}$$

Since f is differentiable on $\left(\frac{1}{b}, \frac{1}{a}\right)$ and

$$F'(t) = f\left(\frac{1}{t}\right) - \frac{1}{t} f'\left(\frac{1}{t}\right), \tag{2.3}$$

then applying the mean value theorem to F on the interval $[x,y] \subset \left[\frac{1}{b}, \frac{1}{a}\right]$ we get

$$\frac{F(x) - F(y)}{x - y} = F'(\eta) \tag{2.4}$$

for some $\eta \in (x,y)$.

Let $x_2 = \frac{1}{x}$, $x_1 = \frac{1}{y}$ and $\xi = \frac{1}{\eta}$. Then, since $\eta \in (x,y)$, we have

$$x_1 < \xi < x_2.$$

Now, using (2.2) and (2.3) on (2.4), we have

$$\frac{xf\left(\frac{1}{x}\right) - yf\left(\frac{1}{y}\right)}{x - y} = f\left(\frac{1}{\eta}\right) - \frac{1}{\eta} f'\left(\frac{1}{\eta}\right),$$

that is

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

This completes the proof of the theorem. □

Remark 2.2. Following [7, p. 84 – 85], we will mention here a geometrical interpretation of Pompeiu's theorem. The equation of the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

This line intersects the $y-$axis at the point $(0,y)$, where y is

$$\begin{aligned} y &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(0 - x_1) \\ &= \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2}. \end{aligned}$$

The equation of the tangent line at the point $(\xi, f(\xi))$ is

$$y = (x - \xi) f'(\xi) + f(\xi).$$

The tangent line intersects the $y-$axis at the point $(0, y)$, where

$$y = -\xi f'(\xi) + f(\xi).$$

Hence, the geometric meaning of Pompeiu's mean value theorem is that the tangent of the point $(\xi, f(\xi))$ intersects on the $y-$axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

3. EVALUATING THE INTEGRAL MEAN

The following result holds.

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - \ell f'\|_\infty, \tag{3.1}$$

where $\ell(t) = t$, $t \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Applying Pompeiu's mean value theorem, for any $x, t \in [a, b]$, there is a ξ between x and t such that

$$tf(x) - xf(t) = [f(\xi) - \xi f'(\xi)](t - x)$$

giving

$$\begin{aligned} |tf(x) - xf(t)| &\leq \sup_{\xi \in [a,b]} |f(\xi) - \xi f'(\xi)| \, |x - t| \\ &= \|f - \ell f'\|_\infty |x - t| \end{aligned} \tag{3.2}$$

for any $t, x \in [a, b]$.

Integrating over $t \in [a, b]$, we get

$$\begin{aligned} \left| f(x) \int_a^b t dt - x \int_a^b f(t)\, dt \right| &\leq \|f - \ell f'\|_\infty \int_a^b |x - t|\, dt \\ &= \|f - \ell f'\|_\infty \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \\ &= \|f - \ell f'\|_\infty \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{3.3}$$

and since $\int_a^b t dt = \frac{b^2-a^2}{2}$, we deduce from (3.3) the desired result (3.1).

Now, assume that (3.2) holds with a constant $k > 0$, i.e.,

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{b-a}{|x|} \left[k + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - \ell f'\|_\infty, \tag{3.4}$$

for any $x \in [a, b]$.

Consider $f:[a,b]\to\mathbb{R}$, $f(t)=\alpha t+\beta;\ \alpha,\beta\neq 0$. Then

$$\begin{aligned}\|f-\ell f'\|_{\infty} &= |\beta|,\\ \frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{a+b}{2}\cdot\alpha+\beta,\end{aligned}$$

and by (3.4) we deduce

$$\left|\frac{a+b}{2}\left(\alpha+\frac{\beta}{x}\right)-\left(\frac{a+b}{2}\alpha+\beta\right)\right|\leq\frac{b-a}{|x|}\left[k+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|\beta|$$

giving

$$\left|\frac{a+b}{2}-x\right|\leq(b-a)\,k+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2 \tag{3.5}$$

for any $x\in[a,b]$.

If in (3.5) we let $x=a$ or $x=b$, we deduce $k\geq\frac{1}{4}$, and the sharpness of the constant is proved. $\square$

The following interesting particular case holds.

Corollary 3.2. *With the assumptions in Theorem 3.1, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)}{2\,|a+b|}\,\|f-\ell f'\|_{\infty}. \tag{3.6}$$

4. THE CASE OF WEIGHTED INTEGRALS

We will consider now the weighted integral case.

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. If $w:[a,b]\to\mathbb{R}$ is nonnegative integrable on $[a,b]$, then for each $x\in[a,b]$, we have the inequality:*

$$\begin{aligned}&\left|\int_a^b f(t)\,w(t)\,dt-\frac{f(x)}{x}\int_a^b tw(t)\,dt\right| \tag{4.1}\\ &\qquad\leq\|f-\ell f'\|_{\infty}\left[\operatorname{sgn}(x)\left(\int_a^x w(t)\,dt-\int_x^b w(t)\,dt\right)\right.\\ &\qquad\qquad\left.+\frac{1}{|x|}\left(\int_x^b tw(t)\,dt-\int_a^x tw(t)\,dt\right)\right].\end{aligned}$$

Proof. Using the inequality (3.2), we have

$$
\begin{aligned}
(4.2)\qquad & \left|f(x)\int_a^b tw(t)\,dt - x\int_a^b f(t)\,w(t)\,dt\right| \\
& \le \|f-\ell f'\|_\infty \int_a^b w(t)\,|x-t|\,dt \\
& = \|f-\ell f'\|_\infty \left[\int_a^x w(t)(x-t)\,dt + \int_x^b w(t)(t-x)\,dt\right] \\
& = \|f-\ell f'\|_\infty \left[x\int_a^x w(t)\,dt - \int_a^x tw(t)\,dt + \int_x^b tw(t)\,dt - x\int_x^b w(t)\,dt\right] \\
& = \|f-\ell f'\|_\infty \left[x\left(\int_a^x w(t)\,dt - \int_x^b w(t)\,dt\right) + \int_x^b tw(t)\,dt - \int_a^x tw(t)\,dt\right]
\end{aligned}
$$

from where we get the desired inequality (4.1). □

Now, if we assume that $0<a<b$, then

$$
a \le \frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt} \le b, \tag{4.3}
$$

provided $\int_a^b w(t)\,dt > 0$.

With this extra hypothesis, we may state the following corollary.

Corollary 4.2. *With the above assumptions, we have*

$$
\begin{aligned}
(4.4)\qquad & \left|f\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt}\right) - \frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)\,w(t)\,dt\right| \\
& \qquad \le \|f-\ell f'\|_\infty \left[\frac{\int_a^x w(t)\,dt - \int_x^b w(t)\,dt}{\int_a^b w(t)\,dt} + \frac{\int_x^b w(t)\,tdt - \int_a^x tw(t)\,dt}{\int_a^b tw(t)\,dt}\right].
\end{aligned}
$$

5. A QUADRATURE FORMULA

We assume in the following that $0<a<b$.

Consider the division of the interval $[a,b]$ given by

$$
I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,
$$

and $\xi_i \in [x_i, x_{i+1}]$, $i=0,\dots,n-1$ a sequence of intermediate points. Define the quadrature

$$
\begin{aligned}
(5.1)\qquad S_n(f,I_n,\xi) &:= \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i}\cdot\frac{x_{i+1}^2 - x_i^2}{2} \\
&= \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i}\cdot\frac{x_i + x_{i+1}}{2}\cdot h_i,
\end{aligned}
$$

where $h_i := x_{i+1} - x_i$, $i=0,\dots,n-1$.

The following result concerning the estimate of the remainder in approximating the integral $\int_a^b f(t)\,dt$ by the use of $S_n(f,I_n,\xi)$ holds.

Theorem 5.1. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then we have the representation*

$$\int_a^b f(t)\,dt = S_n(f,I_n,\xi)+R_n(f,I_n,\xi), \tag{5.2}$$

where $S_n(f,I_n,\xi)$ is as defined in (5.1), and the remainder $R_n(f,I_n,\xi)$ satisfies the estimate

$$\begin{aligned} |R_n(f,I_n,\xi)| &\le \|f-\ell f'\|_\infty \sum_{i=0}^{n-1}\frac{h_i^2}{\xi_i}\left[\frac{1}{4}+\left|\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h_i}\right|^2\right] \\ &\le \frac{1}{2}\|f-\ell f'\|_\infty \sum_{i=0}^{n-1}\frac{h_i^2}{\xi_i} \le \frac{\|f-\ell f'\|_\infty}{2a}\sum_{i=0}^{n-1}h_i^2. \end{aligned} \tag{5.3}$$

Proof. Apply Theorem 3.1 on the interval $[x_i,x_{i+1}]$ for the intermediate points ξ_i to obtain

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(t)\,dt-\frac{f(\xi_i)}{\xi_i}\cdot\frac{x_i+x_{i+1}}{2}\cdot h_i\right| \\ &\qquad\le \frac{1}{\xi_i}h_i^2\left[\frac{1}{4}+\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h_i}\right)^2\right]\|f-\ell f'\|_\infty \\ &\qquad\le \frac{1}{2\xi_i}h_i^2\|f-\ell f'\|_\infty \le \frac{1}{2a}h_i^2\|f-\ell f'\|_\infty \end{aligned} \tag{5.4}$$

for each $i\in\{0,\dots,n-1\}$.

Summing over i from 1 to $n-1$ and using the generalised triangle inequality, we deduce the desired estimate (5.3). □

Now, if we consider the mid-point rule (i.e., we choose $\xi_i=\frac{x_i+x_{i+1}}{2}$ above, $i\in\{0,\dots,n-1\}$)

$$M_n(f,I_n):=\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i,$$

then, by Corollary 3.2, we may state the following result as well.

Corollary 5.2. *With the assumptions of Theorem 5.1, we have*

$$\int_a^b f(t)\,dt = M_n(f,I_n)+R_n(f,I_n), \tag{5.5}$$

where the remainder satisfies the estimate:

$$\begin{aligned} |R_n(f,I_n)| &\le \frac{\|f-\ell f'\|_\infty}{2}\sum_{i=0}^{n-1}\frac{h_i^2}{x_i+x_{i+1}} \\ &\le \frac{\|f-\ell f'\|_\infty}{4a}\sum_{i=0}^{n-1}h_i^2. \end{aligned} \tag{5.6}$$

6. APPLICATIONS FOR SPECIAL MEANS

For $0<a<b$, let us consider the means

$$\begin{aligned} A &= A(a,b) := \frac{a+b}{2}, \\ G &= G(a,b) := \sqrt{a\cdot b}, \\ H &= H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \\ L &= L(a,b) := \frac{b-a}{\ln b-\ln a} \quad \text{(logarithmic mean)}, \\ I &= I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \quad \text{(identric mean)} \end{aligned}$$

and the $p-$logarithmic mean

$$L_p = L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \quad p\in\mathbb{R}\setminus\{-1,0\}.$$

It is well known that

$$H \le G \le L \le I \le A,$$

and, denoting $L_0 := I$ and $L_{-1} = L$, the function $\mathbb{R}\ni p\mapsto L_p\in\mathbb{R}$ is monotonic increasing.

In the following we will use the following inequality obtained in Corollary 3.2,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)}{2(a+b)}\left\|f-\ell f'\right\|_{\infty}, \tag{6.1}$$

provided $0<a<b$.

(1) Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$, $f(t)=t^p$, $p\in\mathbb{R}\setminus\{-1,0\}$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= [A(a,b)]^p, \\ \frac{1}{b-a}\int_a^b f(t)\,dt &= L_p^p(a,b), \\ \left\|f-\ell f'\right\|_{[a,b],\infty} &= \begin{cases} (1-p)\,a^p & \text{if } p\in(-\infty,0)\setminus\{-1\}, \\ |1-p|\,b^p & \text{if } p\in(0,1)\cup(1,\infty). \end{cases} \end{aligned}$$

Consequently, by (6.1) we deduce

$$\left|A^p(a,b)-L_p^p(a,b)\right| \le \frac{1}{4}\times\begin{cases} \dfrac{(1-p)\,a^p\,(b-a)}{A(a,b)} & \text{if } p\in(-\infty,0)\setminus\{-1\}, \\ \dfrac{|1-p|\,b^p\,(b-a)}{A(a,b)} & \text{if } p\in(0,1)\cup(1,\infty). \end{cases} \tag{6.2}$$

(2) Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$, $f(t)=\frac{1}{t}$. Then

$$f\left(\frac{a+b}{2}\right)=\frac{1}{A(a,b)},$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{L(a,b)},$$
$$\|f-\ell f'\|_{[a,b],\infty}=\frac{2}{a}.$$

Consequently, by (6.1) we deduce

$$0\le A(a,b)-L_p(a,b)\le\frac{b-a}{2a}L(a,b). \tag{6.3}$$

(3) Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$, $f(t)=\ln t$. Then

$$f\left(\frac{a+b}{2}\right)=\ln[A(a,b)],$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt=\ln[I(a,b)],$$
$$\|f-\ell f'\|_{[a,b],\infty}=\max\left\{\left|\ln\left(\frac{a}{e}\right)\right|,\left|\ln\left(\frac{b}{e}\right)\right|\right\}.$$

Consequently, by (6.1) we deduce

$$1\le\frac{A(a,b)}{I(a,b)}\le\exp\left\{\frac{b-a}{4A(a,b)}\max\left\{\left|\ln\left(\frac{a}{e}\right)\right|,\left|\ln\left(\frac{b}{e}\right)\right|\right\}\right\}. \tag{6.4}$$

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